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On numerical methods for analysis of chaotic phenomena in free boundary problems

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1 Introduction

The free boundary problems are very important, because they often arise from the practical situations. They are nonlinear, so they easily involve chaotic phenomena. Thus investigation of chaotic phenomena in free boundary problems is very important.

The investigation is carried out via analysis of bifurcation and attractors[18]. In the previous work bifurcation phenomena in a free boundary problem related to natural convection were analyzed numerically[20]. Attractors in free boundary problems were analyzed theoretically. Attractors or inertial sets for the phase field model were analyzed in [3], [8] - [14]. Attractors for the problem in which the motion of the free boundary is given explicitly were analyzed in [1]. However, in these papers concrete analysis of attractors was not carried out, because their analysis was based on PDE systems.

Attractors of the ODE system play a very important role. This is because they are useful for concrete analysis[5, 6, 15]. For autonomous ODE systems numerical computation of Lyapunov exponents is easily carried out. If there exist positive Lyapunov exponents, chaotic phenomena exist. However, it is difficult to derive the autonomous ODE system which approximates the PDE system describing a free boundary problem.

In the paper a method for numerical computation of attractors in free boundary problems and Lyapunov exponents is presented. To see the procedure of the method a free boundary problem with some parameters is considered. It is of the type of a two-phase Stefan problem. The method consists of SCM(Spectral Collocation Method) [4], the fixed domain method[11] and transformation from the nonautonomous system into the autonomous one[2].

2 Test problem

We consider the following one-dimensional free boundary problem with some parameters.

Problem 1. For parameters $|\alpha^{\pm}|$, $|\beta|$, $|s_0| < 1$, $0 \le r \le 1$, q and $\omega^{\pm}$, find $u^{\pm}(x,t)$ and $s(t)$ such that

$$
\begin{aligned}
u_t^+(x,t) &= u_{xx}^+(x,t) + g^+(x,t), && 0 < t, && -1 < x < s(t), && (2.1)\\
u^+(-1,t) &= h^+(t), && 0 \le t, && && (2.2)\\
u^+(s(t),t) &= 0, && 0 \le t, && && (2.3)\\
u^+(x,0) &= f^+(x), && && -1 < x < s_0, && (2.4)\\
u_t^-(x,t) &= u_{xx}^-(x,t) + g^-(x,t), && 0 < t, && s(t) < x < 1, && (2.5)\\
u^-(1,t) &= h^-(t), && 0 \le t, && && (2.6)\\
u^-(s(t),t) &= 0, && 0 \le t, && && (2.7)\\
u^-(x,0) &= f^-(x), && && s_0 < x < 1, && (2.8)
\end{aligned}
$$

$$
\frac{d}{dt}s(t) = -k^+(t)\ u_x^+(s(t),t) + k^-(t)\ u_x^-(s(t),t), \qquad 0 < t, \tag{2.9}
$$

$$
s(0) = s_0 \tag{2.10}
$$

where

$$
k^{\pm}(t) = r + (1-r)\,\frac{1}{2}\,\frac{1 \pm \beta \sin t}{\pm 1 + \alpha^{\pm} \sin t}\,\beta \cos t, \tag{2.11}
$$

$$
h^{\pm}(t) = \pm 1 + \alpha^{\pm} \sin(\omega^{\pm} t), \tag{2.12}
$$

$$
g^{\pm}(x,t) = q\left(\pm\frac{(\beta - \alpha^{\pm})\cos t}{(1 \pm \beta \sin t)^2}(x - \beta \sin t) \pm \frac{\pm 1 + \alpha^{\pm} \sin t}{1 \pm \beta \sin t}\beta \cos t\right), \tag{2.13}
$$

$$
f^+(x) = (x - s_0)\left(a(x+1) - \frac{1}{s_0 + 1}\right), \tag{2.14}
$$

$$
f^-(x) = (x - s_0)\left(b(x-1) + \frac{1}{s_0 - 1}\right). \tag{2.15}
$$

Parameters a, b should be determined such that $f^+(x) \ge 0$, $f^-(x) \le 0$.

Remark 1. *For $a = b = s_0 = r = 0$, $\omega^{\pm} = 1$ and $q = 1$, there are exact solutions as follows:*

$$
s(t) = s_p(t) \ \equiv\ \beta \sin t, \tag{2.16}
$$

$$
u^{\pm}(x,t) = \frac{\mp h^{\pm}(t)}{1 \pm s_p(t)}(x - s_p(t)) = \mp\frac{\pm 1 + \alpha^{\pm} \sin t}{1 \pm \beta \sin t}(x - \beta \sin t). \tag{2.17}
$$

3 Our method

In this section a method for derivation of the ODE system which approximates the PDE system describing a free boundary problem is presented. It consists of the fixed domain method and SCM. For numerical computation of Lyapunov exponents transformation from the nonautonomous system into the autonomous one is also necessary. To see the procedure the method is applied to Problem 1.

