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<td>Author(s)</td>
<td>Yamazaki, Susumu</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1211: 86-95</td>
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<tr>
<td>Issue Date</td>
<td>2001-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41126">http://hdl.handle.net/2433/41126</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Microlocalization
of Topological Boundary Value Morphism
and Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation
of the boundary value problems for hyperfunction or microfunction solutions to a system
of linear partial differential equations with analytic coefficients (that is, a coherent (left)
$\mathcal{D}$-Module, here in this article, we shall write Module with a capital letter, instead of
sheaf of modules). If the system is regular-specializable, the nearby-cycle of the system
can be defined in the theory of $\mathcal{D}$-Modules. After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 2], [Sc 3], for any hyperfunction solutions to
regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value mor-
phism which takes values in hyperfunction solutions to the nearby-cycle of the system
instead of the induced system. This morphism is injective (cf. [MF 2]) and a general-
ization of the non-characteristic boundary value morphism (for the non-characteristic
case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). More-
over recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value
morphism and discussed the solvability under a kind of hyperbolicity condition (the near-
hyperbolicity). However, since this morphism is defined only for hyperfunction solutions,
a microlocal boundary value problem is not considered. Therefore in this article, we
shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-
Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

*Research Fellow of The Japan Society for The Promotion of Science.
1 Notation

We denote the set of integers, of real numbers and of complex numbers by \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \) respectively as usual. Moreover we set \( \mathbb{N} := \{ n \in \mathbb{Z}; n \geq 1 \} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{ 0 \} \).

All the manifolds are assumed to be paracompact. Let \( \tau: E \to Z \) a vector bundle over a manifold \( Z \). Then, set \( \dot{E} := E \setminus Z \) and \( \tau \) the restriction of \( \tau \) to \( \dot{E} \). Let \( M \) be an \( (n+1) \)-dimensional real analytic manifold and \( N \) a one-codimensional closed real analytic submanifold of \( M \). Let \( X \) and \( Y \) be complexifications of \( M \) and \( N \) respectively such that \( Y \) is a closed submanifold of \( X \) and that \( Y \cap M = N \). Moreover, we assume the existence of a partial complexification of \( M \) in \( X \), that is, there exists a \( (2n+1) \)-dimensional real analytic submanifold \( L \) of \( X \) containing both \( M \) and \( Y \) such that the triplet \( (N, M, L) \) is locally isomorphic to \( (\mathbb{R}^n \times \{ 0 \}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R}) \) by a local coordinate system \( (z, \tau) = (x + \sqrt{-1} y, t + \sqrt{-1} s) \) of \( X \) around each point of \( N \). We say such a coordinate system admissible. We shall mainly follow the notation in Kashiwara-Schapira [K-S]; we denote the normal deformations of \( N \) and \( Y \) in \( M \) and \( L \) by \( \overline{M}_N \) and \( \tilde{L}_Y \) respectively and regard \( \overline{M}_N \) as a closed submanifold of \( \tilde{L}_Y \). We have the following commutative diagram:

![Diagram](attachment:image.png)

and by admissible coordinates we have locally the following relation:

\[
\begin{align*}
N &= \mathbb{R}^n_x \times \{ 0 \} & M &= \mathbb{R}^n_x \times \mathbb{R}_t \\
Y &= \mathbb{C}^n_z \times \{ 0 \} & L &= \mathbb{C}^n_z \times \mathbb{R}_t & X &= \mathbb{C}^n_z \times \mathbb{C}_\tau.
\end{align*}
\]

With these coordinates, we often identify \( T_Y X \) and \( T_Y L \) with \( X \) and \( L \) respectively.

