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Numerical Methods for Computing Discontinuous Solutions of a Class of Hamilton-Jacobi Equations Using a Level Set Method

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Abstract

In this article, we first introduce a Lax-Friedrichs type finite difference method to compute the L-solution, following its original definition recently proposed by the second auther in[12] using level sets. We then generalize our numerical methods to compute the proper viscosity solution proposed in [11] for a more general class of HJ equations that includes conservation laws. We couple our numerical methods with a singular diffusive term of essential importance. With this singular viscosity, our numerical methods do not require the divergence structure of equations and do apply to more general equations developing shocks other than conservation laws. These numerical methods are generalized to higher order accuracy using WENO Local Lax-Friedrichs methods [17]. We verify that our numerical solutions approximate the proper viscosity solutions of [11] and, in particular, the entropy solutions in case of conservtion laws.

1 Introduction

Nonlinear Hamilton-Jacobi Equations arise in many different fields, including control theory, and differential games. Because of the nonlinearity, the Cauchy problems usually have non-classical solutions due to the crossing of characteristic curves.

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For scalar equations of conservation law type, there is a well known theory regarding the existence and uniqueness of a weak solution called entropy solution, using the special integral structure of equation [19]. Advanced numerical methods, e.g. [13][14][24][26], have been developed and widely used to compute approximations that converge to the correct entropy solutions.

Nevertheless, this notion of weak solution cannot be applied to many fully nonlinear equations, e.g the eikonal equation $u_t + |\nabla u| = 0$. In 1983, Crandall and Lions [5] first introduced the notion of viscosity solution for this type of equations, based on a maximum principle and the order preserving property of parabolic equations. In general, for any given Hamilton-Jacobi equation of the form

$$u_t + H(x, t, u, Du) = 0,$$

where H is a continuous function from $\Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n$, non-decreasing in u, and Ω is an open subset of $\mathbb{R}^n$, there exists a unique uniformly continuous viscosity solution if the initial data is bounded, uniformly continuous.[1]

Correspondingly, Crandall and Lions in [4] proved the convergence of two approximations to the viscosity solution of equations whose Hamiltonians only depend on Du. This was generalized by Souganidis to equations with variable coefficients in [25]. Many sophisticated numerical methods have since been developed [17][21][22].

However, there are problems in control theory and differential games which demand discontinuous solutions. The notion of semi-continuous viscosity solution has been introduced first by Ishii [15, 16] using an extension of Perron's method. Because of the non-uniqueness in Ishii's result, other notions of semi-continuous solutions were proposed by various authors [1][2] with different kinds of additional properties imposed on the Hamiltonian. Nevertheless these notions, in their original definitions, do not facilitate the construction of their numerical approximations.

Finally, for the class of equations with Hamiltonians $H(x, u, Du)$ nondecreasing in u, M.-H. Sato and the second author [12] introduced a new notion for semi-continuous solution. This notion of solution is defined by the evolution of the zero level curve of the auxiliary level set equation which embeds the original HJ equation. It is thus called the L-solution. In this article, we will device a Lax-Friedrichs type scheme to compute approximation of the L-solution in its original formulation (i.e. level set). We will also show that with suitable CFL condition, our schemes keep the discrete version of an important property of this class of HJ equations.

When the Hamiltonians $H(t, x, u, Du)$ are not nondecreasing in u, the solutions may develop shocks in finite time even if the initial data is continuous. Recently, a new notion called the proper viscosity solution is introduced by the second

[1]Notice that the conservation laws do not fall into this category because the corresponding H might not be monotone in u; e.g. shocks may develop from smooth initial data.

author [11] to track the whole evolution. This notion is consistent with the entropy solution when the equation is a conservation law. In order to approximate the proper viscosity solution of a class of more general HJ equations, we introduce a singular diffusive term in the vertical direction to the auxiliary level set equations so that the level curves will not overturn. We will show numerically that the singular diffusive term regularizes the shock solutions of conservation laws such that the "equal area" entropy condition is satisdied and thus demonstrate its validity.

We remark that a simple monotone Lax-Friedrichs scheme seems to produce convergent approximations of the L-solution for the first class of HJ equations in their original form, even though the scheme does not follow the original definition of the L-solution. However, for the second class of equations, it is likely that the numerical approximations obtained this way converge to the wrong weak solution. This is a well known fact for monotone schemes for conservation laws in nonconservative form. In contrast, our numerical approximations for the corresponding "nonconservative" level set equations appear to converge to the right weak solution; i.e. the proper viscosity solution and, in case of conservation laws, the entropy solution.

In the following sections, we first review briefly the previous work on using level sets as a tool to analyze and compute solution of given PDEs. We then derive the level set equation from a given HJ equation. We then devise numerical methods for the level set equations for the computation of the HJ equations solutions according to the behavior of H_u. We extend each type of our numerical schemes to higher order accuracy using the WENO schemes devised in [17].

1.1 Analysis by the level set function

Osher [20] rediscovered a method of Jacobi [3] to study the Cauchy problem of general first order nonlinear equations through the aid of the level set equations. In that paper, Osher derived from the general first order equation

$$F(x, y, u, u_x, u_y) = 0$$

a time dependent Hamilton-Jacobi equation

$$\phi_t + H(x, y, t, \phi_x, \phi_y) = 0$$

and proved that the zero level set of its viscosity solution at time t is the set $\{(x, y) : u(x, y) = t\}$. With continuous initial values, the viscosity solution theory gives the existence and uniqueness of the solution to the time-dependent Hamilton-Jacobi equations provided that H does not change sign.

