数理解析研究所講究録 1211 巻 2001 年 21-33

ASYMPTOTIC ALGEBRAS

INGO WITT

ABSTRACT. The concept of asymptotic type that primarily appears in singular and asymptotic analysis is developed. Especially, asymptotic algebras are introduced.

1. INTRODUCTION

Let $\mathfrak{M}$ be a unital algebra and let $\rho_0 \colon \mathfrak{M} \to L(\mathfrak{F})$ be a faithful representation of $\mathfrak{M}$ on some linear space, $\mathfrak{F}$. Henceforth, we shall (often) identify (via ρ_0) elements of $\mathfrak{M}$ with operators on $\mathfrak{F}$, i.e., we shall write P instead of $\rho_0(P)$. We intend to assign to the elements of $\mathfrak{F}$ certain "asymptotics" with respect to (the representation ρ_0 of) $\mathfrak{M}$. First of all, there is a distinguished linear subspace, $\mathfrak{F}_0$ — the subspace of all "flat" elements of $\mathfrak{F}$. (In this abstract setting, "flatness" means nothing but belonging to the subspace $\mathfrak{F}_0$.) The subspace $\mathfrak{F}_0$, however, is in general no left invariant under the action of operators in $\mathfrak{M}$; thus leading to the concept of asymptotic type. In a sense, asymptotic types measure the "deviation" of $P\mathfrak{F}_0$, $P \in \mathfrak{M}$, from the flat subspace $\mathfrak{F}_0$.

Thus an asymptotic type is a linear subspace of the quotient space $\mathfrak{F}/\mathfrak{F}_0$. The set of all asymptotic types, $\mathfrak{J}$, should be a sublattice of the complete lattice $\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ of all linear subspaces of $\mathfrak{F}/\mathfrak{F}_0$. Furthermore, it is required:

- $\mathcal{O} \in \mathfrak{J}$ (where $\mathcal{O} = \mathfrak{F}_0/\mathfrak{F}_0$ is the empty asymptotic type);
- for each $P \in \mathfrak{M}$ and all $J \in \mathfrak{J}$, there is a $K \in \mathfrak{J}$ such that $J^P \subseteq K$ (where $J^P = (P\mathfrak{F}_0 + \mathfrak{F}_0)/\mathfrak{F}_0$ is the push-forward of $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ under the action of $P \in \mathfrak{M}$);
- $\bigcap_{\iota \in \mathcal{I}} J_\iota \in \mathfrak{J}$ for each non-empty family $\{J_\iota\}_{\iota \in \mathcal{I}} \subset \mathfrak{J}$.

Date: January 5, 2001.

1991 *Mathematics Subject Classification.* Primary: 35A27; Secondary: 35C20, 06B30.

The subset $\mathfrak{J} \subseteq \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ is called a lattice of asymptotic types (abbreviated l.a.t.) and $(\mathfrak{M}, \rho_0, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is called an asymptotic algebra. Further, $u \in \mathfrak{F}$ is said to have asymptotics of type J if and only if $u \in \mathfrak{F}_J = \pi^{-1}(J)$, where $\pi \colon \mathfrak{F} \to \mathfrak{F}/\mathfrak{F}_0$ is the canonical projection. In particular, the flat subspace $\mathfrak{F}_0 = \mathfrak{F}_{\mathcal{O}}$ is the space of all elements having no asymptotics. Given the l.a.t. $\mathfrak{J}$, we get the ideal

$$\mathfrak{S}_{\mathfrak{J}} = \{S \in \mathfrak{M};\ S(\mathfrak{F}) \subseteq \mathfrak{F}_J \text{ for some } J \in \mathfrak{J}\}$$

of residual operators and the multiplicatively closed set

$$\mathfrak{E}_{\mathfrak{J}} = \{P \in \mathfrak{M};\ \text{there is a } Q \in \mathfrak{M} \text{ such that } PQ - 1,\ QP - 1 \in \mathfrak{S}_{\mathfrak{J}}\}$$

of elliptic operators. Note that $\mathfrak{E}_{\mathfrak{J}}/\mathfrak{S}_{\mathfrak{J}}$ is the group of invertible elements in the algebra $\mathfrak{M}/\mathfrak{S}_{\mathfrak{J}}$. For operators $P \in \mathfrak{E}_{\mathfrak{J}}$ elliptic regularity holds: For each $K \in \mathfrak{J}$, there is a $J \in \mathfrak{J}$ (depending on P, K) such that $u \in \mathfrak{F}$ and $Pu \in \mathfrak{F}_K$ implies $u \in \mathfrak{F}_J$. In fact, let $Q \in \mathfrak{E}$ be a parametrix to P, i.e., we have $PQ - 1$, $QP - 1 \in \mathfrak{S}_{\mathfrak{J}}$. Then $u = Q(Pu) - (QP - 1)u$, hence the claim.

Remark. The third property implies that every non-empty subset $\mathcal{S} \subseteq \mathfrak{J}$ possesses a meet (= greatest lower bound) $\bigwedge \mathcal{S} = \bigcap_{J \in \mathcal{S}} J$ and every bounded subset $\mathcal{T} \subseteq \mathfrak{J}$ possesses a join (= least upper bound) $\bigvee \mathcal{T} = \bigwedge \{K;\ K \supseteq J \text{ for all } J \in \mathcal{T}\}$. In particular, $\mathfrak{J}$ topped with $\mathfrak{F}/\mathfrak{F}$, i.e., $\mathfrak{J} \cup \{\mathfrak{F}/\mathfrak{F}\}$, is a complete lattice. Note further that $\bigwedge_{\mathfrak{J}} \mathcal{S} = \bigwedge_{\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)} \mathcal{S}$, while, in general, we only have $\bigvee_{\mathfrak{J}} \mathcal{T} \supseteq \bigvee_{\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)} \mathcal{T}$.