The projection \( \tau_Y: T_Y L \to Y \) and \( s_L: T_Y L \to \tilde{L}_Y \) induce natural mappings:

\[
T^*_N Y \leftarrow T^*_N M \times T^*_N Y \xrightarrow{\tau_Y^*} T^*_N M T_Y L \xleftarrow{i_Y^*} T^*_N M \times T^*_M \tilde{L}_Y \xrightarrow{i^*_L} T^*_M \tilde{L}_Y,
\]

and by these mappings, we identify \( T^*_N M T_Y L \) with \( T_N M \times T^*_N Y \) and \( T_N M \times T^*_M \tilde{L}_Y \).
$T_Y L \setminus T_Y Y$ has two components with respect to its fiber. We denote one of them by $T_Y L^+$ and represent (at least locally) by fixing an admissible coordinate system

$$T_Y L^+ = \{(z, t) \in T_Y L; \ t > 0\}.$$  

Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Set an open embedding $f : T_Y L^+ \hookrightarrow T_Y L$ and $f_N := f\big|_{T_N M^+} : T_N M^+ \hookrightarrow T_N M$. We regard $T_N M^+ \times T_N^* Y$ as an open set of $T_{T_N M}^* T_Y L$. Moreover $f$ induces mappings:

$$T_{T_N M}^* T_Y L^+ \dashv T_N^* M^+ \times T_N^* Y \dashv f_N \times \text{id}.$$ 

Hence we identify $T_{T_N M}^* T_Y L^+ \times T_{T_N M}^* T_Y L$ with $T_N^* M^+ \times T_N^* Y$, and $f_N$ with $f_N \times \text{id}$.

Let $\pi_{N,M} : T_Y L \rightarrow \tilde{L}_Y \rightarrow \overline{M}_N$ and $\pi_{N|M} : T_{T_N M}^* T_Y L \rightarrow T_N M$, be the natural projections. We denote as usual by $\nu$ and $\mu$ the Sato specialization and microlocalization functors respectively.

## 2 General Boundary Values

By using an admissible coordinate system we define a continuous section $\sigma : Y \rightarrow \tilde{T}_Y X$ by $z \mapsto (z, 1)$. Similarly we define $\mathcal{S} : Y \rightarrow \tilde{T}_Y X$ by $z \mapsto (z, 1)$. In general, let $Z$ be a complex manifold, $\tau : E \rightarrow Z$ a complex vector bundle. Then, denote by $\mathcal{D}_c^b(E)$ the subcategory of $\mathcal{D}^b(E)$ consisting of $C^\infty$-conic objects.

### 2.1 Theorem.

For any object $\mathcal{F}$ of $\mathcal{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \text{Ob}(\mathcal{D}_c^b(T_Y X))$, there exists the following natural isomorphism:

$$f_{\pi}^{-1} \mu_{T_N M}(\nu_Y(i_{i'} \mathcal{F})) \cong f_{\pi}^{-1} \tau_{\pi}^{-1} \mu_N(\sigma^{-1} \nu_Y(\mathcal{F})) \otimes \omega_{L/X}.$$

### 2.2 Definition.

For any object $\mathcal{F}$ of $\mathcal{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \text{Ob}(\mathcal{D}_c^b(T_Y X))$, we define by virtue of Kashiwara-Schapira [K-S] and Theorem 2.1:

$$\beta : f_{\pi}^{-1} s_{L\pi}^{-1} \mu_{\tilde{M}_N}(Rj_L \tilde{p}_L^{-1} i_L \mathcal{F}) \rightarrow f_{\pi}^{-1} \mu_{T_N M}(\nu_Y(i_{i'} \mathcal{F})) \cong f_{\pi}^{-1} \tau_{\pi}^{-1} \mu_N(\sigma^{-1} \nu_Y(\mathcal{F})) \otimes \omega_{L/X}.$$

### 2.3 Definition (Laurent-Monteiro Fernandes [L-MF 2]).

We say an object $\mathcal{F}$ of $\mathcal{D}^b(X)$ is near-hyperbolic at $x_0 \in N$ (in dt-codirection) if there exist positive constants $C$ and $\epsilon_1$ such that

$$\mathcal{S}(\mathcal{F}) \cap \{(z, \tau; z^*, \tau^*) \in T^* X; |z - x_0|, |	au| < \epsilon_1, 0 < \text{Re} \tau\}$$

$$\subset \{(z, \tau; z^*, \tau^*) \in T^* X; |\text{Re} \tau^*| < C(|\text{Im} z^*|(|\text{Im} z| + |\text{Im} \tau|) + |\text{Re} z^*|)\}$$