3.1 Spectral collocation method

The spectral methods are superior in accuracy[4]. In particular, the application of SCM is similar to that of FDM. So, it is easily applied to nonlinear problems. In the paper, SCM using Chebyshev Polynomials and Chebyshev-Gauss-Lobatto case's collocation points are used. In SCM it is easy to increase the order of the approximation by increasing the number of collocation points. This feature is quite remarkable and different from other discretization methods.

3.2 Fixed domain method

SCM can not be applied directly to a free boundary problem due to the unknown shape of the domain. To avoid this difficulty, we use the fixed domain method[7, 11]. Mapping functions are introduced for mapping the unknown domain to the fixed rectangular domain.

We use the following mapping function (variable transformation) : $(x,t) \to (\xi,\tilde{t})$ such that

$$t = t(\tilde{t}) = \tilde{t}, \qquad 0 \leq t, \tag{3.18}$$

$$x = x(\xi,\tilde{t}) = \begin{cases} \dfrac{\tilde{s}(\tilde{t})+1}{2}(\xi+1)-1, & 0 \leq t, \ -1 \leq x \leq s(t), \\ \dfrac{1-\tilde{s}(\tilde{t})}{2}(\xi-1)+1, & 0 \leq t, \ s(t) \leq x \leq 1. \end{cases} \tag{3.19}$$

Using these mapping functions, we define

$$\tilde{s}(\tilde{t}) = s(t(\tilde{t})), \tag{3.20}$$

$$\tilde{u}^+(\xi,\tilde{t}) = u^+(x(\xi,\tilde{t}),t(\tilde{t})), \tag{3.21}$$

$$\tilde{u}^-(\xi,\tilde{t}) = u^-(x(\xi,\tilde{t}),t(\tilde{t})). \tag{3.22}$$

Then, Problem 1 is transformed into the following fixed boundary problem.

Problem 2. Find $\tilde{u}^{\pm}(\xi,\tilde{t})$ and $\tilde{s}(\tilde{t})$ such that

$$
\begin{aligned}
\tilde{u}^{+}_{\tilde{t}}(\xi,\tilde{t}) &= -k^{+}(\tilde{t})\frac{2(\xi+1)}{\left(\tilde{s}(\tilde{t})+1\right)^{2}}\tilde{u}^{+}_{\xi}(1,\tilde{t})\tilde{u}^{+}_{\xi}(\xi,\tilde{t}) \\
&\quad - k^{-}(\tilde{t})\frac{2(\xi+1)}{\tilde{s}(\tilde{t})^{2}-1}\tilde{u}^{-}_{\xi}(-1,\tilde{t})\tilde{u}^{+}_{\xi}(\xi,\tilde{t}) + \frac{4}{\left(\tilde{s}(\tilde{t})+1\right)^{2}}\tilde{u}^{+}_{\xi\xi}(\xi,\tilde{t}) \\
&\quad + q\left\{\frac{(\beta-\alpha^{+})\cos\tilde{t}}{(1+\beta\sin\tilde{t})^{2}}\left(\frac{\tilde{s}(\tilde{t})+1}{2}(\xi+1)-1-\beta\sin\tilde{t}\right)\right. \\
&\quad \left. +\frac{(1+\alpha^{+}\sin\tilde{t})\beta\cos\tilde{t}}{1+\beta\sin\tilde{t}}\right\}, \qquad 0<\tilde{t}, \quad -1<\xi<1, && (3.23)\\
\tilde{u}^{+}(-1,\tilde{t}) &= 1+\alpha^{+}\sin(\omega^{+}\tilde{t}), \qquad 0\le\tilde{t}, && (3.24)\\
\tilde{u}^{+}(1,\tilde{t}) &= 0, \qquad 0\le\tilde{t}, && (3.25)\\
\tilde{u}^{+}(\xi,0) &= \frac{1}{4}(\xi-1)\{a(s_0+1)^{2}(\xi+1)-2\}, \qquad -1<\xi<1, && (3.26)\\
\tilde{u}^{-}_{\tilde{t}}(\xi,\tilde{t}) &= -k^{+}(\tilde{t})\frac{2(\xi-1)}{\tilde{s}(\tilde{t})^{2}-1}\tilde{u}^{+}_{\xi}(1,\tilde{t})\tilde{u}^{-}_{\xi}(\xi,\tilde{t}) \\
&\quad - k^{-}(\tilde{t})\frac{2(\xi-1)}{\left(\tilde{s}(\tilde{t})-1\right)^{2}}\tilde{u}^{-}_{\xi}(-1,\tilde{t})\tilde{u}^{-}_{\xi}(\xi,\tilde{t}) + \frac{4}{\left(\tilde{s}(\tilde{t})-1\right)^{2}}\tilde{u}^{-}_{\xi\xi}(\xi,\tilde{t}) \\
&\quad + q\left\{-\frac{(\beta-\alpha^{-})\cos\tilde{t}}{(1-\beta\sin\tilde{t})^{2}}\left(\frac{1-\tilde{s}(\tilde{t})}{2}(\xi-1)+1-\beta\sin\tilde{t}\right)\right. \\
&\quad \left. +\frac{(1-\alpha^{-}\sin\tilde{t})\beta\cos\tilde{t}}{1-\beta\sin\tilde{t}}\right\}, \qquad 0<\tilde{t}, \quad -1<\xi<1, && (3.27)\\
\tilde{u}^{-}(-1,\tilde{t}) &= 0, \qquad 0\le\tilde{t}, && (3.28)\\
\tilde{u}^{-}(1,\tilde{t}) &= -1+\alpha^{-}\sin(\omega^{-}\tilde{t}), \qquad 0\le\tilde{t}, && (3.29)\\
\tilde{u}^{-}(\xi,0) &= \frac{1}{4}(\xi+1)\{b(s_0-1)^{2}(\xi-1)-2\}, \qquad -1<\xi<1, && (3.30)\\
\frac{d}{d\tilde{t}}\tilde{s}(\tilde{t}) &= -k^{+}(\tilde{t})\frac{2}{\tilde{s}(\tilde{t})+1}\tilde{u}^{+}_{\xi}(1,\tilde{t}) - k^{-}(\tilde{t})\frac{2}{\tilde{s}(\tilde{t})-1}\tilde{u}^{-}_{\xi}(-1,\tilde{t}), \qquad 0<\tilde{t}, && (3.31)\\
\tilde{s}(0) &= s_0. && (3.32)
\end{aligned}
$$