For a quadruple $(\mathfrak{M}, \rho_0, \mathfrak{F}, \mathfrak{F}_0)$, the basic question concerns the appropriate choice of the l.a.t., $\mathfrak{J}$. Here the answer will, of course, strongly depend on the given context. The most obvious answer, namely to take for $\mathfrak{J}$ the least subset of $\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ that contains $\mathcal{O}$ and that is closed under push-forwards by elements of $\mathfrak{M}$ as well as under forming non-empty intersections and finite sums, yields, in general, a l.a.t. that is too large (and, therefore, inadequate). Instead, one wishes (with the help of the asymptotic types) to reflect a certain "internal" structure. In this paper, we shall consider a situation in which it is possible to overcome the difficulty of introducing an appropriate l.a.t. by reducing to a so-called symbol algebra.

A key observation in that respect is that, when defining concepts like the push-forward J^P for $P \in \mathfrak{M}$, $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, it is sufficient to know the action of P on $\mathfrak{F}$

"only modulo $\mathfrak{F}_0$ in the image", i.e., it is sufficient to know the composed map

$$\mathfrak{F} \xrightarrow{\rho_0(P)} \mathfrak{F} \xrightarrow{\pi} \mathfrak{F}/\mathfrak{F}_0.$$

Hence instead of a linear representation $\rho_0 \colon \mathfrak{M} \to L(\mathfrak{F})$ we shall consider a linear map $\rho \colon \mathfrak{M} \to L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ such that, for all P, $Q \in \mathfrak{M}$, the diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{\rho(QP)} & \mathfrak{F}/\mathfrak{F}_0 \\ {\scriptstyle\rho(P)}\downarrow & & \downarrow \\ \mathfrak{F}/\mathfrak{F}_0 & \xrightarrow[\rho(Q)]{} & (\mathfrak{F}/\mathfrak{F}_0)/Q\mathfrak{F}_0 \end{array} \tag{1.1}$$

commutes. (Especially, it is possible to choose $\rho(P) = \pi\rho_0(P)$ for a linear representation ρ_0.) Here $\rho(Q)$ in the second horizontal line is the map induced by $\rho(Q) \colon \mathfrak{F} \to \mathfrak{F}/\mathfrak{F}_0$, and the second vertical line is the canonical quotient map.

The plan of the paper is as follows: In Section 2, we study quasi-invertible operators which are, in particular, elliptic operators for which explicit calculations on asymptotic types are possible (upon an appropriate choice of $\mathfrak{J}$), see Proposition 2.2. In Section 3, we then deal with "linear maps" from $\mathfrak{F}$ to the affine subspaces of $\mathfrak{F}/\mathfrak{F}_0$. (Sending the flat subspace $\mathfrak{F}_0$ to linear subspace of $\mathfrak{F}/\mathfrak{F}_0$ corresponds to the "production of asymptotics" by the elements of $\mathfrak{M}$.) This is preparatory for the introduction of asymptotic algebras followed in Section 4. Then, in Section 5, we describe the reduction of the the quadruple $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$ to a symbol algebra, see Propositions 5.4, 5.5. Finally, in Section 6, we show that wave front sets arise as asymptotic types, and also discuss cone algebras. Further examples will be provided elsewhere.

2. Quasi-invertible Operators

For $P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$, $J \in \operatorname{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, the push-forward J^P of J under P is defined to be the space $J^P = P\mathfrak{F}_J$. In particular, $\mathcal{O}^P = P\mathfrak{F}_0$ characterizes the amount of asymptotics "produced" by P. Furthermore, let L_P for $P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ the largest subspace of $\mathfrak{F}/\mathfrak{F}_0$ such that $(L_P)^P = \mathcal{O}^P$, i.e., $L_P = (P^{-1}(\mathcal{O}) + \mathfrak{F}_0)/\mathfrak{F}_0$. Thus L_P characterizes the amount of asymptotics "annihilated" by P.

For operators P, $Q \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$, we define the composition $Q \circ P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^Q})$ in a manner predicted by (1.1), i.e.,

$$Q \circ P \colon \mathfrak{F} \xrightarrow{P} \mathfrak{F}/\mathfrak{F}_0 \xrightarrow{Q} \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^Q} \cong (\mathfrak{F}/\mathfrak{F}_0)/\mathcal{O}^Q,$$

where, again, the second arrow is induced by $Q\colon \mathfrak{F} \to \mathfrak{F}/\mathfrak{F}_0$.

Definition 2.1. An operator $P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ is called quasi-invertible if there is an operator $Q \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ such that

$$P \circ Q = 1 \text{ in } L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^P}), \quad Q \circ P = 1 \text{ in } L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^Q}). \tag{2.1}$$

Q is called a quasi-inverse to P. (Q is then also quasi-invertible, and P is a quasi-inverse to Q.)

Remark. Under the mere conditions (1.1) and $\rho(1) = 1$, invertible operators in $\mathfrak{M}$ are mapped (by ρ) to quasi-invertible ones.

It is readily seen that the set of all quasi-inverses to P is the affine space

$$Q + \{S \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0);\ S(\mathfrak{F}) \subseteq \mathcal{O}^Q\}. \tag{2.2}$$

Hence we can regard a quasi-invertible operator $P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ as being determined only modulo operators sending $\mathfrak{F}$ to $\mathcal{O}^P$. Even with this freedom of choice, the push-forward J^P for $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ is uniquely determined.