In [6], Evans used the level set method described in [20] to obtain the *level surface heat equation.* He gave the geometrical interpretation to the instant "unfolding" of multi-valued initial data of the solution of linear heat equation. By considering the viscous Burgers' equation

$$u_t + uu_x = \varepsilon u_{xx}, \quad \varepsilon > 0$$

as a lower order perturbation to the heat equation, Evans provided further analysis and a geometrical explanation as to how the term εu_{xx} keeps the solution from becoming multi-valued.

Recently, M.-H. Sato and the second author proposed to characterize the semi-continuous solutions of HJ equations using a similar approach. In this paper [12], they define the L-solution and prove existence and uniqueness of the L-solution with a class of Hamiltonian. We remark that the L-solution is equivalent to the conventional viscosity solution if the hypothesis are identical.

The idea is to represent the "graph" of a semi-continuous function $u(x)$ as the zero level set of a function $\phi : \mathbb{R}^2 \to \mathbb{R}^1$ with the requirement that every level set of ϕ is the graph of some function of x. More precisely, we define the subgraph of a function u be be $\mathrm{sg}(u) := \{(x, y) \in \mathbb{R}^2 : y \leq u(x)\}$ and the curve $\Gamma(t)$ to be the upper boundary of $\mathrm{sg}(u)$. For smooth functions $u(x, t)$, $\Gamma(t)$ is simply the graph of u at time t. Consider the general first order equation:

$$u_t + H(t, x, u, u_x) = 0 \tag{1}$$

where u is a function from $\mathbb{R} \to \mathbb{R}$. We embed $\Gamma(t)$ as the zero level set of a function $\phi : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ (i.e. $\phi(t, x, y) = 0$ for all $(x, y) \in \Gamma(t)$ for each $t \in \mathbb{R}^+$) and derive the level set equation:

$$\phi_t - \phi_y H(t, x, y, -\frac{\phi_x}{\phi_y}) = 0. \tag{2}$$

In the following sections, we will use H_u to denote the partial derivative of $H(x, u, u_x)$ with respect to u for the original HJ equations and H_y to denote the partial derivative of $H(x, y, \phi_x, \phi_y)$ with respect to y, where $H(x, y, \phi_x, \phi_y)$ is the Hamiltonian derived from the original one. Finally, the level set function ϕ is set up to be non-decreasing in y initially in the examples of this paper.

2 Model Equations

We first consider the scalar 1D equation

$$u_t + H(x, u, u_x) = 0$$

with the Hamiltonian $H(x, u, u_x)$ satisfying the following properties:

1. H is Lipschitz in all its arguments
2. $\lim_{\lambda\to 0} \lambda H(x, u, p/\lambda)$ exists.

In addition, we are concerned with the following two classes of equations: 1)Equations with $H_u \geq 0$ but with discontinuous initial data; 2)Equations such as conservation laws that do not belong in the first class.

Let us consider the following two model equations, both of which can be in either the first or the second class depending on the parameters:

- Equations that contains both terms from conservation laws and fully nonlinear first order terms:

$$u_t + uu_x + a\,u|u_x| = 0,\; a \in \mathbb{R}; \tag{3}$$

The associated level set equation reads

$$\phi_t - y \cdot (a\,\mathrm{sign}(\phi_y)\,|\phi_x| - \phi_x) = 0. \tag{4}$$

If $a \geq 1$, H_u will be non-decreasing. Equation (3) then falls into the first class of equations. For $0 \leq |a| < 1$, H_u changes signs according to the value of u_x. Equation (3) then belongs to the second class. Notice that if $a = 0$, we have the inviscid Burgers' equation.

- Equations that prescribe the normal motion of the graph of u:

$$u_t - v(u)\sqrt{1 + u_x^2} = 0; \tag{5}$$

The corresponding level set equation is

$$\phi_t + \mathrm{sign}(\phi_y)\,v(y)|\nabla\phi| = 0. \tag{6}$$

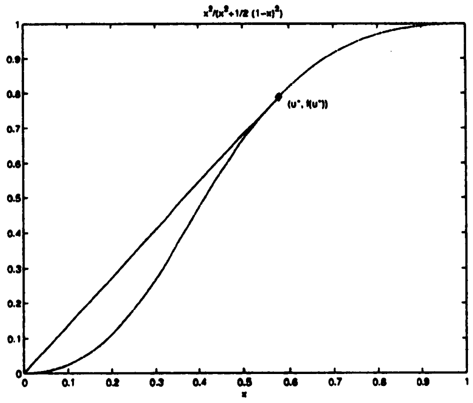
The function v is the normal velocity of the graph of u, or the level sets of ϕ. If v ever decreases, then $H_u \leq 0$ and the equation fails to be in the first class.

2.1 Geometrical Explanation of the Non-overturning Conditions

As mentioned earlier, we need to pay special attention in order to prevent the overturning of the level curves of ϕ. One equivalent criterion is to demand the minimum principle: $\phi_y(x, y, t) \geq 0$ for $t \geq 0$.

