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A reaction–diffusion system approximation

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1 Introduction

The theory of reaction–diffusion systems has been developed in the last two decades. Especially, a singular limit analysis reveals various behavior of solutions to us. It has mainly been used for purpose of studying the dynamics of the specific reaction–diffusion system.

In this paper, the following type of equations is called a *reaction–diffusion system*:

$$\boldsymbol{u}_t = \boldsymbol{D}\triangle\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}), \tag{1.1}$$

in $x \in \Omega \subset \mathbf{R}^N$, $t > 0$ with the Neumann homogeneous boundary condition and initial condition where

$$\boldsymbol{u} = {}^t(u_1, \cdots, u_M), \quad \boldsymbol{f}(\boldsymbol{u}) = {}^t(f_1(\boldsymbol{u}), \cdots, f_M(\boldsymbol{u})),$$

and $\boldsymbol{D}$ is a diagonal matrix whose elements are positive (or non-negative). A reaction–diffusion system consists of two parts: one is a kinetic term $\boldsymbol{f}$; the other is a diffusion one $\boldsymbol{D}\triangle$. One might think that the diffusion term makes the solution spatially homogenize. After discovering Turing's instability, it turns out that one's intuition might not apply all the reaction–diffusion system (cf. [14, 8, 12]).

We encounter the questions: How wide is a class of reaction–diffusion systems?, or How rich are the dynamics of reaction–diffusion systems? Thus we study the relationship between reaction–diffusion systems and the following two systems:

(i) the two–phase Stefan problem,

(ii) the cross-diffusion system.

Obviously, these systems do not belong to a class of reaction–diffusion systems. In this paper, new types of reaction–diffusion systems with a small parameter are proposed, which converge to the above two system (i) and (ii) respectively. That is, by the singular limit analysis, it is shown that any solution of these systems converges to that of (i) or (ii) respectively. In other word, the above two system (i) and (ii) can be embedded in the class of reaction–diffusion systems.

2 Main Results

2.1 A reaction–diffusion approximation to the two–phase Stefan problem

We consider the two–phase Stefan problem with reaction terms:

$$\begin{cases} u_t = d_1 \triangle u + f(u) & \text{in } \Omega_u(t), \\ v_t = d_2 \triangle v + g(v) & \text{in } \Omega_v(t), \\ \lambda V_n = -\dfrac{d_1}{s_1}\dfrac{\partial u}{\partial n} - \dfrac{d_2}{s_2}\dfrac{\partial v}{\partial n} & \text{on } \Gamma(t), \\ u = v = 0 & \text{on } \Gamma(t), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

for $t \in (0, T]$ where d_i, s_i $(i = 1, 2)$ and λ are positive constants and ν is the outward normal vector to $\partial\Omega$. The region Ω is divided into two subregion $\Omega_u(t)$ and $\Omega_v(t)$. In $\Omega_u(t)$, $u > 0$ and $v = 0$ and in $\Omega_v(t)$, $v > 0$ and $u = 0$. The boundary between $\Omega_u(t)$ and $\Omega_v(t)$ are denoted by $\Gamma(t)$, which is called an *interface*. Namely,

$$\Omega = \Omega_u(t) \cup \Omega_v(t) \cup \Gamma(t).$$

The unit normal vector on $\Gamma(t)$ oriented from $\Omega_u(t)$ to $\Omega_v(t)$ is denoted by n and the normal velocity of the interface $\Gamma(t)$ is by V_n. The assumptions of reaction terms f and g are specified later. If we choose $s_1 = s_2 = 1, f = g = 0$ and regard $u - v$ as the temperature, the system (2.1) is called the *classical two–phase Stefan problem.*

To approximate the system (2.1), Hilhorst, Iida, Ninomiya and Mimura have proposed the following reaction–diffusion system in [7]:

$$\begin{cases} u_t = d_1 \triangle u + f(u) - \dfrac{s_1 uv}{\epsilon} - \dfrac{\lambda s_1 (1-w)u}{\epsilon}, & x \in \Omega,\ t > 0, \\ v_t = d_2 \triangle v + g(v) - \dfrac{s_2 uv}{\epsilon} - \dfrac{\lambda s_2 wv}{\epsilon}, & x \in \Omega,\ t > 0, \\ w_t = \dfrac{(1-w)u}{\epsilon} - \dfrac{wv}{\epsilon}, & x \in \Omega,\ t > 0, \end{cases} \tag{2.2}$$

with the Neumann boundary condition and the initial condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T], \tag{2.3}$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x) \qquad \text{in } \Omega. \tag{2.4}$$