Proposition 2.2. *Let $P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ be quasi-invertible with quasi-inverse Q. Then:*

(a) *$L_P = \mathcal{O}^Q$ and $L_Q = \mathcal{O}^P$;*

(b) *The mapping $J \mapsto (J^P)^Q$ on the complete lattice $\{J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0);\ J \supseteq L_P\}$ is order-preserving. Furthermore, $J \supseteq (J^P)^Q \supseteq L_P$ for all $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ satisfying $J \supseteq L_P$, and $(J^P)^Q = J$ if and only if $J/L_P = J^{Q\circ P}$ (where $J^{Q\circ P} = (Q \circ P)\mathfrak{F}_0$);*

(c) *The mapping*

$$\begin{aligned} \{J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0);\ J \supseteq L_P,\ J/L_P = J^{Q\circ P}\} \\ \longrightarrow \{K \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0);\ K \supseteq L_Q,\ K/L_Q = K^{P\circ Q}\},\quad J \mapsto J^P, \end{aligned} \tag{2.3}$$

is an order-isomorphism with inverse $K \mapsto K^Q$.

Proof. First we show (a). Since P, Q are quasi-inverse to each other, it suffices to prove that $L_P = \mathcal{O}^Q$. Let $u \in \mathfrak{F}$ such that $Pu = \mathcal{O}$. Then $u = Q \circ Pu = \mathcal{O}$ in $\mathfrak{F}/\mathfrak{F}_{\mathcal{O}^Q}$, i.e., $u \in \mathfrak{F}_{\mathcal{O}^Q}$ and $L_P \subseteq \mathcal{O}^Q$. Conversely, for $v \in \mathfrak{F}_0$, we have $v = P \circ Qv = \mathcal{O}$ in $\mathfrak{F}/\mathfrak{F}_{\mathcal{O}^P}$, i.e., $Qv \in L_P$ and $L_P \supseteq \mathcal{O}^Q$.

Further, by Lemma 2.3 below, we have $(J^P)^Q = J^{QP} \vee \mathcal{O}^Q$ for any $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, where $QP \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ is a lifting of $Q \circ P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^Q})$. On the one hand, we get $J^{QP} \vee \mathcal{O}^Q \subseteq J \vee \mathcal{O}^Q$, since $QP = 1 + R$, where $R(\mathfrak{F}) \subseteq \mathcal{O}^Q$, by the quasi-invertibility of P. Hence $(J^P)^Q \subseteq J \vee L_P$ by (a). On the other hand, $J^P \supseteq \mathcal{O}$. Therefore, $(J^P)^Q \supseteq \mathcal{O}^Q = L_P$, again by (a).

To complete the proof of (b), (c), it now suffices (by symmetry) to show that, for each $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, we have $J^P = ((J^P)^Q)^P$. We already know that $J^P \supseteq ((J^P)^Q)^P$. So let $u \in \mathfrak{F}$, $\pi u \in J^P$. Then $\pi u = Pv$ for some $v \in \mathfrak{F}_J$. Now $PQ = 1 + S$ with $S(\mathfrak{F}) \subseteq \mathcal{O}^P$, where $PQ \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ is a lifting of $P \circ Q \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^P})$. We get that $(PQ) \circ Pv - Pv \in \mathcal{O}^P$ and $(PQ) \circ P = (P \circ Q) \circ P = P \circ (Q \circ P)$ in view of the quasi-invertibility of P. Hence $\pi u = Pv \in ((J^P)^Q)^P$, which concludes the proof. $\square$

The following result is easy.

Lemma 2.3. *For P, $Q \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$, $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, we have*

$$(J^P)^Q = J^{QP} \vee \mathcal{O}^Q, \tag{2.4}$$

where $QP \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ is any map that covers $Q \circ P \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^Q})$, i.e., $Q \circ Pu = QPu + \mathcal{O}^Q$ holds for all $u \in \mathfrak{F}$. (In particular, if $\rho\colon \mathfrak{M} \to L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ is a linear map for which (1.1) *is fulfilled, then one can take $\rho(QP)$ for $\rho(Q)\rho(P)$.)*

3. Linear Maps with Values in Affine Spaces

The following section is a technical one. Let $\mathrm{Aff}(\mathfrak{F})$ denote the space of all affine subspaces of $\mathfrak{F}$. Further let

$$\mathrm{Aff}(\mathfrak{F}, \mathfrak{F}_0) = (\mathfrak{F} \times \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0))/\!\sim, \tag{3.1}$$

where the equivalence relation $(v, L) \sim (v', L')$ for $(v, L), (v', L') \in \mathfrak{F} \times \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ is defined by $L = L'$ and $v - v' \in \mathfrak{F}_L$. The subspace $\mathrm{Lin}(\mathfrak{F}, \mathfrak{F}_0) \subseteq \mathrm{Aff}(\mathfrak{F}, \mathfrak{F}_0)$ is given by pairs (v, L) satisfying $v \in \mathfrak{F}_L$. Upon sending the pair (v, L) to $\pi v + L$, the spaces $\mathrm{Aff}(\mathfrak{F}, \mathfrak{F}_0)$ and $\mathrm{Lin}(\mathfrak{F}, \mathfrak{F}_0)$, respectively, are identified with $\mathrm{Aff}(\mathfrak{F}/\mathfrak{F}_0)$ and $\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$.

On $\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0)$, we define an "addition" and a multiplication by scalars by

$$(v,L)+(v',L'):=(v+v',L+L'),$$
$$\lambda(v,L):=(\lambda v,L),\ \ \lambda\in\mathbb{C}.$$

A "linear map" $\mathcal{P}\colon\mathfrak{F}\to\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0)$ is given by a pair $(P,K)\in L(\mathfrak{F})\times\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ with respect to the equivalence relation $(P,K)\sim(P',K')$ for (P,K), $(P',K')\in L(\mathfrak{F})\times\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ when $K=K'$ and $(P-P')(\mathfrak{F})\subseteq\mathfrak{F}_K$. (Thus a "linear map" $\mathcal{P}\colon\mathfrak{F}\to\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0)$ can likewise be given by a pair $(P,K)\in L(\mathfrak{F},\mathfrak{F}/\mathfrak{F}_0)\times\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ with respect to the same equivalence relation: $(P,K)\sim(P',K')$ when $K=K'$ and $(P-P')(\mathfrak{F})\subseteq K$.) To the pair (P,K) we then assign the map $\mathfrak{F}\to\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0)$, $v\mapsto(Pv,\mathcal{O}^P+K)$. The space of all these mappings is denoted by $L(\mathfrak{F},\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0))$.

