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Stationary solutions of a bistable equation with clustering layers and spikes

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Abstract

We consider stationary solutions of a spatially inhomogeneous Allen-Cahn type nonlinear diffusion equation in one space dimension. The equation involves a small parameter ϵ, and its nonlinearity has the form $h(x)^2 f(u)$, where $h(x)$ represents the spatial inhomogeneity and $f(u)$ is derived from a double-well potential with equal well-depth. When ϵ is very small, stationary solutions develop transition layers that can possibly cluster in the spatial region. We first show that those transition layers can appear only near the local minimum and local maximum points of the coefficient $h(x)$ and that at most a single layer can appear near each local minimum point of $h(x)$. We then discuss the stability of layered stationary solutions and prove that the Morse index of a solution coincides with the total number of its layers that appear near the local maximum points of $h(x)$. We also show the existence of a stationary solution that has layers at any given set of local minimum and local maximum points of $h(x)$ with the multiplicity of layers being arbitrary at the local maximum points.

1 Introduction

Some classes of reaction-diffusion systems give rise to sharp transition layers when the diffusion coefficients are very small. Such phenomena have long been known in physics, biology and other areas of science. Since the middle of 1980's, they have become a subject of intensive mathematical study, and the nature of those layers — their motion, location and stability — is now well understood by using various techniques of singular perturbation theory. However, most of those studies have been focused on isolated layers whose interaction with other layers (if they exist) is negligible, and little is known about the situation in which many layers appear within a relatively small distance from one another.

We consider multi-layered stationary solutions for a spatially inhomogeneous Allen-Cahn equation of the form

$$\begin{cases} \epsilon u_t = \epsilon u_{xx} + \dfrac{1}{\epsilon} h(x)^2 f(u) & (0 < x < 1,\ t > 0) \\ u_x(0,t) = u_x(1,t) = 0 & (t > 0) \\ u(x,0) = u_0(x) & (0 < x < 1). \end{cases} \tag{1.1}$$

Here ϵ is a small parameter and the coefficient $h(x) > 0$ represents spatial inhomogeneity of the diffusive medium. f satisfies the following conditions.

(F1) f has precisely three zeros $\alpha^- < 0 < \alpha^+$ and satisfies

$$f'(\alpha^-) < 0, \quad f'(0) > 0, \quad f'(\alpha^+) < 0,$$

(F2) $\displaystyle\int_{\alpha-}^{\alpha^+} f(u)du = 0,$

(F3) $\dfrac{f(u)}{u} > f'(u)$ $(u \neq 0)$.

Note that (F1) implies that (1.1) has a double-well potential (namely $W(x,u)$ defined by (1.4)) and (F2) implies that the two wells are of equal depth.

Our goal is :

(a) to find out where stationary layers appear;

(b) to show that in certain circumstances multiple stationary layers appear within a very small distance from one another (clustering layers);

(c) to study the stability of multi-layered stationary solutions; in particular, to determine the Morse index of such solutions from the information about the location of the layers.

Let us formulate our problem more precisely. The stationary problem for (1.1) is written in the following form:

$$\begin{cases} \epsilon u'' + \dfrac{1}{\epsilon} h(x)^2 f(u) = 0 & (0 < x < 1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.2}$$

In an earlier work [7] the author has shown the existence of stable layered solutions of (1.2) whose layers appear near the local minimum points of $h(x)$. More precisely, we have shown the following theorem in [7].

Theorem 1.1 *Let* $\{x_1, x_2, \cdots, x_m\}$ *be an arbitrary subset of the set of strict local minimum points of* $h(x)$. *Then there exists a stable solution of* (1.2) *which has one layer near each* x_k $(k = 1, 2, \cdots, m)$ *and has no layer in the rest of the interval* $(0, 1)$.

Let us explain briefly why the local minima of $h(x)$ are relevant. Solutions of (1.2) are critical points of the functional

$$E(u) = \int_0^1 \left(\frac{\epsilon}{2}(u')^2 + \frac{1}{\epsilon}W(x,u)\right) dx, \tag{1.3}$$

where the potential $W(x,u)$ is given by

$$W(x,u) = -\int_{\alpha^-}^{u} h(x)^2 f(v)dv. \tag{1.4}$$

This potential has two minimal values at $u = \alpha^-$ and $u = \alpha^+$, so it is so-called a double-well potential. And since (F2) implies

$$W(x,\alpha^-) = W(x,\alpha^+) \ (= 0),$$

this potential well has equal depth at its minima.