Since $w(x, t)$ behaves like an indicator of $\Omega_u(t)$, it is natural to assume that

$$\begin{aligned} v_0(x) = 0, \quad w_0(x) = 1, \qquad & \text{if } u_0(x) > 0, \\ u_0(x) = 0, \quad w_0(x) = 0, \qquad & \text{if } v_0(x) > 0. \end{aligned}$$

By introducing the new variable $p = 1 - w$, (2.1) is replaced by

$$\begin{cases} u_t = d_1 \Delta u + f(u) - \dfrac{s_1 uv}{\epsilon} - \dfrac{\lambda s_1 pu}{\epsilon}, & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v + g(v) - \dfrac{s_2 uv}{\epsilon} - \dfrac{\lambda s_2 wv}{\epsilon}, & x \in \Omega,\ t > 0, \\ w_t = \dfrac{pu}{\epsilon} - \dfrac{wv}{\epsilon}, & x \in \Omega,\ t > 0, \\ p_t = -\dfrac{pu}{\epsilon} + \dfrac{wv}{\epsilon}, & x \in \Omega,\ t > 0. \end{cases} \tag{2.5}$$

This system can be regard as the model one of the following chemical reaction:

$$\begin{aligned} U + P &\longrightarrow W + Z_1, \\ V + W &\longrightarrow P + Z_2, \\ U + V &\longrightarrow Z_3, \end{aligned}$$

where U, V, W, P represent the chemical substances whose densities are u, v, w, p respectively and Z_i $(i = 1, 2, 3)$ means that expect for U, V, W and P.

To state our main result, we impose the assumptions on f, g and the initial datum (u_0, v_0, w_0).

A1 (Assumption on f and g)
There exist C^1-functions $\tilde{f}(u)$ and $\tilde{g}(u)$ and positive constants K_1 and K_2 such that

$$\begin{aligned} &f(u) = \tilde{f}(u)u, \quad g(u) = \tilde{g}(u)u, \\ &\tilde{f}(u) \le 0 \text{ for } u \ge K_1, \quad \tilde{g}(u) \le 0 \text{ for } u \ge K_2. \end{aligned}$$

A2 (Assumption on the initial datum)

$$\begin{aligned} &(u_0, v_0, w_0) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega), \\ &0 \le u_0(x) \le \alpha, \quad 0 \le v_0(x) \le \beta, \quad 0 \le w_0(x) \le 1 \qquad \text{in } \Omega, \\ &u_0 v_0 = (1 - w_0)u_0 = w_0 v_0 = 0 \qquad \text{in } \Omega, \end{aligned}$$

for some positive constants α and β.

Set

$$\begin{aligned} \Gamma_0 &:= \{x \in \Omega \mid\ u_0(x) = v_0(x) = 0\}, \\ Q_T &:= \Omega \times [0, T]. \end{aligned}$$

THEOREM 2.1. *Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \ge 1)$ with C^2-boundary $\partial\Omega$. Assume* **A1** *and* **A2**. *Let $(u^\epsilon, v^\epsilon, w^\epsilon)$ be the solution of* (2.2) *in Q_T with* (2.3)–(2.4).

Then there exists $(u, v, w) \in L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega)) \times L^2(Q_T)$ *such that*

$$\begin{array}{ll} u^\epsilon \longrightarrow u, & v^\epsilon \longrightarrow v \qquad \text{in } L^2(Q_T), \\ w^\epsilon \longrightarrow w & \text{weakly in } L^2(Q_T) \end{array}$$

as $\epsilon \to +0$, *and*

$$\Omega_u \cap \Omega_v = \emptyset,$$
$$w = \begin{cases} 1 & \text{in } \Omega_u, \\ 0 & \text{in } \Omega_v, \end{cases}$$

where

$$\Omega_u := \{(x,t) \in Q_T |\ u(x,t) > 0\} = \{(x,t) \in Q_T |\ x \in \Omega_u(t),\ 0 \le t \le T\},$$
$$\Omega_v := \{(x,t) \in Q_T |\ v(x,t) > 0\} = \{(x,t) \in Q_T |\ x \in \Omega_v(t),\ 0 \le t \le T\}.$$