Lemma 3.1. *The space $L(\mathfrak{F},\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0))$ consists of all mappings $\mathcal{P}\colon\mathfrak{F}\to\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0)$ satisfying $\mathcal{P}(u+v)=\mathcal{P}(u)+\mathcal{P}(v)$ and $\mathcal{P}(\lambda u)=\lambda\mathcal{P}(u)$ for $u,v\in\mathfrak{F}$, $\lambda\in\mathbb{C}$. In that case, we have $\mathcal{P}=(P,K)$ for some $P\in L(\mathfrak{F})$, where $K=\mathcal{P}(0)\in\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$. Furthermore, $\mathcal{P}$ induces a "linear map"*

$$\mathcal{P}\colon\ \mathrm{Aff}(\mathfrak{F})\to\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0),\ \ (u,L)\mapsto\big(Pu,L^P+K\big)\tag{3.2}$$

such that $\mathcal{P}(\mathrm{Lin}(\mathfrak{F}))\subseteq\mathrm{Lin}(\mathfrak{F},\mathfrak{F}_0)$.

Eventually, for $\mathcal{P}\in\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0)$, $J\in\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, we define the push-forward $J^{\mathcal{P}}$ of J under the action of $\mathcal{P}$ by $J^{\mathcal{P}}=\mathcal{P}\mathfrak{F}_J$. Furthermore, for linear operators $\mathcal{P}$, $\mathcal{Q}\in L(\mathfrak{F},\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0))$, where $\mathcal{P}=(P,K)$, $\mathcal{Q}=(Q,L)$, we define the composition by

$$\mathcal{Q}\circ\mathcal{P}=(QP,K^{\mathcal{Q}}+L)\in L(\mathfrak{F},\mathrm{Aff}(\mathfrak{F},\mathfrak{F}_0)).$$

4. Asymptotic Algebras

Now we are in a position to introduce what we shall call an asymptotic algebra. To do so, we start with pre-asymptotic algebras on which the notion of asymptotic type makes sense.

Definition 4.1. A pre-asymptotic algebra is a quadruple $(\mathfrak{M},\rho,\mathfrak{F},\mathfrak{F}_0)$, where $\mathfrak{M}$ is a unital algebra, $\mathfrak{F}$ is a linear space, $\mathfrak{F}_0$ is its linear subspace, and $\rho\colon\mathfrak{M}\to L(\mathfrak{F},\mathfrak{F}/\mathfrak{F}_0)$ is a linear map such that $\mathfrak{p}$, given by $\mathfrak{p}(P)=(\rho(P),\mathcal{O})$ for $P\in\mathfrak{M}$, is a faithful "representation of $\mathfrak{M}$ on $\mathfrak{F}$ modulo $\mathfrak{F}_0$ in the image." The latter means that

$\mathfrak{p}\colon \mathfrak{M} \to L(\mathfrak{F}, \mathrm{Aff}(\mathfrak{F}, \mathfrak{F}_0))$ is an injective "linear map" such that $\mathfrak{p}(1) = (1, \mathcal{O})$ and, for all P, $Q \in \mathfrak{M}$, we have

$$(4.1) \qquad \mathfrak{p}(QP) = \mathfrak{p}(Q) \circ \mathfrak{p}(P) \ \text{ in } L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_{\mathcal{O}^Q}).$$

Note that (4.1) means that, for all P, $Q \in \mathfrak{M}$, the diagram (1.1) commutes. For $P \in \mathfrak{M}$, $J \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, the push-forward $J^{\mathfrak{p}(P)} = J^{\rho(P)}$ shall also be denoted by J^P. Similarly, we shall write L_P instead of $L_{\rho(P)}$. An operator $P \in \mathfrak{M}$ is called quasi-invertible, and $Q \in \mathfrak{M}$ is called a quasi-inverse to P, if $\rho(P) \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ is quasi-invertible with quasi-inverse $\rho(Q)$. This definition is meaningful due to (1.1).

Remark. The seemingly more general definition $\mathfrak{p}(P) = (\rho(P), K)$ for $P \in \mathfrak{M}$ and some linear map $\rho\colon \mathfrak{M} \to L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_K)$, where $K \in \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ (and $\mathfrak{p}(1) = (1, K)$) is in fact not more general — this is seen upon replacing $\mathfrak{F}_0$ with $\mathfrak{F}_K$.

Definition 4.2. A lattice of asymptotic type (l.a.t.) on a pre-asymptotic algebra $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$ is a sublattice of $\mathfrak{J}$ of $\mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ such that

- $\mathcal{O} \in \mathfrak{J}$;
- for each $P \in \mathfrak{M}$ and all $J \in \mathfrak{J}$, there is a $K \in \mathfrak{J}$ such that $J^P \subseteq K$;
- $\bigcap_{\iota \in \mathcal{I}} J_\iota \in \mathfrak{J}$ for each non-empty family $\{J_\iota\}_{\iota \in \mathcal{I}} \subset \mathfrak{J}$.