holds by an admissible coordinate system. Here $\mathcal{S}(\mathcal{F})$ denotes the microsupport of $\mathcal{F}$.
2.4 Theorem. Let $\mathcal{F}$ be a object of $\mathcal{D}^{b}(X)$. Assume that $\nu_{Y}(\mathcal{F}) \in \text{Ob}(\mathcal{D}^{b}_{C^{\ast}}(T_{Y}X))$ and $\mathcal{F}$ is near-hyperbolic at $x_{0} \in N$. Then, for any $p^{\ast} \in T_{N,M}^{\ast}T_{Y}L^{\ast}$

$$\beta: s_{L^{\ast}}^{-1} \mu_{M_{N}}^{-1}(Rj_{L\ast} \tilde{\rho}_{L}^{-1} i_{L}^{\ast} \mathcal{F})_{p^{\ast}} \to \mu_{N}(\sigma^{-1} \nu_{Y}(\mathcal{F}))_{\nu_{Y}(p^{\ast})} \otimes \omega_{L/X}$$

is an isomorphism.

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable $\mathcal{D}$-Module and its nearby-cycle.

As usual, we denote by $\mathcal{D}_{X}$ the sheaf on $X$ of holomorphic differential operators, and by $\{\mathcal{D}_{X}^{(m)}\}_{m \in \mathbb{N}_{0}}$ the usual order filtration on $\mathcal{D}_{X}$.

3.1 Definition. Denote by $\mathcal{I}_{Y}$ the defining Ideal of $Y$ in $\mathcal{O}_{X}$ with a convention that $\mathcal{I}_{Y}^{j} = \mathcal{O}_{X}$ for $j \leq 0$. The $V$-filtration $\{V_{Y}^{k}(\mathcal{D}_{X})\}_{k \in \mathbb{Z}}$ (along $Y$) is a filtration on $\mathcal{D}_{X}|_{Y}$ defined by

$$V_{Y}^{k}(\mathcal{D}_{X}) := \bigcap_{j \in \mathbb{Z}} \{P \in \mathcal{D}_{X}|_{Y}; P\mathcal{I}_{Y}^{j} \subset \mathcal{J}_{Y}^{j-k}\}.$$

Let us denote by $\vartheta$ the Euler operator. Note that $\vartheta \in V_{Y}^{0}(\mathcal{D}_{X}) \setminus V_{Y}^{-1}(\mathcal{D}_{X})$ and that $\vartheta$ can be represented by $\tau \partial_{r}$ by admissible coordinates.

3.2 Definition. A coherent $\mathcal{D}_{X}|_{Y}$-Module $\mathcal{M}$ is said to be regular-specializable (along $Y$) if there exist locally a coherent $\mathcal{O}_{X}$-sub-Module $\mathcal{M}_{0}$ of $\mathcal{M}$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that the following conditions are satisfied:

1. $\mathcal{M}_{0}$ generates $\mathcal{M}$ over $\mathcal{D}_{X}$; that is, $\mathcal{M} = \mathcal{D}_{X} \mathcal{M}_{0}$;
2. $b(\vartheta) \mathcal{M}_{0} \subset (\mathcal{D}_{X}^{(m)} \cap V_{Y}^{-1}(\mathcal{D}_{X})) \mathcal{M}_{0}$, where $m$ is the degree of $b(\alpha)$.

In what follows, we shall omit the phrase "along $Y$" since $Y$ is fixed.

3.3 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_{X}|_{Y}$-Module for which $Y$ is non-characteristic. Then, it is easy to see that $\mathcal{M}$ is regular-specializable.

(2) Kashiwara-Kawai [K-K] proved that every regular-holonomic $\mathcal{D}_{X}|_{Y}$-Module is regular-specializable.

3.4 Proposition. If $\mathcal{M}$ is a regular-specializable $\mathcal{D}_{X}|_{Y}$-Module, $R\text{Hom}_{\mathcal{D}_{X}}(\mathcal{M}, \mathcal{O}_{X})$ and $R\text{Hom}_{\mathcal{D}_{X}}(\mathcal{M}, \mathcal{O}_{X})$ are objects of $\mathcal{D}^{b}_{C^{\ast}}(T_{Y}X)$ and $\mathcal{D}^{b}_{C^{\ast}}(T_{Y}X)$ respectively.

Let $\iota: Y \to X$ be the natural inclusion. Then the induced system, or the inverse image in the sense of $\mathcal{D}$-Modules is defined by $D\iota^{\ast} \mathcal{M} := \mathcal{O}_{Y} \otimes_{\mathcal{O}_{X}} L_{\iota}^{-1} \mathcal{M}$.