Moreover, if

$$\Gamma := Q_T \setminus (\Omega_u \cup \Omega_v) = \{(x,t) \in Q_T \mid x \in \Gamma(t),\ 0 \le t \le T\}$$

is a smooth hypersurface satisfying $\Gamma(t) \subset\subset \Omega$ *for* $0 \le t \le T$ *and if* u *(resp.* v*) is smooth on* $\overline{\Omega_u}$ *(resp.* $\overline{\Omega_v}$*), then* (Γ, u, v) *is the unique solution of the free boundary problem* (2.1) *and the initial conditions*

$$\Gamma(0) = \Gamma_0,$$
$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in } \Omega.$$

This theorem implies that we can derive the classical two–phase Stefan problem from the reaction–diffusion system (2.2) in the absence of the reaction terms f and g, taking the limit ϵ tends to zero. The parameter λ in (2.1) corresponds to the latent heat in the Stefan problem. For sufficiently small ϵ, one can expect that u and v exhibit corner layers on the interface $\Gamma(t)$, while w has a sharp transition layer. Along the same line, the one-phase Stefan problem can be discussed. We refer to the papers by Hilhorst, van der Hout and Peletier [4, 5, 6] and by Eymard, Hilhorst, van der Hout and Peletier [3].

2.2 A reaction–diffusion approximation to the cross–diffusion system

The following cross–diffusion system is considered in this subsection:

$$\begin{cases} u_t^* = d_1 \Delta u^* + \Delta a_1(u^*) + b_1(u^*, v^*), & x \in \Omega, t > 0 \\ v_t^* = d_2 \Delta v^* + \Delta a_2(u^*, v^*) + b_2(u^*, v^*), & x \in \Omega, t > 0, \end{cases} \tag{2.6}$$

with the boundary condition

$$\frac{\partial u^*}{\partial n} = 0, \ \frac{\partial v^*}{\partial n} = 0 \quad \text{for } x \in \partial\Omega, t > 0. \tag{2.7}$$

and the initial condition

$$u^*(x,0) = u_0(x), \ v^*(x,0) = v_0(x) \quad \text{for } x \in \Omega, \tag{2.8}$$

where d_1 and d_2 are positive constants. Next, we consider the auxiliary parabolic system:

$$\begin{cases} u_t = d_1\Delta u + \Delta w + \tilde{b}_1(u,v), & x \in \Omega, t > 0, \\ v_t = d_2\Delta v + \Delta z + \tilde{b}_2(u,v), & x \in \Omega, t > 0, \\ w_t = 2d_1\Delta w + \dfrac{1}{\epsilon}(\tilde{a}_1(u) - w), & x \in \Omega, t > 0, \\ z_t = 2d_2\Delta z + \dfrac{1}{\epsilon}(\tilde{a}_2(u,v) - z), & x \in \Omega, t > 0, \end{cases} \tag{2.9}$$

with the boundary condition

$$\frac{\partial u}{\partial n} = 0, \ \frac{\partial v}{\partial n} = 0, \ \frac{\partial w}{\partial n} = 0, \ \frac{\partial z}{\partial n} = 0 \quad \text{for } x \in \partial\Omega, t > 0, \tag{2.10}$$

and the initial condition

$$\begin{cases} u(x,0) = u_0(x), \ v(x,0) = v_0(x) \\ w(x,0) = \tilde{a}_1(u_0(x)), \ z(x,0) = \tilde{a}_2(u_0(x), v_0(x)) \quad \text{for } x \in \Omega. \end{cases} \tag{2.11}$$

It is easy to expect that the solutions of (2.9) converge to those of (2.6) as $\epsilon \downarrow 0$, if $\tilde{a}_i = a_i$ and $\tilde{b}_i = b_i$ $(i = 1, 2)$. By the technical reason, however, a_i and b_i $(i = 1, 2)$ should be modified outside of the large rectangle in the u-v plane.