For the implications of the third condition, see Remark in Section 1. For a given pre-asymptotic algebra $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$, there is an obvious choice of a l.a.t., namely the least subset $\mathfrak{J}_0 \subseteq \mathrm{Lin}(\mathfrak{F}/\mathfrak{F}_0)$ that contains $\mathcal{O}$ and that is closed under push-forwards by operators in $\mathfrak{M}$ as well as under forming non-empty intersections and finite sums. For most applications, however, the l.a.t. $\mathfrak{J}_0$ will be too large. Therefore, one always has the problem of choosing the l.a.t. $\mathfrak{J}$ (in general, $\mathfrak{J} \neq \mathfrak{J}_0$) in an appropriate way.

Definition 4.3. An asymptotic algebra $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is a pre-asymptotic algebra $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$ equipped with a l.a.t. $\mathfrak{J}$.

For an asymptotic algebra $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$, we have the ideal

$$\mathfrak{S}_{\mathfrak{J}} = \{S \in \mathfrak{M};\ \rho(S)(\mathfrak{F}) \subseteq J \text{ for some } J \in \mathfrak{J}\}$$

of residual operators and the multiplicatively closed set containing the group of all invertible elements

$$\mathfrak{E}_{\mathfrak{J}} = \{P \in \mathfrak{M};\ \text{there is a } Q \in \mathfrak{M} \text{ such that } PQ - 1,\ QP - 1 \in \mathfrak{S}_{\mathfrak{J}}\}$$

of elliptic operators, see Section 1.

Remark. It is very likely that, in a final version of the definition of a l.a.t., the condition that the join-irreducible elements are join-dense in $\mathfrak{J}$ has also to be added. This would provide "irreducible elements" as building blocks for asymptotic types.

5. Reduction to Symbol Algebras

In pseudodifferential analysis, a symbol algebra $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is characterized by the property that each elliptic operator $F \in \mathfrak{E}_{\mathfrak{N},\mathfrak{J}}$ is invertible in $\mathfrak{N}$. One possibility in finding an appropriate l.a.t. $\mathfrak{J}$ for the pre-asymptotic algebra $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$ is to reduce it to a symbol algebra $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ (if it exists).

Definition 5.1. An asymptotic algebra $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ is called reduced if $\mathfrak{S}_{\mathfrak{J}} = 0$.

For a reduced asymptotic algebra $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$, each elliptic element $F \in \mathfrak{E}_{\mathfrak{J}}$ is invertible (and its parametrix G is uniquely determined). In fact, $FG - 1 \in \mathfrak{S}_{\mathfrak{J}}$ and $GF - 1 \in \mathfrak{S}_{\mathfrak{J}}$ implies $FG = GF = 1$ in $\mathfrak{N}$.

Definition 5.2. An asymptotic algebra $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J}_{0,\mathfrak{N}})$ is called a symbol algebra for the pre-asymptotic algebra $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$ if $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J}_{0,\mathfrak{N}})$ is reduced and if there is a surjective homomorphism $\Theta\colon \mathfrak{M} \to \mathfrak{N}$ of unital algebras such that

$$(\rho(P) - \sigma(\Theta P))\,(\mathfrak{F}) \subseteq J \tag{5.1}$$

for all $P \in \mathfrak{M}$ and some $J \in \mathfrak{J}_{0,\mathfrak{N}}$ depending on P. Here $\mathfrak{J}_{0,\mathfrak{N}}$ is the l.a.t. $\mathfrak{J}_0$ formed in the pre-asymptotic algebra $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0)$, see the previous section immediately after Definition 4.2.

Lemma 5.3. *For* $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$, $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J}_{0,\mathfrak{N}})$ *being as in* Definition 5.2, *the ideal* $\mathfrak{S}_{\mathfrak{M},\mathfrak{J}_{0,\mathfrak{N}}}$ *is the kernel of the algebra homomorphism* $\Theta\colon \mathfrak{M} \to \mathfrak{N}$.

Proof. Let $S \in \mathfrak{M}$, $\Theta S = 0$. Then $\rho(S)(\mathfrak{F}) \subseteq K$ for some $K \in \mathfrak{J}_{0,\mathfrak{N}}$, by (5.1), i.e., $S \in \mathfrak{S}_{\mathfrak{M},\mathfrak{J}_{0,\mathfrak{N}}}$. Vice versa, if $S \in \mathfrak{S}_{\mathfrak{M},\mathfrak{J}_{0,\mathfrak{N}}}$, then $\sigma(\Theta S)(\mathfrak{F}) \subseteq J$ for some $J \in \mathfrak{J}_{0,\mathfrak{N}}$, again by (5.1). But this implies $\Theta S = 0$, since $\Theta S \in \mathfrak{N}$ and $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J}_{0,\mathfrak{N}})$ is reduced. □

Proposition 5.4. *If* $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J}_{0,\mathfrak{N}})$ *is a symbol algebra for the pre-asymptotic algebra* $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$, *then* $\mathfrak{J}_{0,\mathfrak{N}}$ *is a l.a.t. on* $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0)$.

Proof. Let $P \in \mathfrak{M}$, $J \in \mathfrak{J}_{0,\mathfrak{N}}$. Then $\rho(P) = \sigma(\Theta P) + S$, where $S(\mathfrak{F}) \subseteq J'$ for some $J' \in \mathfrak{J}_{0,\mathfrak{N}}$, by (5.1). But then $J^P \subseteq J^{\Theta P} \vee J' \in \mathfrak{J}_{0,\mathfrak{N}}$. All the other properties are obviously fulfilled. $\square$

Proposition 5.5. *Let* $(\mathfrak{M}, \rho, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ *be an asymptotic algebra with symbol algebra* $(\mathfrak{N}, \sigma, \mathfrak{F}, \mathfrak{F}_0, \mathfrak{J})$ *(with algebra homomorphism* $\Theta\colon \mathfrak{M} \to \mathfrak{N}$*), where* $\mathfrak{J} = \mathfrak{J}_{0,\mathfrak{N}}$*. Then:*