3.3 ODE system

Numerical computation of attractors can be carried out by the application of SCM in space and time to Problem 2[7, 16]. However, this procedure is not proper for numerical computation of Lyapunov exponents which are computed for the ODE system. The ODE system is very important not only in numerical computation of Lyapunov exponents but also in theoretical analysis. For its derivation SCM not in time but in space is first applied.

By applying SCM in space with the following Chebyshev-Gauss-Lobatto points:

$$\xi_i = \cos\frac{i\pi}{N_x}, \qquad i = 0, 1, \cdots, N_x \tag{3.33}$$

to Problem 2, we obtain the following ODE system : Problem 3. For simplicity we substitute the symbol t for the symbol $\tilde{t}$.

Problem 3. Find $\tilde{u}_i^{\pm}(t)$, $i = 1, 2, \cdots, N_x - 1$ and $\tilde{s}(t)$ such that

$$\begin{aligned}
\frac{d}{dt}\tilde{u}_i^+(t) = &-k^+(t)\frac{2(\xi_i+1)}{\left(\tilde{s}(t)+1\right)^2}\Big(\sum_{k=1}^{N_x-1}(D_x)_{0,k}\,\tilde{u}_k^+(t) + (D_x)_{0,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\Big)\\
&\Big(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\,\tilde{u}_k^+(t) + (D_x)_{i,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\Big)\\
&-k^-(t)\frac{2(\xi_i+1)}{\tilde{s}(t)^2-1}\Big(\sum_{k=1}^{N_x-1}(D_x)_{N_x,k}\,\tilde{u}_k^-(t) + (D_x)_{N_x,0}\big(\alpha^-\sin(\omega^-t)-1\big)\Big)\\
&\Big(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\,\tilde{u}_k^+(t) + (D_x)_{i,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\Big)\\
&+\frac{4}{\left(\tilde{s}(t)+1\right)^2}\Big(\sum_{k=1}^{N_x-1}(D_{xx})_{i,k}\,\tilde{u}_k^+(t) + (D_{xx})_{i,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\Big)\\
&+q\left\{\frac{(\beta-\alpha^+)\cos t}{(1+\beta\sin t)^2}\left(\frac{\tilde{s}(t)+1}{2}(\xi_i+1)-1-\beta\sin t\right)\right.\\
&\left.+\frac{(1+\alpha^+\sin t)\beta\cos t}{1+\beta\sin t}\right\}, \qquad 0 < t,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\frac{d}{dt}\tilde{u}_i^-(t) = &-k^+(t)\frac{2(\xi_i-1)}{\tilde{s}(t)^2-1}\Big(\sum_{k=1}^{N_x-1}(D_x)_{0,k}\,\tilde{u}_k^+(t) + (D_x)_{0,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\Big)\\
&\Big(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\,\tilde{u}_k^-(t) + (D_x)_{i,0}\big(\alpha^-\sin(\omega^-t)-1\big)\Big)\\
&-k^-(t)\frac{2(\xi_i-1)}{\left(\tilde{s}(t)-1\right)^2}\Big(\sum_{k=1}^{N_x-1}(D_x)_{N_x,k}\,\tilde{u}_k^-(t) + (D_x)_{N_x,0}\big(\alpha^-\sin(\omega^-t)-1\big)\Big)\\
&\Big(\sum_{k=1}^{N_x-1}(D_x)_{i,k}\,\tilde{u}_k^-(t) + (D_x)_{i,0}\big(\alpha^-\sin(\omega^-t)-1\big)\Big)\\
&+\frac{4}{\left(\tilde{s}(t)-1\right)^2}\Big(\sum_{k=1}^{N_x-1}(D_{xx})_{i,k}\,\tilde{u}_k^-(t) + (D_{xx})_{i,0}\big(\alpha^-\sin(\omega^-t)-1\big)\Big)\\
&+q\left\{-\frac{(\beta-\alpha^-)\cos t}{(1-\beta\sin t)^2}\left(\frac{1-\tilde{s}(t)}{2}(\xi_i-1)+1-\beta\sin t\right)\right.\\
&\left.+\frac{(1-\alpha^-\sin t)\beta\cos t}{1-\beta\sin t}\right\}, \qquad 0 < t,
\end{aligned} \tag{3.35}$$