For any regular-specializable $\mathcal{D}_{X}$-Module $\mathcal{M}$, the nearby-cycle $\Psi_{Y}(\mathcal{M})$ of $\mathcal{M}$ and the vanishing-cycle $\Phi_{Y}(\mathcal{M})$ of $\mathcal{M}$ in the theory of $\mathcal{D}$-Modules can be defined. For the definitions of $\Psi_{Y}(\mathcal{M})$ and $\Phi_{Y}(\mathcal{M})$, we refer to Laurent [L], Mebkhout [Me]. We shall recall the following two results:
3.5 Proposition (Laurent [L], Mebkhout [Me]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, $\Phi_Y(\mathcal{M})$, $\Psi_Y(\mathcal{M})$ and each cohomology of $D^*_L \mathcal{M}$ are coherent $\mathcal{D}_Y$-Modules. Moreover, there exists the following distinguished triangle:

$$\Phi_Y(\mathcal{M}) \xrightarrow{\text{Var}} \Psi_Y(\mathcal{M}) \to D^*_L \mathcal{M} \to 1.$$

Here, $\text{Var} := \varphi(\partial) \tau$ with $\varphi(\zeta) := (e^{2\pi \sqrt{-1} \zeta} - 1)/\zeta$.

3.6 Theorem (Laurent [L]). Let $\mathcal{E}^R_{Y|X}$ be the sheaf of real holomorphic microfunctions on $T^*_Y X$ as usual. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, there exists the following isomorphism of distinguished triangles:

$$\mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_X}(M, \mathcal{O}_X)|_Y \to \mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_X}(M, \sigma^{-1} \nu_Y(\mathcal{O}_X)) \to \mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_X}(M, \sigma^{-1} \mathcal{E}^R_{Y|X}) \to 1.$$

$$\mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_Y}(D^*_L \mathcal{M}, \mathcal{O}_Y) \to \mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{O}_Y) \to \mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{O}_Y) \to 1.$$

3.7 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$\mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_Y}(D^*_L \mathcal{M}, \mathcal{O}_Y) \simeq \mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_{\mathcal{D}_X}(M, \mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

4 Boundary Values for Regular-Specializable System

We denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{E}_M$ the sheaf of holomorphic functions on $X$, of hyperfunctions on $M$ and of microfunctions on $T^*_M X$ respectively.

4.1 Definition (Oaku [Oa 2], Oaku-Yamazaki [O-Y]). We set:

$$\mathcal{E}_{N|M} := s_{L*}^{-1} \mu_{\overline{M}_N}(Rj_L \ast \overline{p}_L^{-1} i_L \mathcal{O}_X) \otimes \mathcal{O}_{M/X}[n + 1].$$

We can regard $\mathcal{E}_{N|M}$ as a microlocalization of $\nu_N(\mathcal{B}_M)$.

4.2 Proposition. (1) $\mathcal{E}_{N|M}$ is concentrated in degree zero; that is, $\mathcal{E}_{N|M}$ is regarded as a sheaf on $T^*_N M \nu_Y L$. Further $\mathcal{E}_{N|M}|_{T^*_N M} = \nu_N(\mathcal{B}_M)$ holds.

(2) There exists the following exact sequence on $T^*_N M$:

$$0 \to \nu_Y(\mathcal{O}_L)|_{T^*_N M} \to \nu_N(\mathcal{B}_M) \to \hat{\mathcal{E}}_{N|M} \to 0.$$

Here $\mathcal{B}_L := \mathcal{H}_L(\mathcal{O}_X) \otimes \mathcal{O}_{L/X}$ is the sheaf of hyperfunctions with holomorphic parameters on $L$. Note that $\nu_Y(\mathcal{B}_L)$ is concentrated in degree zero.
4.3 Definition. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. By Proposition 3.4, $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ satisfies the assumption of Theorem 2.1. Thus, by Definition 2.2 and Theorem 3.6, we define:

$$\beta: f_\pi^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \rightarrow f_\pi^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

4.4 Theorem. (1) The morphism $\beta$ gives a monomorphism

$$\beta^0: f_\pi^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \rightarrow f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

(2) The restriction of $\beta^0$ to the zero-section $T_NM^+$ coincides with the boundary value morphism in the sense of Monteiro Fernandes [MF1].