Now, stable solutions of (1.2) are local minimizers of $E(u)$. In order to minimize the energy $E(u)$, the function $u(x)$ tends to have values very close to α^- or α^+ in most of the spatial region, while at those places where $u(x)$ has a transition layer, $h(x)$ plays an important role since the quantity $W(x,u)$ is no longer small on the transition layer. Roughly speaking, $h(x)$ represents the density of the 'localized energy' at each transition layer. As the integrand in (1.3) is almost zero in the intervals away from the transition layers, the energy $E(u)$ is almost equal to the sum of these localized energies at the transition layers. Thus, in order to locally minimize the energy $E(u)$, the function $u(x)$ can have transition layers only near the local minimum points of $h(x)$. (This claim will be proved rigorously in Section 5 of the present paper; see Corollary 4.) What Theorem 1.1 shows is that there are indeed such stable solutions that have transition layers at any prescribed minimum points of $h(x)$.

Conversely, if a critical point $u(x)$ has transition layers only near the local minimum points of $h(x)$, then the above intuitive argument suggests that $u(x)$ is a local minimizer. The results by Ei, Iida and Yanagida [4] on the motion of interfaces for equation (1.1) rigorously justifies this observation, at least partially (they deal with even higher dimensional cases). The paper by Norbury and Yeh [10] also confirms the above observation through a formal asymptotic analysis and numerical simulations. Let us also mention a related paper by do Nascimento [9], who makes further stability analysis of layered solutions in several space dimensions.

The above arguments and results are basically concerened with solutions whose layers appear at isolated locations. However, if some layers cluster at certain points, then the situation becomes more complicated, and one needs to do more delicate analysis.

The contents of this article is as follows: Section 2 is a preliminary section, where we state basic properties of n-mode solutions. In Section 3, we show that layers can appear only in a small neighborhood of the set of local minimum and local maximum points of $h(x)$ (Theorem 1) and that at most a single layer can appear near each local minimum point of $h(x)$ (Theorem 2).

In Section 4, we discuss the stability of layered solutions and show that the Morse index of any such solution coincides with the number of its layers that appear near the local maximum points of $h(x)$ (Theorem 3). This theorem not only completely characterizes the stability of solutions in terms of the location of its layers, but also it implies one important fact: *the linearized operator associated with any layered stationary solution has no zero eigenvalue.*

Because of this remarkable property, layered solutions with a given qualitative profile persist under a small perturbation of the equation. Using this property, we will prove in Section 5 the existence of a solution of (1.2) having an arbitrary number of clustering layers at an arbitrarily chosen set of local maximum points of $h(x)$ while having a single layer at an arbitrarily chosen set of minimum points of $h(x)$ (Theorem 5). The strategy for proving this theorem is to use a homotopy argument. Details of the above arguments are given in [8].

Lastly we should mention some related results on multi-layered solutions. When the present work was nearly completed, the author was informed that Ai and Hastings were also completing a paper [1] showing the existence of solutions with clustering layers for a different type of nonlinearity of the form

$$f(x,u) = -\lambda u + u^3 + \cos x.$$

Unlike the present paper, their result deals with the "unbalanced" case, namely the case where the corresponding potential $W(x,u)$ has two wells of generally unequal depth with their balance varying from place to place. This situation is essentially the same as the equation studiend by Angenent, Maret-Paret and Peletier [3], who proved the existence of stable solutions with multiple (but non-clustering) layers. In this unbalanced case, so-called spikes can also appear. In a forthcoming paper [6], we will also deal with the unbalanced case, but in a more general form, and study the existence and stability of multi-layered solutions.

2 Preliminaries

In this section we will state basic properties of n-mode solutions.

Definition 2.1 *u is called an n-mode solution if u is a solution of* (1.2) *that has precisely n zeros in the interval* $(0,1)$.

One can show by using bifurcation techniques the following proposition:

Proposition 2.2 *For each $n \in \mathbf{N}$, there exists an $\epsilon_n > 0$ such that for $0 < \epsilon < \epsilon_n$, problem* (1.2) *has at least two n-mode solutions.*

In the rest of this article $n \in \mathbf{N}$ is fixed arbitrarily and any n-mode solution for sufficiently small ϵ is denoted by u_ϵ.