(a) $P \in \mathfrak{M}$ *is elliptic, i.e.,* $P \in \mathfrak{E}_{\mathfrak{M}}$*, if and only if* $\Theta P \in \mathfrak{N}$ *is invertible.*

(b) *Let* $P \in \mathfrak{M}$ *be elliptic,* Q *be a parametrix to* P*, and assume that* $(\rho(P) - \sigma(\Theta P))(\mathfrak{F}) \subseteq \mathcal{O}^P$, $(\rho(Q) - \sigma(\Theta Q))(\mathfrak{F}) \subseteq \mathcal{O}^Q$*. Then:*

(i) $P \in \mathfrak{M}$ *is quasi-invertible with quasi-inverse* $Q \in \mathfrak{M}$*;*
(ii) $L_P, L_Q \in \mathfrak{J}$*;*
(iii) $J^P \in \mathfrak{J}$ *for all* $J \in \mathfrak{J}$*;*
(iv) $(J^P)^Q = J$ *for all* $J \in \mathfrak{J}$ *satisfying* $J \supseteq L_P$*. Furthermore,*

$$\{J \in \mathfrak{J};\ J \supseteq L_P\} \to \{K \in \mathfrak{J};\ K \supseteq L_Q\},\quad J \mapsto J^P,$$

is an order-isomorphism with inverse $K \mapsto K^Q$*, see also* (2.3).

Proof. (a) By Lemma (5.3), $\mathfrak{N} \cong \mathfrak{M}/\mathfrak{S}_{\mathfrak{M},\mathfrak{J}}$. Thus $\Theta\mathfrak{E}_{\mathfrak{M},\mathfrak{J}} \cong \mathfrak{E}_{\mathfrak{M},\mathfrak{J}}/\mathfrak{S}_{\mathfrak{M},\mathfrak{J}}$ is the group of invertible elements of $\mathfrak{N}$. Also $\mathfrak{E}_{\mathfrak{M},\mathfrak{J}} = \Theta^{-1}(\Theta\mathfrak{E}_{\mathfrak{M},\mathfrak{J}})$.

(b) Now assume that $P \in \mathfrak{M}$ is elliptic, Q is a parametrix to P, and $(\rho(P) - \sigma(\Theta P))(\mathfrak{F}) \subseteq \mathcal{O}^P$, $(\rho(Q) - \sigma(\Theta Q))(\mathfrak{F}) \subseteq \mathcal{O}^Q$. Since ΘP, $\Theta Q \in \mathfrak{N}$ are inverse to each other, by (2.2) and (5.1), $\rho(P)$, $\rho(Q) \in L(\mathfrak{F}, \mathfrak{F}/\mathfrak{F}_0)$ are quasi-inverse to each other, i.e., $P \in \mathfrak{M}$ is quasi-invertible with quasi-inverse Q.

Then (iii) is implied by $J^P = J^{\Theta P}$ which holds for all $J \in \operatorname{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, see the paragraph after (2.2). Then (ii) also follows, since $L_P = \mathcal{O}^Q \in \mathfrak{J}$, similarly for $L_Q = \mathcal{O}^P$.

It remains to show that $J = (J^P)^Q$ for all $J \in \operatorname{Lin}(\mathfrak{F}/\mathfrak{F}_0)$, $J \supseteq L_P$. To this end, it suffices to note that $(J^P)^Q = (J^{\Theta P})^{\Theta Q} = J^{(\Theta P)(\Theta Q)} \vee \mathcal{O}^{\Theta Q} = J^1 \vee \mathcal{O}^Q = J \vee L_P$ for any $J \in \operatorname{Lin}(\mathfrak{F}/\mathfrak{F}_0)$. $\square$

Remark. To discuss the elliptic regularity for the equation $Pu = f$ with right-hand side $f \in K$, where $K \in \mathfrak{J}$, in (b) of the foregoing proposition it is not necessary to assume that P possesses a quasi-inverse Q belonging to $\mathfrak{M}$. Instead, we obtain that all solutions $u \in \mathfrak{F}$ belong to the space $\mathfrak{F}_{K^G}$, where $K^G \in \mathfrak{J}$ and $K^G \supseteq L_P$. Here $G \in \mathfrak{N}$ is the inverse to ΘP, i.e., $(\Theta P)G = G(\Theta P) = 1$ in $\mathfrak{N}$.

6. Examples

6.1. **Wave front sets.** Let X be a C^∞-manifold. For a linear, sequentially continuous operator $A\colon C_0^\infty(X) \to \mathcal{D}'(X)$, let $\mathrm{WF}'(A)$ denote the wave front relation of A, i.e.,

$$\mathrm{WF}'(A) = \{(x,\xi;y,\eta) \in (T^*X \times T^*Y) \setminus 0;\ (x,y,\xi,-\eta) \in \mathrm{WF}(K_A)\},$$

where $(x,\xi) \in T^*X$ and $(y,\eta) \in T^*Y$, respectively, are generic points and $(T^*X \times T^*Y)\setminus 0 \cong T^*(X \times Y)\setminus 0$ are canonically identified; $K_A(x,y)$ is the kernel of A. For details, see [H71]. Further let

$$\mathrm{WF}'_x(A) = \{(x,\xi) \in T^*(X)\setminus 0;\ (x,\xi;y,0) \in \mathrm{WF}'(A) \text{ for some } y \in X\}$$
$$\mathrm{WF}'_y(A) = \{(y,\eta) \in T^*(X)\setminus 0;\ (x,0;y,\eta) \in \mathrm{WF}'(A) \text{ for some } x \in X\}$$