$$\frac{d}{dt}\tilde{s}(t) = -k^+(t)\frac{2}{\tilde{s}(t)+1}\left(\sum_{k=1}^{N_x-1}(D_x)_{0,k}\,\tilde{u}_k^+(t) + (D_x)_{0,N_x}\big(\alpha^+\sin(\omega^+t)+1\big)\right),$$

$$-k^-(t)\frac{2}{\tilde{s}(t)-1}\left(\sum_{k=1}^{N_x-1}(D_x)_{N_x,k}\,\tilde{u}_k^-(t) + (D_x)_{N_x,0}\big(\alpha^-\sin(\omega^-t)-1\big)\right), \quad 0<t, \tag{3.36}$$

$$\tilde{u}_i^+(0) = \left(\frac{a}{4}(s_0+1)^2(\xi_i+1)-\frac{1}{2}\right)(\xi_i-1), \tag{3.37}$$

$$\tilde{u}_i^-(0) = \left(\frac{b}{4}(s_0-1)^2(\xi_i-1)-\frac{1}{2}\right)(\xi_i+1), \tag{3.38}$$

$$\tilde{s}(0) = s_0. \tag{3.39}$$

Of course, it is easy to change N_x. This means original attractors of the PDE system can be approximated arbitrarily by the method. This feature of the method is very important from the theoretical view point. For $N_x = 2$ the ODE system becomes as follows.

Problem 4. Find $\tilde{u}_1^{\pm}(t)$ and $\tilde{s}(t)$ such that

$$\begin{aligned}\frac{d}{dt}\tilde{u}_1^+(t) = &-\frac{k^+(t)}{2\big(\tilde{s}(t)+1\big)^2}\big(4\tilde{u}_1^+(t)-\alpha^+\sin(\omega^+t)-1\big)\big(\alpha^+\sin(\omega^+t)+1\big)\\ &+\frac{k^-(t)}{2\big(\tilde{s}(t)^2-1\big)}\big(4\tilde{u}_1^-(t)-\alpha^-\sin(\omega^-t)+1\big)\big(\alpha^+\sin(\omega^+t)+1\big)\\ &-\frac{4}{\big(\tilde{s}(t)+1\big)^2}\big(2\tilde{u}_1^+(t)-\alpha^+\sin(\omega^+t)-1\big)\\ &+q\left\{\frac{(\beta-\alpha^+)\cos t}{(1+\beta\sin t)^2}\left(\frac{\tilde{s}(t)-1}{2}-\beta\sin t\right)\right.\\ &\left.+\frac{(1+\alpha^+\sin t)\beta\cos t}{1+\beta\sin t}\right\}, \qquad 0<t,\end{aligned} \tag{3.40}$$

$$\begin{aligned}\frac{d}{dt}\tilde{u}_1^-(t) = &-\frac{k^+(t)}{2\big(\tilde{s}(t)^2-1\big)}\big(4\tilde{u}_1^+(t)-\alpha^+\sin(\omega^+t)-1\big)\big(\alpha^-\sin(\omega^-t)-1\big)\\ &+\frac{k^-(t)}{2\big(\tilde{s}(t)-1\big)^2}\big(4\tilde{u}_1^-(t)-\alpha^-\sin(\omega^-t)+1\big)\big(\alpha^-\sin(\omega^-t)-1\big)\\ &-\frac{4}{\big(\tilde{s}(t)-1\big)^2}\big(2\tilde{u}_1^-(t)-\alpha^-\sin(\omega^-t)+1\big)\\ &+q\left\{-\frac{(\beta-\alpha^-)\cos t}{(1-\beta\sin t)^2}\left(\frac{\tilde{s}(t)+1}{2}-\beta\sin t\right)\right.\\ &\left.+\frac{(1-\alpha^-\sin t)\beta\cos t}{1-\beta\sin t}\right\}, \qquad 0<t,\end{aligned} \tag{3.41}$$

$$\frac{d}{dt}\tilde{s}(t) = \frac{k^+(t)}{\tilde{s}(t)+1}\left(4\tilde{u}_1^+(t) - \alpha^+ \sin(\omega^+ t) - 1\right)$$
$$- \frac{k^-(t)}{\tilde{s}(t)-1}\left(4\tilde{u}_1^-(t) - \alpha^- \sin(\omega^- t) + 1\right), \qquad 0 < t, \tag{3.42}$$

$$\tilde{u}_1^+(0) = \frac{1}{2} - \frac{a}{4}(s_0+1)^2, \tag{3.43}$$

$$\tilde{u}_1^-(0) = -\frac{1}{2} - \frac{b}{4}(s_0-1)^2, \tag{3.44}$$

$$\tilde{s}(0) = s_0, \qquad 0 < t. \tag{3.45}$$

3.4 Transformation into the autonomous system

The ODE systems in Problems 3 and 4 are not autonomous. So, transformation into the autonomous system is necessary for numerical computation of Lyapunov exponents. It can be done by introducing a new parameter θ[2]. Problem 4 is transformed into the following autonomous system.