Proposition 2.3 (*Shape of layers*) *Let x_ϵ be an arbitrary zero point of u_ϵ such that $\frac{d}{dx}u_\epsilon(x_\epsilon) > 0$ (resp. $\frac{d}{dx}u_\epsilon(x_\epsilon) < 0$). Set $v_\epsilon(z) = u_\epsilon(\frac{\epsilon}{h(x_\epsilon)}z + x_\epsilon)$. Then $v_\epsilon(z)$ converges to $\phi(z)$ (resp. $\phi(-z)$) uniformly on every compact set of $\mathbf{R}$, where $\phi(z)$ is the unique solution of the following problem:*

$$\begin{cases} \phi'' + f(\phi) = 0 & (-\infty < z < \infty), \\ \phi'(-\infty) = \alpha^-, \quad \phi'(\infty) = \alpha^+, & \\ \phi(0) = 0. & \end{cases} \tag{2.1}$$

Roughly speaking, the above proposition implies that a transition layer appears around each zero of the solution u_ϵ, and that the shape of the transition layer is given by squeezing ϕ holizontally by the scale $\epsilon/h(x_\epsilon)$.

The following proposition, on the other hand, shows that u_ϵ stays very close to α^+ or α^- in any interval distant from $S_\epsilon = \{x \in (0,1);\ u_\epsilon(x) = 0\}$.

Proposition 2.4 *There exist constants $0 < C_1 < C_2$ and $0 < K_2 < K_1$ such that for sufficiently small $\epsilon > 0$, the following holds:*

$$C_1 \exp(-\frac{K_1 d(x)}{\epsilon}) \le u_\epsilon(x) - \alpha^- \le C_2 \exp(-\frac{K_2 d(x)}{\epsilon}) \quad \textit{for} \quad x \in S^-, \tag{2.2}$$

$$C_1 \exp(-\frac{K_1 d(x)}{\epsilon}) \le \alpha^+ - u_\epsilon(x) \le C_2 \exp(-\frac{K_2 d(x)}{\epsilon}) \quad \textit{for} \quad x \in S^+, \tag{2.3}$$

where $d(x) = dist(x, S_\epsilon)$, $S^- = \{x \in (0,1);\ u(x) < 0\}$ and $S^+ = \{x \in (0,1);\ u(x) > 0\}$.

3 Where do layers appear?

Set

$$\tilde{\mathcal{M}} = \{x \in (0,1);\ h'(x) = 0\},$$

and let $\mathcal{M}$ be the set of local minimum and maximum points of $h(x)$ in $(0,1)$. Clearly we have $\mathcal{M} \subset \tilde{\mathcal{M}}$. We assume that

(M1) $\tilde{\mathcal{M}}$ is a finite set;

(M2) $h''(x) \neq 0$ for each $x \in \mathcal{M}$.

In what follows we fix $n \in \mathbf{N}$ arbitrarily and discuss the behavior of n-mode solutions when ϵ is very small.

Propositions 2.3 and 2.4 imply that there is one-to-one correspondence between layers of u_ϵ and zeros of u_ϵ. The following theorem tells where transition layers appear.

Theorem 1 (*location*) *Assume (M1) and (M2). Then there exists a constant $C_0 > 0$ such that for $\epsilon > 0$ sufficiently small, any zero point of u_ϵ lies in $C_0\epsilon|\log\epsilon|$ neighbourhood of $\mathcal{M} \cup \{0,1\}$.*

The above theorem shows that transition layers can appear only near the internal local extremum points of $h(x)$ or near the boundary.

Theorem 2 (*multiplicity of internal layers*) *Assume (M1) and (M2). Then for $\epsilon > 0$ sufficiently small, there appears at most a single layer of u_ϵ near each local minimum point of $h(x)$ in $(0,1)$. Here the term "near" means a distance not exceeding $C_0\epsilon|\log\epsilon|$.*

As a consequence of the above theorems, we see that, if n is larger than the total number of local minimum points and maximum points of $h(x)$, then multiple layers appear near some of the local maximum points of $h(x)$. The following corollary follows from the proof of Theorem 2.