4.5 Definition. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$-Module. Then we say $\mathcal{M}$ is near-hyperbolic at $x_0 \in N$ (in $dt$-codirection) if $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is near-hyperbolic in the sense of Definition 2.3. Here, we remark that $SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \text{char} \langle \mathcal{M} \rangle$.

The following theorem is a direct consequence of Theorem 2.4:

4.6 Theorem. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Assume that $\mathcal{M}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* \in T_{T_NM^+}^*T_YL^+$

$$\beta: R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})_{p^*} \rightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N)_{\tau_{Y\pi}(p^*)}$$

is an isomorphism.

4.7 Remark. Let $\mathcal{C}_{N|M}^F$ be the sheaf of $F$-mild microfunctions on $T_{T_NM^+}^*T_YL$, and set $\mathcal{C}_{N|M}^A := \mathcal{H}^n(\mu_N(\mathcal{O}_X|_Y)) \otimes \mathcal{O}_{N/Y}$ (see Oaku [Oa 1], [Oa 2], and Oaku-Yamazaki [O-Y]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Set $\mathcal{M}_Y := \mathcal{H}^0(D\iota^*\mathcal{M}) = \mathcal{O}_Y \otimes \iota^{-1}\mathcal{M}$.

By the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

$$\begin{array}{ccc}
\rightarrow & \rightarrow & \rightarrow \\
\downarrow & \downarrow & \downarrow \\
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) & \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N)
\end{array}$$

that is, the boundary value morphism

$$\gamma^F: f_\pi^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \rightarrow f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N)$$

and $\beta^0$ are compatible. In particular, if $Y$ is non-characteristic for $\mathcal{M}$, then it is known that $\Psi_Y(\mathcal{M}) \simeq D\iota^*\mathcal{M} \simeq \mathcal{M}_Y$ and by Oaku [Oa 2] (cf. Oaku-Yamazaki [O-Y]) we have:

$$\tilde{\gamma}_{N|M}: R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \simeq \tau_{Y\pi}^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N).$$
In this case we see that $\beta^{0}$ is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa 2]). In particular, the restriction of $\beta^{0}$ to the zero-section $T_{Y}M^{+}$ is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Further, if $Y$ is non-characteristic for $M$ and $\pm dt \in T_{Y}M$ is hyperbolic for $M$, then the nearly-hyperbolic condition is satisfied and $\beta$ is an isomorphism.

4.8 Example. Assume that $X = \mathbb{C}^{n+1}$ and so on by an admissible coordinate system.

(1) Let $b(\alpha)$ be a non-zero polynomial with degree $m$, and $Q \in \mathcal{D}_{X}^{(m)} \cap V_{Y}^{-1}(\mathcal{D}_{X})$. Set $M := \mathcal{D}_{X}/\mathcal{D}_{X}(b(\vartheta) + Q)$. Then $M$ is regular-specializable. Assume that $b(\alpha) = \prod_{j=1}^{\mu}(\alpha - \alpha_{j})^{\nu_{j}}$. Then a direct calculation shows that $\mathcal{Y}(M) \simeq \mathcal{Y}^{m}$, and $\beta^{0}$ is equivalent to $\gamma$ in Oaku [Oa 2]: Let $p^{*} = (x_{0}, f_{0}; -\sqrt{-1}\langle \xi_{0}, dx \rangle)$ be a point of $T_{T_{N}M}^{*}T_{Y}L^{+}$, and $f(x, t)$ a germ of $\mathcal{H}om_{\mathcal{D}_{X}}(M, \mathcal{C}_{N|M})$ at $p^{*}$. Then, we can see that $f(x, t)$ has a defining function

$$F(z, \tau) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_{j}} F_{jk}(z, \tau) \tau^{k-j} (\log \tau)^{k-1}.$$ 

Here each $F_{jk}(z, \tau)$ is holomorphic on a neighborhood of $\{(z, 0) \in X; |x_{0} - z| < \epsilon, \text{Im} z \in \Gamma\}$ with a positive constant $\epsilon$ and an open convex cone $\Gamma$ such that $\xi_{0} \in \text{Int}(\Gamma)$ (the interior of the dual cone $\Gamma^{o}$ of $\Gamma$). Then, $\beta^{0}(f)$ is equivalent to $\{\text{sp}_{N}(F_{jk}(x, \partial_{z})) | 1 \leq k \leq j \leq \mu\}$. Moreover, if the principal symbol of $b(\vartheta) + Q$ is written as $\tau^{m}P(z, \tau; z^{*}, \tau^{*})$ for a hyperbolic polynomial $P$ at $dt$-codirection, the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [T].