Problem 5. Find $\tilde{u}_1^{\pm}(t), \tilde{s}(t)$ and $\theta(t)$ such that

$$\frac{d}{dt}\tilde{u}_1^+(t) = -\frac{k^+(t)}{2\big(\tilde{s}(t)+1\big)^2}\left(4\tilde{u}_1^+(t) - \alpha^+ \sin\{\omega^+\theta(t)\} - 1\right)\left(\alpha^+ \sin\{\omega^+\theta(t)\} + 1\right)$$
$$+ \frac{k^-(t)}{2\big(\tilde{s}(t)^2-1\big)}\left(4\tilde{u}_1^-(t) - \alpha^- \sin\{\omega^-\theta(t)\} + 1\right)\left(\alpha^+ \sin\{\omega^+\theta(t)\} + 1\right)$$
$$- \frac{4}{\big(\tilde{s}(t)+1\big)^2}\left(2\tilde{u}_1^+(t) - \alpha^+ \sin\{\omega^+\theta(t)\} - 1\right)$$
$$+ q\left\{\frac{(\beta-\alpha^+)\cos\{\theta(t)\}}{(1+\beta\sin\{\theta(t)\})^2}\left(\frac{\tilde{s}(t)-1}{2} - \beta\sin\{\theta(t)\}\right)\right.$$
$$\left.+\frac{(1+\alpha^+\sin\{\theta(t)\})\beta\cos\{\theta(t)\}}{1+\beta\sin\{\theta(t)\}}\right\}, \qquad 0 < t, \tag{3.46}$$

$$\frac{d}{dt}\tilde{u}_1^-(t) = -\frac{k^+(t)}{2\big(\tilde{s}(t)^2-1\big)}\left(4\tilde{u}_1^+(t) - \alpha^+ \sin\{\omega^+\theta(t)\} - 1\right)\left(\alpha^- \sin\{\omega^-\theta(t)\} - 1\right)$$
$$+ \frac{k^-(t)}{2\big(\tilde{s}(t)-1\big)^2}\left(4\tilde{u}_1^-(t) - \alpha^- \sin\{\omega^-\theta(t)\} + 1\right)\left(\alpha^- \sin\{\omega^-\theta(t)\} - 1\right)$$
$$- \frac{4}{\big(\tilde{s}(t)-1\big)^2}\left(2\tilde{u}_1^-(t) - \alpha^- \sin\{\omega^-\theta(t)\} + 1\right)$$
$$+ q\left\{-\frac{(\beta-\alpha^-)\cos\{\theta(t)\}}{(1-\beta\sin\{\theta(t)\})^2}\left(\frac{\tilde{s}(t)+1}{2} - \beta\sin\{\theta(t)\}\right)\right.$$

$$+\frac{(1-\alpha^{-}\sin\{\theta(t)\})\beta\cos\{\theta(t)\}}{1-\beta\sin\{\theta(t)\}}\Bigg\}, \qquad 0<t, \tag{3.47}$$

$$\frac{d}{dt}\tilde{s}(t) = \frac{k^{+}(t)}{\tilde{s}(t)+1}\left(4\tilde{u}_1^{+}(t)-\alpha^{+}\sin\{\omega^{+}\theta(t)\}-1\right)$$

$$-\frac{k^{-}(t)}{\tilde{s}(t)-1}\left(4\tilde{u}_1^{-}(t)-\alpha^{-}\sin\{\omega^{-}\theta(t)\}+1\right), \qquad 0<t, \tag{3.48}$$

$$\frac{d}{dt}\theta(t) = 1, \qquad 0<t, \tag{3.49}$$

$$\tilde{u}_1^{+}(0) = \frac{1}{2}-\frac{a}{4}(s_0+1)^2, \tag{3.50}$$

$$\tilde{u}_1^{-}(0) = -\frac{1}{2}-\frac{b}{4}(s_0-1)^2, \tag{3.51}$$

$$\tilde{s}(0) = s_0, 0<t. \tag{3.52}$$

Of course, this procedure is applicable to the general system : Problem 3.

3.5 Linearized equations

Linearization of Problem 5 is necessary for numerical computation of Lyapunov exponents[17]. Problem 5 can be rewritten in following general form:

$$\frac{d}{dt}\tilde{u}_1^{+}(t) = F_1\left(\tilde{u}_1^{+}(t),\ \tilde{u}_1^{-}(t),\ \tilde{s}(t),\ \theta(t)\right), \qquad 0<t,$$

$$\frac{d}{dt}\tilde{u}_1^{-}(t) = F_2\left(\tilde{u}_1^{+}(t),\ \tilde{u}_1^{-}(t),\ \tilde{s}(t),\ \theta(t)\right), \qquad 0<t,$$

$$\frac{d}{dt}\tilde{s}(t) = F_3\left(\tilde{u}_1^{+}(t),\ \tilde{u}_1^{-}(t),\ \tilde{s}(t),\ \theta(t)\right), \qquad 0<t,$$