Let $\pi\colon X \times X \to X$, $(x,y) \mapsto x$, be the projection onto the first component. Then we consider

$$\begin{aligned} \mathfrak{M} = \{A\colon C_0^\infty(X) \to \mathcal{D}'(X);\ &A \text{ is linear, sequentially continuous,} \\ &\mathrm{WF}'_y(A) = \emptyset, \text{ and the mapping } \pi\colon \operatorname{supp} K_A \to X \text{ is proper}\}. \end{aligned} \tag{6.1}$$

$\mathfrak{M}$ is a unital algebra. Note that, for all $A \in \mathfrak{M}$, in fact, $A\colon \mathcal{D}'(X) \to \mathcal{D}'(X)$ by continuous extention. Thus we can choose $\mathfrak{F} = \mathcal{D}'(X)$, $\mathfrak{F}_0 = C^\infty(X)$, and $\rho\colon \mathfrak{M} \hookrightarrow L(\mathcal{D}'(X))$ is the canonical embedding. Then $(\mathfrak{M}, \rho, \mathcal{D}'(X), C^\infty(X))$ becomes a pre-asymptotic algebra.

Now, for $A \in \mathfrak{M}$, both of the subsets $\pi(\mathrm{WF}'(A)) \subseteq T^*X\setminus 0$ and $\mathrm{WF}'_x(A) \subseteq T^*X\setminus 0$ are closed and conic. We obtain $A(\mathcal{D}'(X)) \subseteq \mathcal{D}'_{\pi(\mathrm{WF}'(A))}(X)$ and $\mathfrak{F}_{\mathcal{O}^A} = \mathcal{D}'_{\mathrm{WF}'_x(A)}(X)$, where $\mathcal{D}'_\Gamma = \{u \in \mathcal{D}'(X);\ \mathrm{WF}(u) \subseteq \Gamma\}$ for a closed, conic subset $\Gamma \subseteq T^*X \setminus 0$.

Therefore, the appropriate choice for the l.a.t. $\mathfrak{J}$ is

$$\mathfrak{J} = \{\Gamma \subseteq T^*X \setminus 0;\ \Gamma \text{ conic, closed}\}, \tag{6.2}$$

where $\Gamma \in \mathfrak{J}$ is identified with the linear subspace $\mathcal{D}'_\Gamma(X)/C^\infty(X) \subseteq \mathcal{D}'(X)/C^\infty(X)$. Obviously, $\mathfrak{J}$ is order-isomorphic to the lattice of all closed subsets of the cosphere bundle S^*X.

Remark. In this example, $\mathfrak{S}_\mathfrak{J} = \mathfrak{M}$, as is always the case if $\mathfrak{F}/\mathfrak{F} \in \mathfrak{J}$. Especially, each $A \in \mathfrak{M}$ is elliptic, where every $B \in \mathfrak{M}$ serves as a parametrix to A.

6.2. **Cone algebras.** Let X be a manifold with conical singularity, x_0. Hence $X \setminus \{x_0\}$ is a C^∞-manifold and there are a neighbourhood $U \ni x_0$ in X and a homeomorphism $\chi\colon U \to \big([0,1) \times Y\big)/\big(\{0\} \times Y\big)$ for some closed C^∞-manifold Y that restricts to a C^∞-diffeomorphism $\chi|_{U\setminus\{x_0\}}\colon U \setminus \{x_0\} \to (0,1) \times Y$. Close to x_0, χ splits the coordinates $U \setminus \{x_0\} \to (0,1) \times Y$, $x \mapsto (t,y)$, into a radial component $t \in [0,1)$ and a point $y \in Y$ on the cone base. On $X \setminus \{x_0\}$, we have Schulze's algebra $C^0(X,(\delta,\delta,(-\infty,0]))$ of cone pseudodifferential operators on X of order 0 with respect to some fixed conormal order $\delta \in \mathbb{R}$ and asymptotic information carried on the weight strip $\{z \in \mathbb{C};\ \operatorname{Re} z < \dim X/2 - \delta\}$. For details, see [S98].

Let $C^{\infty,\delta}_{\mathrm{as}}(X)$ be the space of all functions $u \in C^\infty(X \setminus \{x_0\})$ that possess a conormal asymptotic expansion as $x \to x_0$ of conormal order at least δ. The latter means that

$$u(x) \sim \sum_{j=1}^{M} \sum_{k=1}^{m_j} t^{-p_j} \log^{m_j-k} t\, c_{jk}(y) \ \text{ as } t \to +0, \tag{6.3}$$

where $M \in \mathbb{N}_0 \cup \{\infty\}$, $p_j \in \mathbb{C}$, $\operatorname{Re} p_j < \dim X/2 - \delta$, $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ when $M = \infty$, $m_j \in \mathbb{N}_0$, and $c_{jk} \in C^\infty(Y)$. Asymptotics are understood in an increasing order of flatness. They are uniquely determined provided that they exist (and $c_{j1}(y) \not\equiv 0$ holds for each j). The space $C^{\infty,\delta}_{\mathrm{as}}(X)$ only depends on the conormal order δ, but not on the chosen splitting of coordinates $x \mapsto (t,y)$ close to x_0. Recall that

$$A\colon C^{\infty,\delta}_{\mathrm{as}}(X) \to C^{\infty,\delta}_{\mathrm{as}}(X) \tag{6.4}$$

for all $A \in C^0(X,(\delta,\delta,(-\infty,0]))$. We further have the space $C^\infty_{\mathcal{O}}(X) \subset C^{\infty,\delta}_{\mathrm{as}}(X)$ of all functions $u \in C^\infty(X \setminus \{x_0\})$ such that $u(x) \sim 0$ as $x \to x_0$ (i.e., $M = 0$ in (6.3)). Hence

$$\big(C^0(X,(\delta,\delta,(-\infty,0])),\rho,C^{\infty,\delta}_{\mathrm{as}}(X),C^\infty_{\mathcal{O}}(X)\big), \tag{6.5}$$

where the embedding $\rho\colon C^0(X,(\delta,\delta,(-\infty,0])) \to L(C^{\infty,\delta}_{\mathrm{as}}(X))$ is supplied by (6.4), is a pre-asymptotic algebra. Note that, in general, $A(C^\infty_{\mathcal{O}}(X)) \not\subseteq C^\infty_{\mathcal{O}}(X)$ for $A \in C^0(X,(\delta,\delta,(-\infty,0]))$.