Corollary 3.1 (*boundary layers*) *If the boundary point $x = 0$ is a local minimum point of $h(x)$, then for sufficiently small $\epsilon > 0$, no layer of u_ϵ appears near $x = 0$. The same statement holds true for $x = 1$.*

4 Stability of layered solutions

In this section we mainly consider solution without boundary layers, but we can allow boundary layers provided that

$$(H1) \qquad\qquad h'(0) = h'(1) = 0.$$

Throughout this section we will assume (H1). The following theorem completely characterizes the stability of multi-layered solutions in terms of the location of layers:

Theorem 3 *Let u_ϵ be an $n-$mode solution of (1.2) for sufficiently small $\epsilon > 0$ and let m denote the total number of layers that appear near the local maximum points of $h(x)$. Then u_ϵ is nondegenerate and m coincides with the Morse index of u_ϵ.*

Corollary 4 *Let u_ϵ be an $n-$mode solution of* (1.2) *for $\epsilon > 0$ sufficiently small. Then u_ϵ is stable, or equivalently, a local minimizer of $E(u)$ in* (1.3) *if and only if its layers appear only near the local minimum points of $h(x)$.*

Here the Morse index of u is defined as follows: Set

$$Lw = \epsilon^2 w_{xx} + h(x)^2 f'(u)w, \tag{4.1}$$

and consider the following linearized eigenvalue problem:

$$\begin{cases} Lw = -\lambda w & \text{in} \quad (0,1), \\ u_y(0) = u_y(1) = 0. \end{cases} \tag{4.2}$$

Definition 4.1 *We say that the Morse index of u is m if* (4.2) *has m negative eigenvalues and all the other eigenvalues are positive.*

The proof of Theorem 3 will be given in Section 6 below.

5 Existence of layered solutions

In this section we will show the existence of solutions that have layers near arbitrary (internal) local minimum and local maximum points of $h(x)$.

Let $\mathcal{M}_0 = \{x_i\}_{i=1}^k$ be an any subset of $\mathcal{M}$. Set

$$I^+ = \{i \in \{1, 2, \cdots, k\};\ x_i \text{ is a local maximum point of } h\ \},$$

$$I^- = \{i \in \{1, 2, \cdots, k\};\ x_i \text{ is a local minimum point of } h\ \}.$$

Define

$$\begin{aligned} \mathcal{P} = \{(p_i)_{i=1}^k \in (\mathrm{N})^k;\ \ p_i &= 1 \quad \text{if} \quad i \in I^- \quad \text{and} \\ p_i &\quad \text{is a positive integer if } i \in I^+\}. \end{aligned}$$

Theorem 5 *For any $(p_i)_{i=1}^k \in \mathcal{P}$, there exists an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$* (1.2) *has a solution with exactly p_i layers near x_i $(i = 1, 2, \cdots, k)$,*

Remark 5.1 *The term 'near' in Theorem 5 indicates $O(\epsilon|\log\epsilon|)$ neighbourhood of the mentioned points (see Proposition 2.4). Note that, by Theorem 2, at most one layer can appear near local minimum point of $h(x)$. Therefore the set $\mathcal{P}$ defined above is an optimal class for the multiplicity of layers.*

Remark 5.2 *For each $(p_i)_{i=1}^k \in P$, there exist at least two solutions that satisfy the properties in Theorem 5. One of these solutions is nearly equal to α^- at $x = 0$, while the other is nearly equal to α^+ at $x = 0$.*

6 Proof of Theorem 3

For the proof of Theorem 3, we use the min-max characterization of eigenvalues and the Sturm-Liouville theory for second order ODE's. The following propositions briefly explain these theories. The proofs of these propositions are rather standard, so we omit the proof.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the eigenvalues of (4.2).

Proposition 6.1 (*min-max principle*) *λ_n is characterized as follows:*

$$\lambda_k = \sup_{\psi_1,\cdots,\psi_{k-1} \in L^2(0,1)} \inf_{w \in X[\psi_1,\cdots,\psi_{k-1}]} \frac{\mathcal{H}(w)}{||w||^2_{L^2(0,1)}}, \tag{6.1}$$

where

$$X[\psi_1,\cdots,\psi_{k-1}] = \{\, w \in H^1(0,1)\backslash\{0\};\ w \perp \psi_j \quad for \quad j = 1,\cdots,k-1\,\},$$

$$\mathcal{H}(w) = \int_0^1 \left(\epsilon^2|\nabla w|^2 - h(x)^2 f'(u)w^2\right) dx$$

and $\perp$ denotes orthogonality in $L^2(0,1)$.