(2) Take an operator $A(z; \partial_{z}) \in \mathcal{D}_{Y}^{(1)}$ at the origin and set $A^{0} := \text{id}$ and $A^{(j)} := \frac{1}{j!} A \circ A^{(j-1)} \in \mathcal{D}_{Y}^{(j)}$ for $j \geq 1$. Let $p^{*} = (0, 1; \sqrt{-1}\langle \xi, dx \rangle)$ be a point of $T_{T_{N}M}^{*}T_{Y}L^{+}$ and set $p_{0} := (0; \sqrt{-1}\langle \xi, dx \rangle) \in T_{N}^{*}Y$. Set $P := (\vartheta - \alpha_{1})(\vartheta - \alpha_{2}) - \tau A(z; \partial_{z}) \vartheta \in \mathcal{D}_{X}|_{Y}$, where $(\alpha_{1}, \alpha_{2}) \in \mathbb{C}^{\oplus 2}$. Consider $M := \mathcal{D}_{X}/\mathcal{D}_{X}P = \mathcal{D}_{X}u$, where $u := 1 \mod P$. Let $f(x, t)$ be a germ of $\mathcal{H}om_{\mathcal{D}_{X}}(M, \mathcal{C}_{N|M})$ at $p^{*}$. Then:

(i) If $(\alpha_{1}, \alpha_{2}) = (-1, 0)$, then

$$\Phi_{Y}(M) \simeq \mathcal{D}_{Y}^{[\vartheta (\partial_{\vartheta} + 1)u]} \simeq \mathcal{D}_{Y}^{[\vartheta (\partial_{\vartheta} + 1)u]}$$

and $\vartheta: [(\vartheta + 1)u] \mapsto [u]$. Hence $M_{Y} \simeq \mathcal{D}_{Y}[(\vartheta + 1)u] \simeq \mathcal{D}_{Y}$. In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = U_{0}(z) + \frac{U_{-1}(z)}{\tau} - \sum_{j=1}^{\infty} \frac{A^{(j)}U_{-1}(z)}{j-1} \tau^{j-1} - AU_{-1}(z) \log \tau,$$
and $\beta^0(f(x,t))$ is given by $\{\text{sp}_N(U_i)(x)\}_{i=-1,0}$ at $p_0$. If $f(x,t)$ is $F$-mild at $p_0$, then $U_{-1}(z) = 0$ and $\gamma^F(f(x,t)) = \{f(x,0)\} = \{\text{sp}_N(U_0)(x)\}$.

(ii) If $(\alpha_1, \alpha_2) = (0,1)$, then:

$$
\begin{align*}
\Phi_Y(M) &= \frac{V_Y^1(D_X)u + V_Y^2(D_X)\partial u}{V_Y^0(D_X)u} = D_Y[\partial^2 u] + D_Y[\partial^2(\theta - 1)u] \simeq D_Y^{\oplus 2}, \\
\Psi_Y(M) &= \frac{V_Y^0(D_X)u + V_Y^1(D_X)\partial u}{V_Y^{-1}(D_X)u + V_Y^0(D_X)\partial u} = D_Y[u] + D_Y[\partial u] \simeq D_Y^{\oplus 2},
\end{align*}
$$

and $\text{Var} [\partial u] = \text{Var} [\partial^2 u] = 0$. Hence $M_Y \simeq D_Y[u] + D_Y[\partial u] \simeq D_Y^{\oplus 2}$. In this case $f(x,t)$ has the following defining function:

$$
F(z,\tau) = U_0(z) + \sum_{j=0}^{\infty} \frac{A^{(j)}U_1(z)}{j+1} \tau^{j+1},
$$

and $f(x,t)$ is always $F$-mild. Hence $\beta^0(f(x,t))$ at $p_0$ coincides with $\gamma^F(f(x,t)) = \{\partial^i f(x,0)\}_{i=0,1} = \{\text{sp}_N(U_i)(x)\}_{i=0,1}$ (if $\tau \neq 0$, $M$ is isomorphic to $D_X/\langle \partial^2 - A(z;\partial)\partial \rangle$ for which $Y$ is non-characteristic).