In light of the level set equation (6), we have a more geometrical requirement on the speed function v. By the method of characteristics, we know that $v(y)$ prescribes the normal velocity of the level sets of ϕ. On the vertical segments of the

Figure 1: The concavification of the flux in Buckley-Leverett equation



level sets, which correspond to jumps in u, $v(y)$ prescribes the horizontal velocity according to y. Overturning will happen if $v(y)$ is increasing, since the upper part of the jump of u moves faster than the lower part.

Consider the primitive function of v:

$$V(y) = \int v(s)ds.$$

The non-increasing condition of v translates to the *concavity* of V! This fact reminds us of one of the entropy conditions for conservation laws with non-concave flux function. It says that the entropy solution of the conservation law with non-convex flux f is the classical solution of the conservation law with the flux f^*, where f^* is the minimal concavification of f over the increasing jump interval. This, in turns, provides us a hint on the regularization of HJ equations (6) — we need to impose a regularization that concavifies the primitive function on the vertical segments of the level sets and nowhere else. We shall demonstrate numerically that our proposed singular diffusive regularization term does exactly that in a later part of this article.

2.2 Equations with Hamiltonian $H_u \geq 0$

We first consider the equations $u_t + H(t, x, u, u_x) = 0$ for which $H_u \geq 0$ and the corresponding level set equation equation (2).

The minimum principle

For this class of equation, one can show that if $\phi_y(x, y, t = 0) \geq 0$ initially, then $\phi_y(x, y, t) \geq 0$ for all time! (See [12]) This implies that $\{\phi = c\}$ *will remain as a* *graph throughout the evolution.* Therefore, we can remove the $\text{sign}(\phi_y)$ term from the derived level set equation (2) of this class of equation.

The Lax-Friedrichs schemes for the level set equation

For the level set equations with H independent of ϕ_y (after removing $\text{sign}(\phi_y)$), we introduce for this class of Hamiltonians a monotone Lax-Friedrichs scheme:

$$\phi_{i,j}^{n+1} = \frac{1}{2}(\phi_{i+1,j}^n + \phi_{i-1,j}^n) + \Delta t\, H(x_i, y_j, D_x^0 \phi_{i,j}^n). \tag{7}$$

For equations such as equation (5), since the Hamiltonians depend on ϕ_y, we use

$$\phi_{i,j}^{n+1} = \frac{1}{4}(\phi_{i+1,j}^n + \phi_{i-1,j}^n + \phi_{i,j+1}^n + \phi_{i,j-1}^n) + \Delta t\, H(x_i, y_j, D_x^0 \phi_{i,j}^n, D_y^0 \phi_{i,j}^n). \tag{8}$$

Here, $\phi_{i,j}^n := \phi(x_i, y_j, t_n)$, $D_x^0 \phi_{i,j}^n$ and $D_y^0 \phi_{i,j}^n$ are the central differencing of $\phi_{i,j}^n$ in the x- and y- direction respectively.

Because of the hypothesis that $H_y \geq 0$, our schemes preserve the minimum principle discretely (i.e. given $\Delta_y^+ u_{i,j}^n \geq 0$ for all $i, j \in Z_d$, then $\Delta_y^+ u_{i,j}^{n+1} \geq 0$ for all $i, j \in Z_d$) if

$$\frac{\Delta t}{\Delta x} \leq C \min(1/||H_{\phi_x}||_\infty, 1/||H_{\phi_y}||_\infty),$$

where $C = 1$ for equation (7) and $C = 2$ for equation (8).

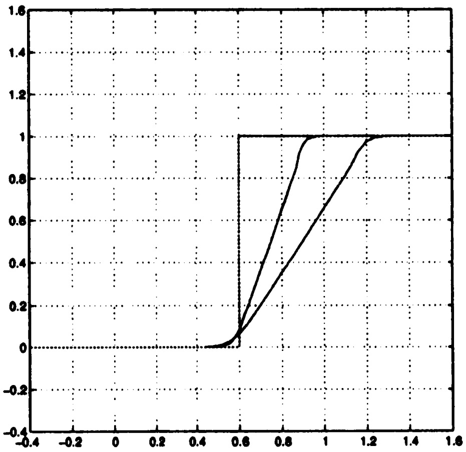
Extension to higher order of accuracy

Following the methods originally conceived for HJ equations $\phi_t + H(D\phi) = 0$ in [22], see also [21], it is possible extend the above methods to higher order accuracy. In this paper, we use an direct extension of the Local Lax-Friedrichs method [22] together with the WENO schemes described in [17] for approximating the partial derivatives of ϕ.

2.2.1 Examples

We provide here some numerical computations for some equations that belong to the class we are considering.

Figure 2: Numerical solution using third order WENO-LLF to the Riemann problem for equation (3) with $u_L = 0.0$, $u_R = 0.1$, and $a = 2.0$. We plotted the zero level set at time $t = 0, 0.1$, and 0.2.



Constant motion along the normal

Consider the equation

$$u_t + c\sqrt{1 + u_x^2} = 0. \tag{9}$$

Given a continuous initial data, it is well-known that the following equation corresponds to motion of the graph with constant normal velocity c.