$$\frac{d}{dt}\theta(t) = 1, \qquad 0<t.$$

The linearized problem for this system is written in following form:

$$\frac{d}{dt}\delta u^+(t) = \frac{\partial F_1}{\partial \tilde{u}_1^+}\delta u^+(t) + \frac{\partial F_1}{\partial \tilde{u}_1^-}\delta u^-(t) + \frac{\partial F_1}{\partial \tilde{s}}\delta s(t) + \frac{\partial F_1}{\partial \theta}\delta\theta(t), \qquad 0 < t,$$

$$\frac{d}{dt}\delta u^-(t) = \frac{\partial F_2}{\partial \tilde{u}_1^+}\delta u^+(t) + \frac{\partial F_2}{\partial \tilde{u}_1^-}\delta u^-(t) + \frac{\partial F_2}{\partial \tilde{s}}\delta s(t) + \frac{\partial F_2}{\partial \theta}\delta\theta(t), \qquad 0 < t,$$

$$\frac{d}{dt}\delta s(t) = \frac{\partial F_3}{\partial \tilde{u}_1^+}\delta u^+(t) + \frac{\partial F_3}{\partial \tilde{u}_1^-}\delta u^-(t) + \frac{\partial F_3}{\partial \tilde{s}}\delta s(t) + \frac{\partial F_3}{\partial \theta}\delta\theta(t), \qquad 0 < t,$$

$$\frac{d}{dt}\delta\theta(t) = 0, \qquad 0 < t$$

where

$$\tilde{u}_1^+(t+\delta t) \approx \tilde{u}_1^+(t) + \delta u^+(t),$$

$$\tilde{u}_1^-(t+\delta t) \approx \tilde{u}_1^-(t) + \delta u^-(t),$$

$$\tilde{s}(t+\delta t) \approx \tilde{s}(t) + \delta s(t),$$

$$\theta(t+\delta t) \approx \theta(t) + \delta\theta(t).$$

By this linearization, Problem 5 becomes

Problem 6.

$$\frac{d}{dt}\delta u^+(t) = c_{11}\,\delta u^+(t) + c_{12}\,\delta u^-(t) + c_{13}\,\delta s(t) + c_{14}\,\delta\theta(t), \qquad 0 < t,$$

$$\frac{d}{dt}\delta u^-(t) = c_{21}\,\delta u^+(t) + c_{22}\,\delta u^-(t) + c_{23}\,\delta s(t) + c_{24}\,\delta\theta(t), \qquad 0 < t,$$

$$\frac{d}{dt}\delta s(t) = c_{31}\,\delta u^+(t) + c_{32}\,\delta u^-(t) + c_{33}\,\delta s(t) + c_{34}\,\delta\theta(t), \qquad 0 < t,$$

$$\frac{d}{dt}\delta\theta(t) = 0, \qquad 0 < t$$

where

$$c_{11} = -\frac{\sin(\omega^+\theta(t)) + 10}{(\tilde{s}(t)+1)^2},$$

$$c_{12} = \frac{\sin(\omega^+\theta(t)) + 2}{\tilde{s}(t)^2 - 1},$$

$$c_{13} = \frac{1}{4(\tilde{s}(t)+1)^3}\left(8\tilde{u}_1^+(t) - \sin(\omega^+\theta(t)) - 2\right)\left(\sin(\omega^+\theta(t)) + 2\right)$$

$$- \frac{\tilde{s}(t)}{4(\tilde{s}(t)^2-1)^2}\left(8\tilde{u}_1^-(t) - \sin(\omega^-\theta(t)) + 2\right)\left(\sin(\omega^+\theta(t)) + 2\right)$$

$$+\frac{4}{(\tilde{s}(t)+1)^3}\left(4\tilde{u}_1^+(t)-\sin(\omega^+\theta(t))-2\right),$$

$$c_{14}=-\frac{\omega^+\cos(\omega^+\theta(t))}{4(\tilde{s}(t)+1)^2}\left(4\tilde{u}_1^+(t)-\sin(\omega^+\theta(t))-2\right)$$
$$+\frac{\omega^+\cos(\omega^+\theta(t))}{8(\tilde{s}(t)^2-1)}\left(8\tilde{u}_1^-(t)-\sin(\omega^-\theta(t))+2\right)$$
$$-\frac{\omega^-\cos(\omega^-\theta(t))}{8(\tilde{s}(t)^2-1)}\left(\sin(\omega^+\theta(t))+2\right)+\frac{2\omega^+\cos(\omega^+\theta(t))}{(\tilde{s}(t)+1)^2},$$