Corollary 6.2 *If there exists an m dimensional subspace Y of $H^1(0,1)$ such that*

$$\mathcal{H}(w) < 0 \quad for\ every \quad w \in Y\backslash\{0\},$$

then $\lambda_m < 0$.

Proposition 6.3 (*Comparison of eigenvalues*) *Let $A(x)$ be a continuous function satisfying $A(x) \geq 0$, $A(x) \not\equiv 0$ on $[0,1]$ and define L by (4.1). If the eigenvalues of the following problem:*

$$\begin{cases} (L - A(x))w = -\lambda w & in \quad (0,1), \\ w'(0) = w'(1) = 0, \end{cases} \tag{6.2}$$

are denoted by $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \tilde{\lambda}_3 < \cdots$, then

$$\tilde{\lambda}_k > \lambda_k \qquad (k = 1,2,3,\cdots).$$

On the other hand, if $A(x) \leq 0$, $A(x) \not\equiv 0$ on $[0,1]$, then

$$\tilde{\lambda}_k < \lambda_k \qquad (k = 1,2,3,\cdots).$$

Corollary 6.4 *If there exist positive functions $w(x)$ and $B(x)$ satisfying*

$$\begin{cases} \epsilon^2 w'' + h(x)^2 f'(u)w = -B(x)w & in \quad (0,1), \\ w'(0) = w'(1) = 0, \end{cases} \tag{6.3}$$

then it holds that $\lambda_1 > 0$.

Proposition 6.5 (*Sturm-Liouville theorem*) *If there exist* $w(x)$ *and* λ *satisfying* (4.2), *and if* $w(x)$ *changes sign precisely* $n-1$ *times in* $(0,1)$, *then* $\lambda = \lambda_n$.

Proof of Theorem 3. What we have to show is the following two inequalities:

$$\lambda_m < 0, \tag{6.4}$$

$$\lambda_{m+1} > 0. \tag{6.5}$$

The inequality (6.4) follows from Lemma 6.6 below and Corollary 6.2, by setting

$$Y = span\{w_1, \cdots, w_m\}.$$

The inequality (6.5) follows from Lemma 6.7 below and Proposition 6.1.

In the rest of this section we denote n-mode solution u_ϵ by u for convenience.

Lemma 6.6 *Let* m *be as in Theorem 3. Then there exist* $w_1, w_2, \cdots, w_m$ *such that*

$$\mathcal{H}(c_1 w_1 + \cdots + c_m w_m) < 0 \tag{6.6}$$

for any $c_1, \cdots, c_m$ *with* $|c_1| + \cdots + |c_m| > 0$.

Proof. Let $\xi_1 < \cdots < \xi_n$ be the zeros of $u(x)$ and let $0 = \zeta_0 < \zeta_1 < \cdots < \zeta_{n-1} < \zeta_n = 1$ be the zeros of $u'(x)$. Clearly we have

$$\zeta_{k-1} < \xi_k < \zeta_k \qquad (k = 1, 2, \cdots, n).$$

Let $k_1, \cdots, k_m$ be the set of subscripts such that for each $k = k_i\,(i = 1, \cdots, m)$ the point ζ_k lies in an $O(\epsilon|\log\epsilon|)$ neighborhood of a local maximum point of $h(x)$. Let $v(x) = u'(x)/h(x)$. This function $|v|$ satisfies the equation

$$Lv = \epsilon^2 h(x) \left(\frac{1}{h(x)}\right)'' v. \tag{6.7}$$

Now we define $w_1, \cdots, w_m$ as follows:

$$w_i(x) = \begin{cases} v(x) & \text{in } (\zeta_{k_i-1}, \zeta_{k_i}), \\ 0 & \text{in } [0,1] \setminus (\zeta_{k_i-1}, \zeta_{k_i}). \end{cases}$$

Using (6.7), we see that

$$\mathcal{H}(w_i) = -\epsilon^2 \int_{\zeta_{k_i-1}}^{\zeta_{k_i}} h(x) \left(\frac{1}{h(x)}\right)'' w_i^2 dx \qquad (i = 1, \cdots, m).$$