Using the notion of the L-solution, we can easily describe the motion defined by equation (9), even with piecewise continuous data. The corresponding level set equation is simply

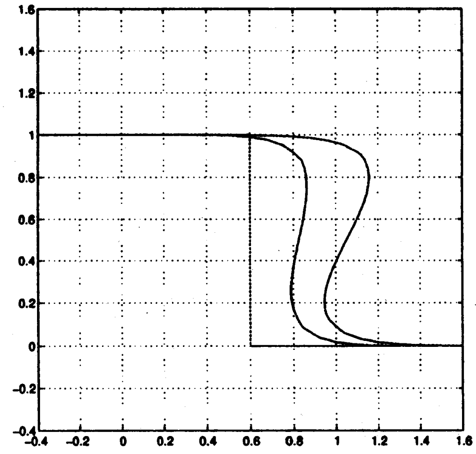
$$\phi_t + c|\nabla\phi| = 0,$$

which describes the constant normal speed motion of each level set of ϕ.

Model equation $u_t + uu_x + a\,u|u_x| = 0$

With $a \geq 1.0$, we know that this model equation retains the property that $\phi_y \geq 0$ for all time. Figure 2 show the computational result using (7) and third order WENO-LLF. The numerical solutions of this equation are computed with $a = 2.0$. Finally, we show that our Lax-Friedrichs type scheme cannot be applied to compute solutions for equation with $a < 1$. See figure3.

Figure 3: Numerical solution to the Riemann problem for equation (4) with $u_L = 1.0$, $u_R = 0.0$, and $a = 0.1$. We plotted the zero level set at time $t = 0, 0.5$, and 1.0.



2.3 Singular Viscosity Regularization

Consider the model equation (4) with $|a| < 1$, and equation (5) with $v(y)$ non-decreasing. We know that it no longer has the minimum principle in ϕ_y, and "over-turning" or "folding" in its solution might develop.

Motivated by the work on a type of singular diffusion in [7, 8, 18], we will add a similar singular diffusion term in the y-direction to both our model equations:

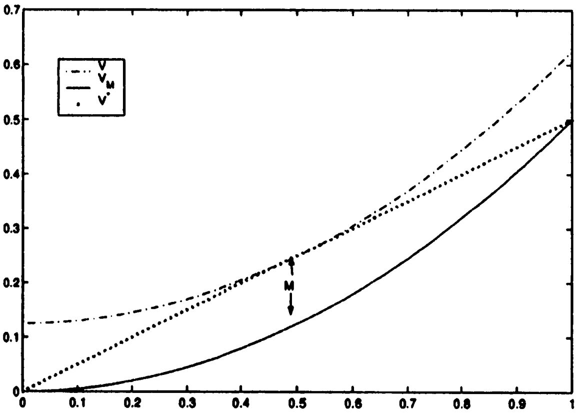
$$M|\nabla\phi|\frac{\partial}{\partial y}(\frac{\phi_y}{|\phi_y|}).$$

We first notice that this viscosity is activated *only when* $\text{sign}(\phi_y) = \phi_y/|\phi_y|$ changes signs! With M sufficiently large, this term $\partial(\text{sign}(\phi_y))/\partial y$ can be shown, at least formally, to concavify the primitive of the speed function on the vertical part of the level sets [10].

We briefly describe how to find the minimum value of M. Consider the primitive function $V(y)$ of the speed function $v(y)$ of equation (5) over $[a, b]$ that is a jump of u. Let V^* be the function whose graph is the upper boundary of the convex hull of V. Let $V_M = V^* + M$. We claim that M has to be large enough such that V_M is tangent to or never crosses V^*. See figure 4 for an example with $V(y) = y^2/2$. Since the purpose of this paper is to provide the numerics, we refer the reader to the recent paper of the second author [10] for a formal reasoning.

Alternatively, we describe another intuitive motivation behind this diffusion term: consider the Heaviside function $y = H(x)$ and the level set function $\phi(x, y)$ for which this is the zero level set. If we treat the zero level set of ϕ locally as

Figure 4: $V(y) = y^2/2$ on $[0, 1]$. The minimum value of M should be $1/8$.



a function of y wherever it is vertical, we see that the "overturning" will increase the total variation of $\{\phi = c\}$ as a function of y. This motivates the following regularization:

$$\min_{\phi} \int |\phi_y| \, dy.$$

The corresponding Euler-Lagrange derivative is

$$\frac{\partial}{\partial y}\left(\frac{\phi_y}{|\phi_y|}\right).$$

To make the diffusion term geometrical, i.e. invariant of the choice of level set function, we multiply it by $|\nabla\phi|$ and arrive at the same diffusion term.

Now, let us go back to our model equation with this viscosity term:

$$\phi_t - y \cdot (a \, \text{sign}(\phi_y) \, |\phi_x| - \phi_x) = M \, |\nabla\phi| \, \frac{\partial}{\partial y}\left(\frac{\phi_y}{|\phi_y|}\right).$$

We use central differencing to approximate the singular diffusion term on the right hand side:

$$\sqrt{(D_x^0\phi_{i,j})^2 + (D_y^0\phi_{i,j})^2} \cdot \frac{\tanh(\gamma \, D_y^+\phi_{i,j}) - \tanh(\gamma \, D_y^-\phi_{i,j})}{\Delta y},$$

where the signum function $\phi_y/|\phi_y|$ is approximated by $\tanh(\gamma\phi_y)$ with $\gamma \to \infty$, and

$$\tanh(\gamma \, D_y^+\phi_{i,j}) = \tanh(\gamma \, \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y})$$

is an approximation of $\phi_y/|\phi_y|$ evaluated at $(x_i, y_{j+1/2})$. Similarly $\tanh(D_y^- \phi_{i,j})$ is an approximation for $\phi_y/|\phi_y|$ at $(x_i, y_{j-1/2})$. The partial derivative ϕ_x on the left hand side is approximated by upwind differencing:

- $|a| < 1:$

$$
\begin{aligned}
y \geq 0: &\quad \phi_x \leftarrow D_x^- \phi \\
y < 0: &\quad \phi_x \leftarrow D_x^+ \phi
\end{aligned}
$$

- $|a| \geq 1:$

$$
\begin{aligned}
\mathrm{sign}(D_y^0 \phi)\, a\, y \leq 0: &\ \phi_x \leftarrow (D_-^x \phi)^+ - (D_+^x \phi)^-. \\
\mathrm{sign}(D_y^0 \phi)\, a\, y > 0: &\ \phi_x \leftarrow -(D_-^x \phi)^- + (D_+^x \phi)^+.
\end{aligned}
$$

Here, p^- denotes the negative part of p (with sign) and p^+ the positive part.

Because of the singular diffusion term, the stability condition is similar to the parabolic equation:

$$
\frac{\Delta t}{\Delta x^2} \leq C_{M,H},
$$

where $C_{M,H}$ is a constant depending on the diffusion coefficient M and the maximum values of H_{ϕ_x} and H_{ϕ_y}.

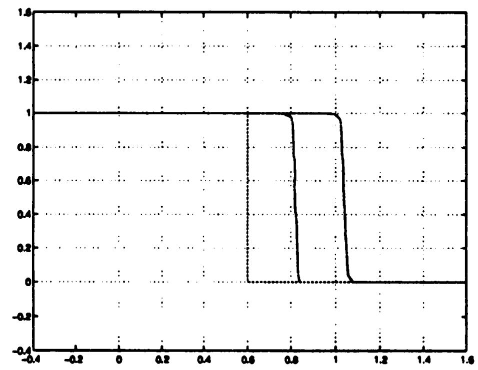
Extension to higher order accuracy

Again, we may combine the central differencing approximation of the viscosity term and the WENO-LLF scheme described in the earlier section for numerical computation. This is needed for future generalization to more complex equations or to system of equations because upwinding is no longer easy.

2.3.1 Test on the model equation: $u_t + uu_x + a\,u|u_x| = 0$

We first test our numerical scheme for the case $a = 0.1$ which cannot be handled by the Lax-Friedrichs scheme (7). Figure 5 shows that the "overturning" is prevented in contrast to the result shown in figure 3.

Figure 5: Numerical solution to the Riemann problem for equation (4) with $u_L = 1.0$, $u_R = 0.0$, $a = 0.1$, and $M = 0.2$. We plotted the zero level set at time $t = 0, 0.5$, and 1.0.



2.3.2 Tests on conservation laws

As we have mentioned earlier, equation (3) with $a = 0$ is equivalent to Burgers' equation in non-conservative form. Here we go one step further to demonstrate numerically that our regularization is equivalent to the entropy condition for conservation laws equations.

We consider the conservation laws

$$u_t + f(u)_x = 0 \tag{10}$$

with $f' \geq 0$ and its corresponding *linear* level set equation

$$\phi_t + f'(y)\phi_x = 0. \tag{11}$$

The numerical results shown in the following examples are obtained by plotting the zero contour of the numerical solution ϕ to the regularized equation:

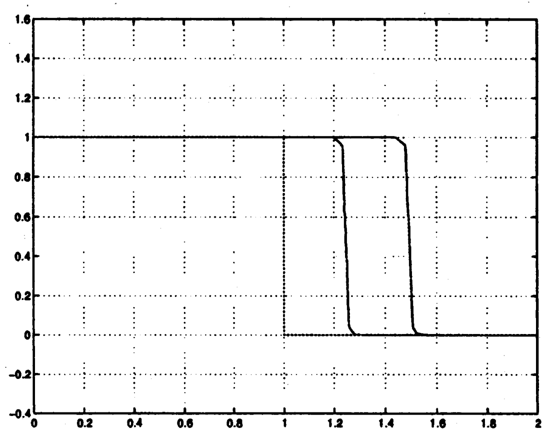
$$\phi_t + f'(y)\phi_x = M\,|\nabla\phi|\,\frac{\partial}{\partial y}\left(\frac{\phi_y}{|\phi_y|}\right). \tag{12}$$

Burgers' Equation

With $f(u) = u^2/2$, we have the inviscid Burgers equation in non-conservative form. We consider the Riemann problem $u(x) = u_L$ for $x < 0.5$ and $u(x) = u_R$ for $x \geq 0.5$. See figure 6.

We first test the case in which $u_L = 1.0$ and $u_R = 0.0$. The result shown in figure 6 verifies the Rankine-Hugoniot shock speed: $s = [f]/[u] = 0.5$. Figure 7 shows a similar computation with two different values of the diffusion coefficients

Figure 6: Numerical solution (WENO5-LLF) to the Riemann problem of Burgers' Equation with $u_L = 1.0$, $u_R = 0.0$, and $M = 0.2$. We plotted the zero level set at time $t = 0, 0.5$, and 1.0.