$$c_{21}=-\frac{\sin(\omega^-\theta(t))-2}{\tilde{s}(t)^2-1},$$

$$c_{22}=\frac{\sin(\omega^-\theta(t))-10}{(\tilde{s}(t)-1)^2},$$

$$c_{23}=\frac{\tilde{s}(t)}{4(\tilde{s}(t)^2-1)^2}\left(8\tilde{u}_1^+(t)-\sin(\omega^+\theta(t))-2\right)\left(\sin(\omega^-\theta(t))-2\right)$$
$$-\frac{1}{4(\tilde{s}(t)-1)^3}\left(8\tilde{u}_1^-(t)-\sin(\omega^-\theta(t))+2\right)\left(\sin(\omega^-\theta(t))-2\right)$$
$$+\frac{4}{(\tilde{s}(t)-1)^3}\left(4\tilde{u}_1^-(t)-\sin(\omega^-\theta(t))+2\right),$$

$$c_{24}=\frac{\omega^+\cos(\omega^+\theta(t))}{8(\tilde{s}(t)^2-1)}\left(\sin(\omega^-\theta(t))-2\right)-\frac{\omega^-\cos(\omega^+\theta(t))}{8(\tilde{s}(t)^2-1)}\left(8\tilde{u}_1^+(t)-\sin(\omega^+\theta(t))-2\right)$$
$$+\frac{\omega^-\cos(\omega^-\theta(t))}{4(\tilde{s}(t)^2-1)}\left(4\tilde{u}_1^-(t)-\sin(\omega^-\theta(t))+2\right)+\frac{2\omega^-\cos(\omega^-\theta(t))}{(\tilde{s}(t)-1)^2},$$

$$c_{31}=\frac{4}{\tilde{s}(t)+1},$$

$$c_{32}=-\frac{4}{\tilde{s}(t)-1},$$

$$c_{33}=-\frac{1}{2(\tilde{s}(t)+1)^2}\left(8\tilde{u}_1^+(t)-\sin(\omega^+\theta(t))-2\right)+\frac{1}{2(\tilde{s}(t)-1)^2}\left(8\tilde{u}_1^-(t)-\sin(\omega^-\theta(t))+2\right)$$

$$c_{34}=-\frac{\omega^+\cos(\omega^+\theta(t))}{2(\tilde{s}(t)+1)}+\frac{\omega^-\cos(\omega^-\theta(t))}{2(\tilde{s}(t)-1)}.$$

Initial conditions are given from orthogonal bases properly.

Then SCM in time[7, 19] is applied for computing Lyapunov exponents. Here we remai that these exponents do not correspond to attractors obtained from the nonautonomoı system Problem 4.

4 Numerical results

In this section, numerical results are shown. We performed numerical simulation for $N_x = 2$, $q = 0$, $r = 1$, $\alpha = \beta = 0.5$ and $\omega^+ = 1$. For time integration we used SCM with 11 Chebyshev-Gauss-Lobatto collocation points in the interval $\Delta t = 0.1$[7, 19].

Figs. 1 - 4 show attractors in the solution space (the three-dimensional space) and Lyapunov exponents. Attractors are computed from Problem 4. Lyapunov exponents are computed from both Problem 5 and its linearized problem[17].

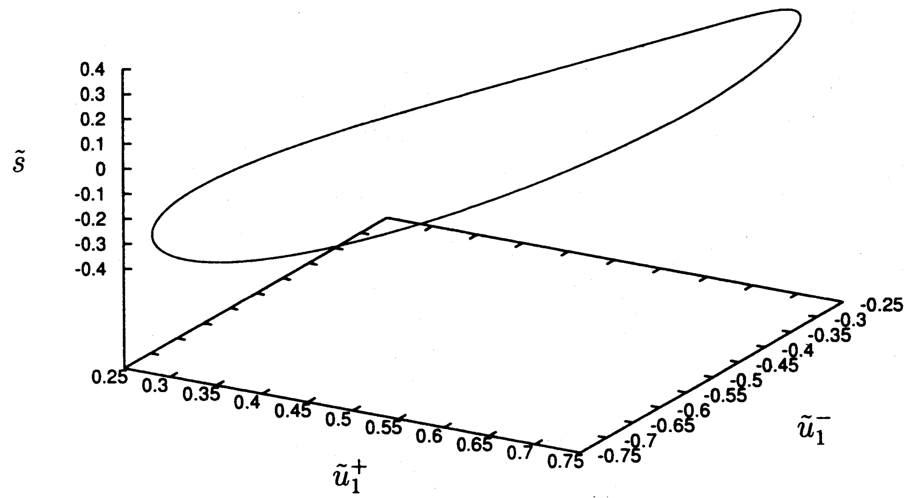

Fig. 1. Attractor in Problem 4 for $\omega^- = 1$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.360$, $\lambda_2 = -6.710$, $\lambda_3 = -19.10$, $\lambda_4 = 0.000$.