THEOREM 2.2. *Assume*

$$\frac{\partial}{\partial u}a_1(r) \geq 0, \quad \frac{\partial}{\partial v}a_2(r,s) \geq 0, \quad \textit{in } (r,s) \in [0, R_1] \times [0, R_2] \tag{2.12}$$

for some positive constants R_1 and R_2. Let $(u^*, v^*) = (u^*(x,t), v^*(x,t))$ be a smooth solution of (2.6)–(2.8) *satisfying*

$$0 \leq u^*(x,t) \leq R_1, \quad 0 \leq v^*(x,t) \leq R_2$$

in $t \in [0,T]$ for some positive T and

$$\|u^*\|_{C^0(\bar{\Omega}\times[0,T])} + \|\nabla u^*\|_{C^0(\bar{\Omega}\times[0,T])} + \|v^*\|_{C^0(\bar{\Omega}\times[0,T])} + \|\nabla v^*\|_{C^0(\bar{\Omega}\times[0,T])} < \infty.$$

Then there exist functions $\tilde{a}_1(u), \tilde{a}_2(u,v), \tilde{b}_1(u,v)$ *and* $\tilde{b}_2(u,v)$ *such that the solution* $(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon) = (u^\epsilon(x,t), v^\epsilon(x,t), w^\epsilon(x,t), z^\epsilon(x,t))$ *to* (2.9)–(2.11) *satisfies*

$$\|u^\epsilon - u^*\|_{C^0([0,T];L^2(\Omega))} \le C\epsilon,$$
$$\|v^\epsilon - v^*\|_{C^0([0,T];L^2(\Omega))} \le C\epsilon.$$

See the details in [9].

REMARK 2.3. The functions $\tilde{a}_i$ and $\tilde{b}_i$ $(i = 1, 2)$ are the modified ones of a_i and b_i satisfying

$$\begin{aligned} \tilde{a}_i &= a_i \quad \text{in } (u,v) \in [0, R_1] \times [0, R_2], \\ \tilde{b}_i &= b_i \quad \text{in } (u,v) \in [0, R_1] \times [0, R_2]. \end{aligned}$$

By applying the linear transformation

$$\tilde{u} = u - \frac{w}{d_1}, \quad \tilde{v} = v - \frac{z}{d_2}, \quad \tilde{w} = w, \quad \tilde{z} = z,$$

i.e.,

$$u = \tilde{u} + \frac{\tilde{w}}{d_1}, \quad v = \tilde{v} + \frac{\tilde{z}}{d_2}, \quad w = \tilde{w}, \quad z = \tilde{z},$$

(2.9) is replaced by the following reaction–diffusion system

$$\begin{cases} \tilde{u}_t = d_1 \Delta \tilde{u} + \tilde{b}_1(\tilde{u} + \frac{\tilde{w}}{d_1}, \tilde{v} + \frac{\tilde{z}}{d_2}) - \frac{1}{d_1 \epsilon}\big(\tilde{a}_1(\tilde{u} + \frac{\tilde{w}}{d_1}) - \tilde{w}\big), \\ \tilde{v}_t = d_2 \Delta \tilde{v} + \tilde{b}_2(\tilde{u} + \frac{\tilde{w}}{d_1}, \tilde{v} + \frac{\tilde{z}}{d_2}) - \frac{1}{d_2 \epsilon}\big(\tilde{a}_2(\tilde{u} + \frac{\tilde{w}}{d_1}, \tilde{v} + \frac{\tilde{z}}{d_2}) - \tilde{z}\big), \\ \tilde{w}_t = 2d_1 \Delta \tilde{w} + \frac{1}{\epsilon}\big(\tilde{a}_1(\tilde{u} + \frac{\tilde{w}}{d_1}) - \tilde{w}\big), \\ \tilde{z}_t = 2d_2 \Delta \tilde{z} + \frac{1}{\epsilon}\big(\tilde{a}_2(\tilde{u} + \frac{\tilde{w}}{d_1}, \tilde{v} + \frac{\tilde{z}}{d_2}) - \tilde{z}\big), \end{cases} \tag{2.13}$$

in $x \in \Omega$, $t > 0$. This implies that the dynamics of cross–diffusion system can be embedded in that of reaction–diffusion system.

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