($M = 0.04$ and $M = 1.0$). We can see that overturning will develop if M is not large enough, and if it is sufficiently large, this coefficient does not affect the shock speed as predicted in [9]. We also compute the approximation obtained with no diffusion term (i.e. $M = 0$) and plot it (green curve) against the one obtained from $M = 0.2$ (blue curve), and show that the "equal-area" entropy condition is satisfied by the latter (blue curve). See figure 8.

Finally, we compute the solution to Burgers' equation starting with a sine curve. Figure 9 shows the resulting well-known N-wave. Our diffusion term successfully keeps the vertical part in the middle from overturning.

Buckley-Leverett Equation

Finally, we test our numerical method for equation (12) to substantiate our assertion that the singular diffusion term minimally concavifies the flux function f over the jump interval. We solve the Riemann problem of equation (10) with

$$f(u) = \frac{u^2}{u^2 + a(1-u)^2}, \quad a > 0, \quad u \in [0, 1].$$

and $u_L = 1.0$, $u_R = 0.0$.

The upper boundary of the convex hull of $\mathrm{sg}(f)$ consists of a straight line segment L from $(0,0)$ to $(u^*, f(u^*))$ followed by $(u, f(u))$ for $u \in [u^*, 1]$, where L is a tangent line of $f(u)$. See figure 1.The slope of L is also the correct shock speed for the Riemann problem. With $a = 0.5$, a simple calculation shows that $u^* = 1/\sqrt{3} \doteq 0.57735$.

Figure 7: Numerical solution to the Riemann problem of Burgers' Equation with $u_L = 1.0$, $u_R = 0.0$. We plotted the zero level sets at time $t = 0$ and 0.5 obtained from $M = 0.04$ and 1.0.

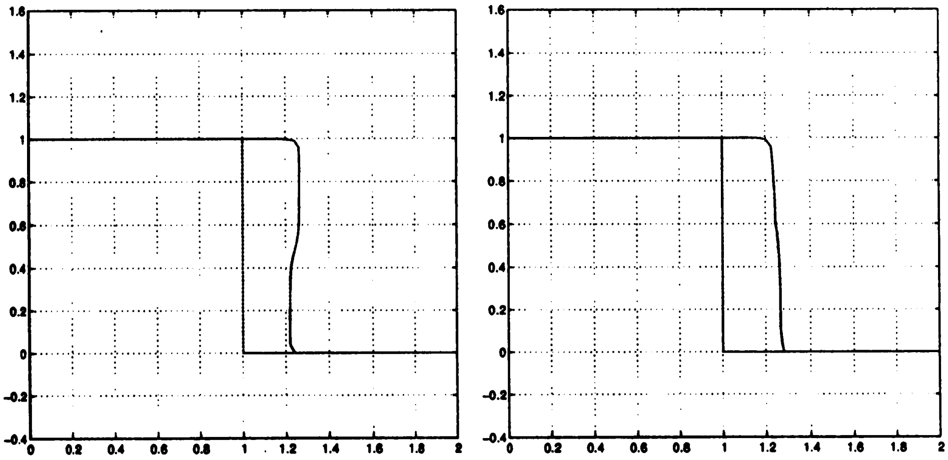


Figure 8: Numerical solution to the Riemann problem of Burgers' Equation with $u_L = 1.0$, $u_R = 0.0$. We plotted the zero level sets at time $t = 0$ and 0.5 obtained from $M = 0.2$ and 0.0.

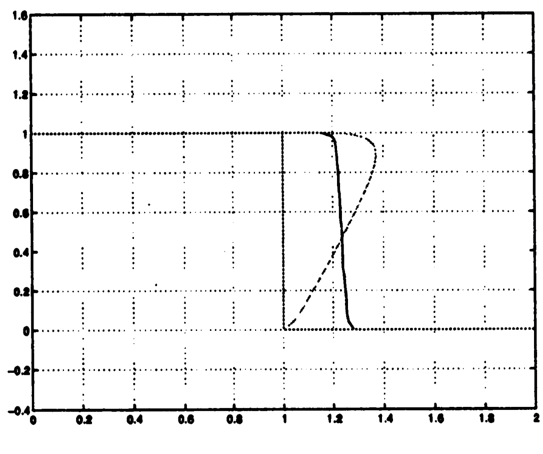


Figure 9: Numerical solution (WENO5-LLF) to the Burgers Equation with sine wave as initial data. We plotted the zero level set at time $t = 0$ and 0.5.

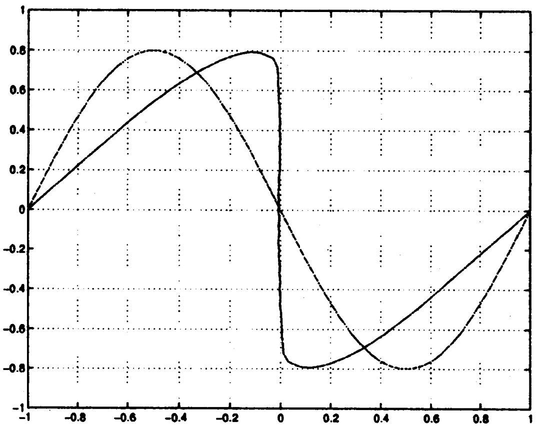


Figure 10 shows the expected rarefaction from u_L to u^* and a shock between u^* and u_R. Figure 11 shows an overlap of the solutions obtained with and without regularization. One can observe that the "equal-area" entropy condition is satisfied.