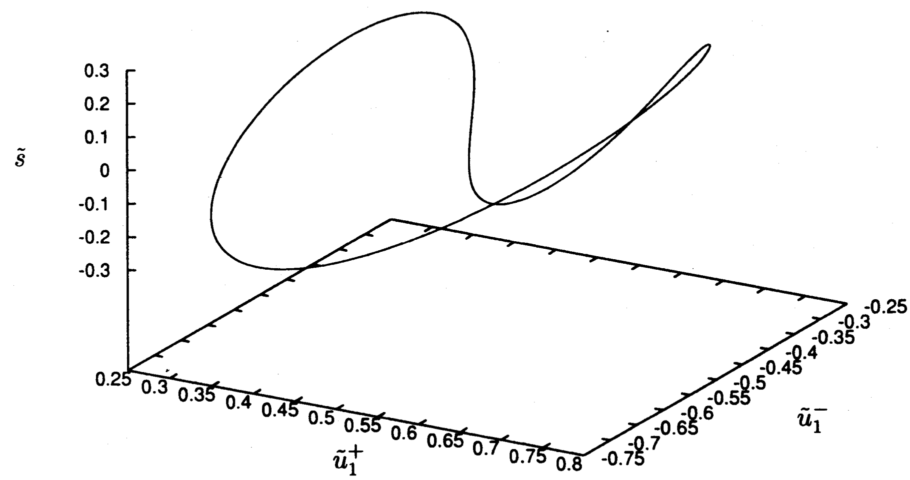

Fig. 2. Attractor in Problem 4 for $\omega^- = 2$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.275$, $\lambda_2 = -7.487$, $\lambda_3 = -14.71$, $\lambda_4 = 0.000$.

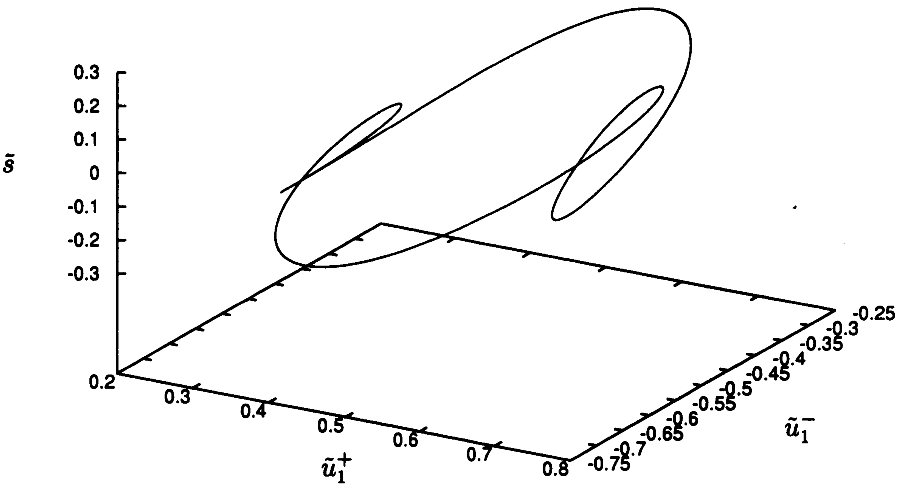

Fig. 3. Attractor in Problem 4 for $\omega^- = 3$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.284$, $\lambda_2 = -7.264$, $\lambda_3 = -15.10$, $\lambda_4 = 0.000$.

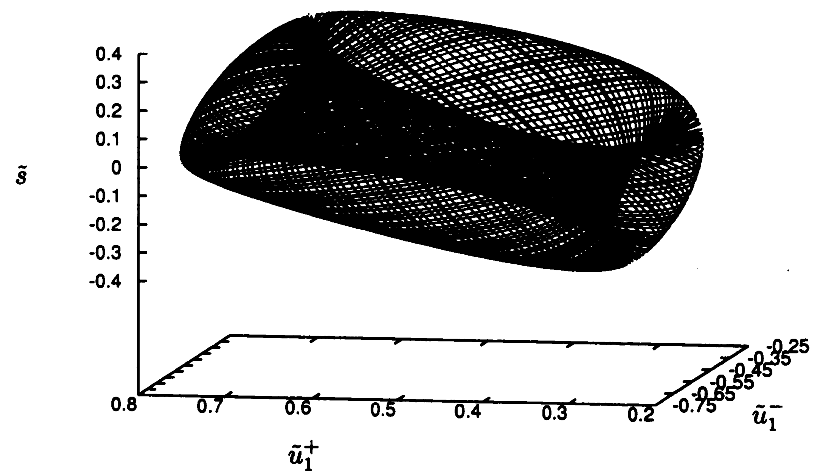

Fig. 4. Attractor in Problem 4 for $\omega^- = \sqrt{2}$. Lyapunov exponents for Problem 5 : $\lambda_1 = -1.289$, $\lambda_2 = -7.228$, $\lambda_3 = -15.75$, $\lambda_4 = 0.000$.

For parameters investigated above attractors are not strange. So, there are no positive Lyapunov exponents.

5 Conclusion

In the paper a method for numerical computation of attractors in free boundary problems and their Lyapunov exponents is presented. The method consists of SCM (Spectral Collocation Method), the fixed domain method and transformation from the nonautonomous system into the autonomous system. To see the procedure of the method it is applied to a free boundary problem with some parameters which is of the type of a two-phase Stefan problem. Various attractors are found in the nonautonomous system and Lyapunov exponents are computed in the autonomous system. The method is based on SCM, so original attractors of the PDE system can be approximated arbitrarily. This means the method plays a very important role in theoretical analysis. Our next goal is to find strange attractors (i.e. positive Lyapunov exponents) in free boundary problems by using our method.

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