(iii) If $(\alpha_1, \alpha_2) = (1,1)$, then

$$
\begin{align*}
\Phi_Y(M) &= \frac{V_Y^2(D_X)u + V_Y^3(D_X)(\theta - 1)u}{V_Y^1(D_X)u + V_Y^2(D_X)(\theta - 1)u} = D_Y[\partial^2 u] + D_Y[\partial^2(\theta - 1)u] \simeq D_Y^{\oplus 2}, \\
\Psi_Y(M) &= \frac{V_Y^1(D_X)u + V_Y^2(D_X)(\theta - 1)u}{V_Y^0(D_X)u + V_Y^1(D_X)(\theta - 1)u} = D_Y[u] + D_Y[\partial u] \simeq D_Y^{\oplus 2},
\end{align*}
$$

and $\text{Var}: ([\partial^2 u], [\partial^3(\theta - 1)u]) \mapsto (2\pi \sqrt{-1} [\partial u], 0)$. Hence $M_Y \simeq D_Y[\partial u] \simeq D_Y$. In this case $f(x,t)$ has the following defining function:

$$
F(z,\tau) = \sum_{j=0}^{\infty} A^{(j)}U_0(z) \tau^{j+1} - \sum_{j=1}^{\infty} \sum_{k=1}^{j} \frac{A^{(j)}U_1(z)}{k} \tau^{j+1} + \sum_{j=0}^{\infty} A^{(j)}U_1(z) \tau^{j+1} \log \tau,
$$

and $\beta^0(f(x,t))$ is given by $\{\text{sp}_N(U_i)(x)\}_{i=0,1}$ at $p_0$. If $f(x,t)$ is $F$-mild at $p_0$, then $U_0(z) = 0$ and $\gamma^F(f(x,t)) = \{\partial x f(x,0)\} = \{\text{sp}_N(U_1)(x)\}$.

(iv) If $(\alpha_1, \alpha_2) = (1,2)$, then:

$$
\begin{align*}
\Phi_Y(M) &= \frac{V_Y^2(D_X)u + V_Y^3(D_X)(\theta - 1)u}{V_Y^1(D_X)u + V_Y^2(D_X)(\theta - 1)u} = D_Y[\partial^2 u] + D_Y[\partial^3(\theta - 1)u] \simeq D_Y^{\oplus 2}, \\
\Psi_Y(M) &= \frac{V_Y^1(D_X)u + V_Y^2(D_X)(\theta - 1)u}{V_Y^0(D_X)u + V_Y^1(D_X)(\theta - 1)u} = D_Y[u] + D_Y[\partial u] \simeq D_Y^{\oplus 2},
\end{align*}
$$

and $\text{Var}: ([\partial^2 u], [\partial^3(\theta - 1)u]) \mapsto (0, 2A[\partial u])$. Hence

$$
M_Y \simeq \frac{D_Y[\partial u] + D_Y[\partial^2(\theta - 1)u]}{D_Y A[\partial u]}. 
$$
In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_2(z) \tau^{j+2} + U_1(z) \tau - \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \frac{jA^{(j)} U_1(z)}{k} \tau^{j+1} + \left( \sum_{j=0}^{\infty} (j+1)A^{(j+1)} U_1(z) \tau^j \right) \tau^2 \log \tau,$$

and $\beta^0(f(x, t))$ is given by $\{\text{sp}_N(U_i)(x)\}_{i=1,2}$ at $p_0$. $f(x, t)$ is $F$-mild under the condition that $A U_1(z) = 0$, and in this case $\gamma^F(f(x, t))$ at $p_0$ is given by $\gamma^F(f_3(x, t)) = \{\partial_t f(x, +0)\}_{i=1,2} = \{\text{sp}_N(U_1)(x), 2\text{sp}_N(U_2)(x)\}$ with $A\partial_t f(x, +0) = A\text{sp}_N(U_1)(x) = 0$.

References