The Vanishing Viscosity Approach

Consider the Lax-Friedrichs type scheme of the following form:

$$u_i^{n+1} = u_i^n - \Delta t H(x_i, u_i^n, D_x^0 u_i^n) + c\Delta_x^+ \Delta_x^- u_i^n/2, \tag{13}$$

where $\Delta_x^{\pm} u_i^n$ denotes the undivided forward/backward difference of u_i^n and $0 \leq c \leq 1$. With suitable CFL condition, the scheme is monotone and seems to yield convergent approximations for the equations with $H_u \geq 0$.

However, this scheme is not suitable for the HJ equations whose solutions develop shocks. Figure 12 shows the numerical approximations using (13) with different values of c and fairly small grid size. The left most curve is the initial data. The remaining curves from left to right are obtained using $c = 0.1, 0.99$, and 0.9 respectively. One can see that the numerical solutions converge to different functions.

We maintain that our level set approach is no less efficient since we can do the computation locally around the zero level curve [23]. Also, the level set approach is more "natural" since it is a part of the theoretical notions of solutions to the HJ equations that we are concerned with.

Figure 10: Numerical solution to the Riemann problem of the Buckley-Leverett Equation with $u_L = 1.0$, $u_R = 0.0$ and $a = 0.5$. We plotted the zero level set at time $t = 0, 0.25$, and 0.5.

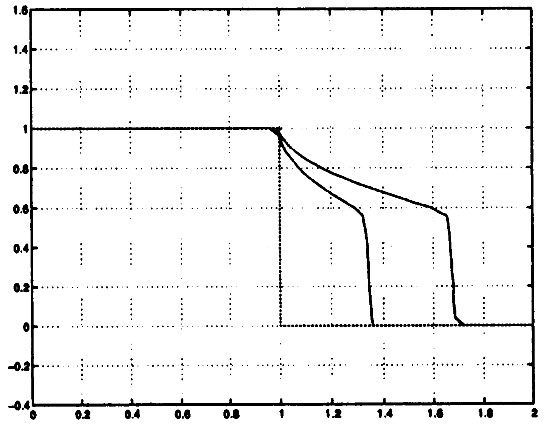


Figure 11: Numerical solution to the Riemann problem of the Buckley-Leverett Equation with $u_L = 1.0$, $u_R = 0.0$ and $a = 0.5$. We plotted the zero level sets at time $t = 0$ and 0.3 obtained from $M = 0.2$ and 0.0. The little fragment of contour at the lower part of the jump is due to the contour plotter.

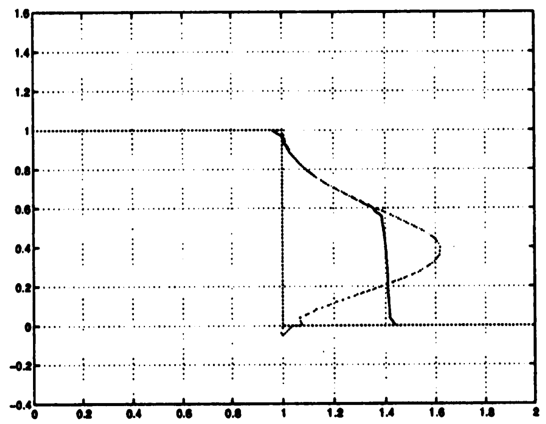
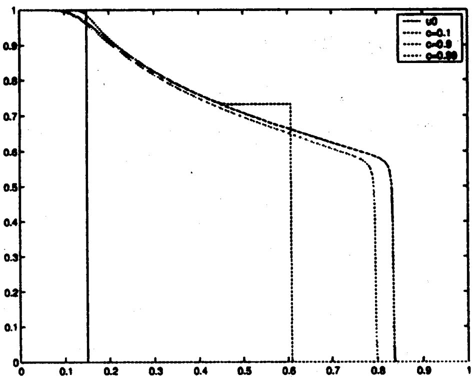


Figure 12: The numerical solutions of the Buckley-Leverett equation in non-conservative form obtained from the monotone Lax-Friedrichs scheme (13). The approximations are computed to $t = 0.1$ on $[0, 1]$ with $2,500$ grid points.



3 Summary

In this paper, we provided two classes of finite difference methods for the computation of the semi-continuous L-solution of a class of HJ equations. By studying the level set equation derived from the HJ equations, we pointed out the necessary condition for the validity of the solution defined as the zero contour line of the level set function. We have also discussed the geometrical interpretation of the motion of the solution embedded in the level set function. The remarks provide hints as how to regularize the zero level curve motion so that it can be interpreted as the graph of a function.

For the class of HJ equations with $H_y \geq 0$, we applied a straightforward Lax-Friedrichs type scheme with possibility of extension to higher order accuracy. We showed numerically that the singular diffusion term $|\nabla\phi|\,\partial(\phi_y/|\phi_y|)/\partial y$ can be applied to compute the shock solution for a more general class of HJ equations. In particular, we numerically verified that our numerical schemes yield approximations compatible with the entropy solution of a conservation laws equation with non-convex flux. Of course, we have also shown the extension of our numerical schemes to higher order WENO-local Lax-Friedrichs schemes.

Lastly, we remark here that our numerical schemes for the derived level set equations can be computed locally around the zero level curve using the technique described in [23] for efficiency.