Note that $h(x)(1/h(x))'' > 0$ near each local maximum point of $h(x)$. Since $w_i(x)$ has a very large value near $x = \xi_{k_i}$ while it decays very rapidly outside a small neighborhood of ξ_{k_i}, we have

$$\mathcal{H}(w_i) < 0 \qquad (i = 1, \cdots, m).$$

Considering that the supports of w_i (i.e. $[\zeta_{k_i-1}, \zeta_{k_i}]$ for $i = 1, \cdots, m$) are mutually disjoint, we obtain

$$\mathcal{H}(c_1 w_1 + \cdots + c_m w_m) = \sum_{i=1}^{m} c_i^2 \mathcal{H}(w_i) < 0,$$

provided that $c_1^2 + \cdots + c_m^2 > 0$. The lemma is proved.

Lemma 6.7 *Let m be as in Theorem 3. Then there exist $\psi_1, \cdots, \psi_m$ such that*

$$\inf_{w \in X[\psi_1, \cdots, \psi_m]} \mathcal{H}(w) > 0\,. \tag{6.8}$$

In order to prove the above lemma, we will prepare two lemmas. For any sub-interval $J = (a, b) \subset (0, 1)$, let

$$\lambda_1^J < \lambda_2^J < \cdots < \lambda_k^J < \cdots$$

be the eigenvalues of the problem

$$\begin{cases} Lw = -\lambda w & \text{in} \quad (a, b), \\ u'(a) = u'(b) = 0, \end{cases} \tag{6.9}$$

and define

$$\mathcal{H}^J(w) = \int_a^b \left(\epsilon^2 |\nabla w|^2 - h(x)^2 f'(u) w^2\right) dx\,.$$

Clearly, the same statements as in Propositions 6.1, 6.3 and 6.5 remain true for λ_k^J and $\mathcal{H}^J(w)$.

Lemma 6.8 *Let $\{\zeta_k\}_{k=0}^n$ be as in the proof of Lemma 6.6. Then $\lambda_{i+1}^J > 0$, where $J = (\zeta_k, \zeta_{k+i})$.*

Proof. Let $\tilde{\lambda}_1^J < \tilde{\lambda}_2^J < \tilde{\lambda}_3^J < \cdots$ be the eigenvalues of the problem

$$\begin{cases} (L + h(x)^2 \left(\dfrac{f(u)}{u} - f'(u) \right))w = -\lambda w & \text{in} \quad (a,b), \\ w'(a) = w'(b) = 0, \end{cases} \tag{6.10}$$

where $a = \zeta_k$, $b = \zeta_{k+i}$. Since $u(x)$ satisfies (1.2), $w = u$ is regarded as an eigenfunction of (6.10) with eigenvalue $\lambda = 0$. Considering that $u(x)$ has i zeros in the interval $J = (a,b)$, we see from Proposition 6.5 that $\lambda = 0$ is the $(i+1)$–th eigenvalue, that is, $\tilde{\lambda}_{i+1}^J = 0$. Applying Proposition 6.3 and using (F.3), we get

$$\lambda_{i+1}^J > \tilde{\lambda}_{i+1}^J = 0.$$

The lemma is proved.

Lemma 6.9 *Let $\{\xi_k\}_{k=1}^n$ and $\{\zeta_k\}_{k=0}^n$ be as in the proof of Lemma 6.6. Let k be such that ξ_k is located within $O(\epsilon|\log\epsilon|)$ neighborhood of a local minimum point of $h(x)$. Then $\lambda_1^J > 0$ for $J = (\zeta_{k-1}, \zeta_k)$. In particular,*

$$\mathcal{H}^J(w) \geq \lambda_1^J \int_J w^2\,dx \geq 0 \quad \textit{for} \quad w \in H^1(J).$$

Proof. As in the proof of Lemma 6.6, we set $v(x) = u'(x)/h(x)$. Then v has a constant sign on J and satisfies (6.7). This function $|v|$ is very large near ξ_k and decays to zero very fast away from this point. If we choose $\varsigma_1, \varsigma_2 > 0$ sufficiently small,then

$$h(x)(1/h(x))'' < 0 \qquad \text{on} \quad [\xi_k - \varsigma_1, \xi_k + \varsigma_2] \tag{6.11}$$

and

$$-f'(u(x)) > -\frac{2}{3}f'(\alpha^+) \qquad \text{on} \quad J \setminus [\xi_k - \varsigma_1, \xi_k + \varsigma_2]. \tag{6.12}$$