There are different notions of asymptotic type associated with (6.5): weakly discrete, discrete, and continuous ones; see again [S98].

Let us discuss the first two of these concepts. A weakly discrete asymptotic type $P \in \underline{\mathrm{As}}^\delta$ is represented by a sequence $\{(p_j,m_j)\}_{j=1}^M \subset \mathbb{C} \times \mathbb{N}_0$, where p_j, m_j are as in (6.3), and two sequences $\{(p_j,m_j)\}_{j=1}^M$, $\{(q_k,n_k\}_{k=1}^N$ represent the same weakly discrete asymptotic type P if, for each j, there is a k such that $p_j = q_k$, $m_j \le n_k$ and if, for each k', there is a j' such that $p_{j'} = q_{k'}$, $m_{j'} \ge n_{k'}$. A function $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ is said to have asymptotics of type P ($u \in C^\infty_P(X)$ for short) if the singular exponents p_j and their multiplicities m_j in (6.3) are given by a sequence representing P (with appropriate coefficients $c_{jk} \in C^\infty(Y)$). Weakly discrete asymptotic types, P, when identified with the linear spaces $C^\infty_P(X)/C^\infty_{\mathcal{O}}(X) \subset C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X)$, form a l.a.t. on (6.5), where the property that, for all $A \in C^0(X,(\delta,\delta,(-\infty,0]))$, $P \in \underline{\mathrm{As}}^\delta$, there is some $Q \in \underline{\mathrm{As}}^\delta$ such that $P^A \subseteq Q$ is part of the theory of cone pseudodifferential operators, while all the other properties are obvious.

A discrete asymptotic type $P \in \underline{\mathrm{As}}^\delta(Y)$ is a given by a sequence $\{(p_j,m_j,L_j)\}_{j=1}^M$, where $\{(p_j,m_j)\}_{j=1}^M$ represents a weakly discrete asymptotic type in $\underline{\mathrm{As}}^\delta$ and L_j is a chain $L_{j1} \subseteq L_{j2} \subseteq \cdots \subseteq L_{jm_j}$ of finite-dimensional subspaces of $C^\infty(Y)$. For $u \in C^{\infty,\delta}_{\mathrm{as}}(X)$ to have asymptotics of type P, it is then additionally required that $c_{jk} \in L_{jk}$ holds for all j, $1 \le k \le m_j$. (For $\dim Y = 0$, there is no distinction between a weakly discrete asymptotic type and a discrete asymptotic type.) The rest of the analysis is as for weakly discrete asymptotic types. In particular, the property that, for all $A \in C^0(X,(\delta,\delta,(-\infty,0]))$, $P \in \underline{\mathrm{As}}^\delta(Y)$, there is some $Q \in \underline{\mathrm{As}}^\delta(Y)$ such that $P^A \subseteq Q$ is again part of the theory of cone pseudodifferential operators.

Remark. In [S98], it has been required that $L_{j1} = L_{j2} = \ldots = L_{jm_j}$. In this case, a "l.a.t." still arises. It is, however, not a sublattice of the complete lattice $\mathrm{Lin}(C^{\infty,\delta}_{\mathrm{as}}(X)/C^\infty_{\mathcal{O}}(X))$.

Weakly discrete asymptotic types $P = \{(p_j,m_j)\}_{j=1}^M$ are invariant under coordinate changes if one additionally requires that, for each j, there is a j' such that $p_j - 1 = p_{j'}$ and $m_j \le m_{j'}$. Although the algebra $C^0(X,(\delta,\delta,(-\infty,0]))$ is coordinate invariant, see [KSW00], the notion of a discrete asymptotic type is not (it explicitly depends on the chosen splitting of coordinates $x \mapsto (t, y)$ close to x_0; when changing these coordinates, the dimensions of the spaces L_{jk} are, in general, getting larger.)

Finally, let us note that the pre-asymptotic algebra (6.5) can be reduced to a symbol algebra $\mathfrak{N}$, namely to the algebra of all complete conormal symbols $\{\sigma_c^{-j}(A)(z);\ j \in \mathbb{N}_0\}$ of operators $A \in C^0(X, (\delta, \delta, (-\infty, 0]))$ under the Mellin translation product:

$$\sigma_c^{-l}(AB)(z) = \sum_{j+k=l} \sigma_c^{-j}(A)(z-k)\, \sigma_c^{-k}(B)(z), \quad l = 0, 1, 2, \ldots$$

Here $\sigma_c^{-j}(A)(z)$ is, among others, a meromorphic function of $z \in \mathbb{C}$ taking values in the classical pseudodifferential operators of order 0 on Y. The resulting l.a.t. $\mathfrak{J}_{0,\mathfrak{N}}$ has been calculated in [LW01]. The resulting refined notion of asymptotic type turns out both to be coordinate invariant (in a well-defined sense) and to be well-adapted to certain aspects of non-linear analysis.

Remark. In [LW01], it has also been proposed to continue to call a weakly discrete asymptotic type weakly discrete, while, henceforth, a discrete asymptotic type should be instead called strongly discrete; to reserve the notion of a discrete asymptotic type for the (new, refined) asymptotic types belonging to $\mathfrak{J}_{0,\mathfrak{N}}$.

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University of Potsdam, Institute of Mathematics, D-14415 Potsdam, Germany

E-mail address: ingo@math.uni-potsdam.de