4 Systems of Conservation Laws

We are generalizing the result of our singular viscosity to study the solution of conservation laws system and the link to Riemann invariants. Here we briefly describe how we are approaching this problem.

Let $\vec{u} = (u, v) \in \mathbb{R}^2$, $\phi(t, x, y) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the vector values level set function such that $\phi(t, x, \vec{u}(t, x)) = 0$. The system

$$\vec{u}_t + A(\vec{u})\vec{u}_x = 0$$

can be formally translated to

$$\phi_t + \phi_y A(y)\phi_y^{-1}\phi_x = 0.$$

We shall use the Riemann invariants for the 2×2 system to diagonalize $A(y)$ and desingularize the term ϕ_y^{-1}.

We propose a singular diffusion term similar to the scalar one we used. With an abuse of notation, this term can be written as

$$|\nabla_{x,y}\phi|\nabla_y \cdot (|\nabla_y\phi|^{-1}\nabla_y\phi),$$

where $\nabla_{x,y}\phi$ is the Jacobian matrix of ϕ with respect to x and y, $\nabla_y\phi = \phi_y$ is the Jacobian matrix of ϕ, and $|A| := \sqrt{AA^*}$ is the Euclidean norm of the matrix A.

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References

[1] Guy Barles. Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit. *Nonlinear Anal.*, 20(9):1123–1134, 1993.

[2] E. N. Barron and R. Jensen. Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians. *Comm. Partial Differential Equations*, 15(12):1713–1742, 1990.

[3] C Caratheodory. *Calculus of varieties of Partial Differential Equations of the first order*. Celsea, 1982.

[4] M.G. Crandall and P.L. Lions. Two approximations of solutions of Hamilton-Jacobi equations. *Mathematics of Computation*, 43:1–19, 1984.

[5] Michael G. Crandall and Pierre-Louis Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983.

[6] Lawrence C. Evans. A geometric interpretation of the heat equation with multivalued initial data. *SIAM J. Math. Anal.*, 27(4):932–958, 1996.

[7] Mi-Ho Giga and Yoshikazu Giga. Crystalline and level set flow—convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane. In *Free boundary problems: theory and applications, I (Chiba, 1999)*, pages 64–79. Gakkōtosho, Tokyo, 2000.

[8] Mi-Ho Giga, Yoshikazu Giga, and Ryo Kobayashi. Very singular diffusion equations. In *Advanced Studies in Pure Mathematics 26, 2000*, pages 1–34, 2000.

[9] Yoshikazu Giga. Unpublished notes.

[10] Yoshikazu Giga. Shocks and very strong vertical diffusion. In *To appear, Proc. of international conference on partial differential equations in celebration of the seventy fifth birthday of Professor Louis Nirenberg, Taiwan*, 2001.

[11] Yoshikazu Giga. Viscosity solutions with shocks. *Hokkaido Univ. Preprint Series in Math.*, (519), 2001.

[12] Yoshikazu Giga and M.-H. Sato. A level set approach to semicontinuous viscosity solutions for Cauchy problems. *To appear in Communication in PDE*, 2000.

[13] Ami Harten. High resolution schemes for hyperbolic conservation laws. *J. Comput. Phys.*, 49(3):357–393, 1983.

[14] Ami Harten, Björn Engquist, Stanley Osher, and Sukumar R. Chakravarthy. Uniformly high-order accurate essentially nonoscillatory schemes. III. *J. Comput. Phys.*, 71(2):231–303, 1987.

[15] Hitoshi Ishii. Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets. *Bull. Fac. Sci. Engrg. Chuo Univ.*, 28:33–77, 1985.

[16] Hitoshi Ishii. Perron's method for Hamilton-Jacobi equations. *Duke Math. J.*, 55(2):369–384, 1987.

[17] Guang-Shan Jiang and Danping Peng. Weighted ENO schemes for Hamilton-Jacobi equations. *SIAM J. Sci. Comput.*, 21(6):2126–2143 (electronic), 2000.

[18] R. Kobayashi and Y. Giga. Equations with singular diffusivity. *J. Statist. Phys.*, 95(5-6):1187–1220, 1999.

[19] Peter D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[20] Stanley Osher. A level set formulation for the solution of the Dirichlet problem for Hamilton-Jacobi equations. *SIAM J Math Anal*, 24(5):1145–1152, 1993.

[21] Stanley Osher and James A. Sethian. Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations. *J. Comput. Phys.*, 79(1):12–49, 1988.

[22] Stanley Osher and Chi-Wang Shu. High-order essentially nonoscillatory schemes for Hamilton-Jacobi equations. *SIAM J. Numer. Anal.*, 28(4):907–922, 1991.

[23] Danping Peng, Barry Merriman, Stanley Osher, Hongkai Zhao, and Myungjoo Kang. A PDE-based fast local level set method. *J. Comput. Phys.*, 155(2):410–438, 1999.

[24] Chi-Wang Shu and Stanley Osher. Efficient implementation of essentially nonoscillatory shock-capturing schemes. II. *J. Comput. Phys.*, 83(1):32–78, 1989.

[25] Panagiotis E. Souganidis. Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations*, 59(1):1–43, 1985.

[26] Bram van Leer. Towards the ultimate conservative difference schemes V. *J. Comput. Phys.*, 32:102–136, 1979.