We can then modify $v(x)$ slightly to obtain a function $w(x)$ satisfying (6.3) with $B \approx -\epsilon^2 h(x)(1/h(x))'' > 0$ and with $(0,1)$ replaced by J. It follows from Corollary 6.4 that $\lambda_1^Q > 0$, where $Q = (\xi_k - \varsigma_1, \xi_k + \varsigma_2)$ (for the details, see [8].)

It follows that

$$\begin{aligned} \mathcal{H}^J(w) &= \mathcal{H}^Q(w) + \mathcal{H}^{J\setminus Q}(w) \\ &\geq \lambda_1^Q \int_Q w^2\,dx + D\int_{J\setminus Q} w^2\,dx \\ &\geq \min\{D, \lambda_1^Q\} \textstyle\int_J w^2\,dx, \end{aligned}$$

where $D = \dfrac{2}{3}|f'(\alpha^+)| \min\limits_{x\in(0,1)} h(x)^2$. This shows that $\lambda_1^J \geq \min\{D, \lambda_1^Q\} > 0$. The lemma is proved.

Now we are ready to prove Lemma 6.7, thereby completing the proof of Theorem 3.

Proof of Lemma 6.7 Let $k_1, \cdots, k_m$ be as in the proof of Lemma 6.6 and set

$$J = \bigcup_{i=1}^{m} [\zeta_{k_i-1}, \zeta_{k_i}].$$

We decompose J into its connected components $J_1, \cdots, J_p$, where $1 \le p \le m$. Let m_i be the number of zeros of $u'(x)$ contained in J_i. By the definition of J, these zeros of $u'(x)$ are all located very near the local maximum points of $h(x)$. Clearly we have

$$m_1 + m_2 + \cdots + m_p = m.$$

Applying Lemma 6.8, we see that

$$\lambda^{J_i}_{m_i+1} > 0.$$

Consequently, by Proposition 6.1, there exist functions $\psi^i_1, \cdots, \psi^i_{m_i} \in H^1(J_i)$ such that

$$\inf_{w \in H^1(J_i)\setminus\{0\},\ w \perp \psi^i_j\ (j=1,\cdots,m_i)} \mathcal{H}^{J_i}(w) \ge \lambda^{J_i}_{m_i+1} \int_{J_i} w^2\, dx,$$

where $\perp$ denotes the orthogonality in $L^2(J_i)$. Now we extend $\psi^i_j(x)$ by setting $\psi^i_j(x) = 0$ outside J_i. Then ψ^i_j belongs to $L^2(0,1)$. It follows that, if $w \in H^1(0,1)$ satisfies $w \perp \psi^i_j$ $(j = 1, \cdots, m_i)$, then

$$\mathcal{H}^{J_i}(w) \ge \lambda^{J_i}_{m_i+1} \int_{J_i} w^2\, dx.$$

Hereafter we set the entire set of function $\{\psi^i_j\}$ as $\{\psi_1, \cdots, \psi_m\}$.

Now let $K = \{1, \cdots, n\} \setminus \{k_1, \cdots, k_m\}$. Then by Lemma 6.9, we have

$$\mathcal{H}^{I_k}(w) \ge \lambda^{I_k}_1 \int_{I_k} w^2\, dx \ge 0$$

for each $k \in K$, where $I_k = (\zeta_{k-1}, \zeta_k)$. Combining the above inequalities, we see that, if $w \in H(0,1)$ satisfies $w \perp \psi_j$ $(j = 1, \cdots, m)$, then

$$\mathcal{H}(w) = \sum_{i=1}^{p} \mathcal{H}^{J_i}(w) + \sum_{k\in K} \mathcal{H}^{I_k}(w) \ge \delta \int_0^1 w^2\, dx,$$

where

$$\delta = \min\left\{\min \lambda^{J_i}_{m_i}, \min \lambda^{I_k}_1, \right\} > 0.$$

Hence by Proposition 6.1, $\lambda_m \ge \delta > 0$. The lemma is proved.

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