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Kyoto University
Microlocalization
of Topological Boundary Value Morphism
and Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation
of the boundary value problems for hyperfunction or microfunction solutions to a system
of linear partial differential equations with analytic coefficients (that is, a coherent (left)
$\mathcal{D}$-Module, here in this article, we shall write Module with a capital letter, instead of
sheaf of modules). If the system is regular-specializable, the nearby-cycle of the system
can be defined in the theory of $\mathcal{D}$-Modules. After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 2], [Sc 3], for any hyperfunction solutions to
regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value morphism which takes values in hyperfunction solutions to the nearby-cycle of the system
instead of the induced system. This morphism is injective (cf. [MF 2]) and a generalization
of the non-characteristic boundary value morphism (for the non-characteristic case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). Moreover recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value morphism and discussed the solvability under a kind of hyperbolicity condition (the near-
hyperbolicity). However, since this morphism is defined only for hyperfunction solutions,
a microlocal boundary value problem is not considered. Therefore in this article, we
shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-
Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

*Research Fellow of The Japan Society for The Promotion of Science.
1 Notation

We denote the set of integers, of real numbers and of complex numbers by \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \) respectively as usual. Moreover we set \( \mathbb{N} := \{ n \in \mathbb{Z}; n \geq 1 \} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{ 0 \} \).

All the manifolds are assumed to be paracompact. Let \( \tau: E \to Z \) a vector bundle over a manifold \( Z \). Then, set \( \dot{E} := E \setminus Z \) and \( \dot{\tau} \) the restriction of \( \tau \) to \( \dot{E} \). Let \( M \) be an \( (n+1) \)-dimensional real analytic manifold and \( N \) a one-codimensional closed real analytic submanifold of \( M \). Let \( X \) and \( Y \) be complexifications of \( M \) and \( N \) respectively such that \( Y \) is a closed submanifold of \( X \) and that \( Y \cap M = N \). Moreover, we assume the existence of a partial complexification of \( M \) in \( X \); that is, there exists a \( (2n+1) \)-dimensional real analytic submanifold \( L \) of \( X \) containing both \( M \) and \( Y \) such that the triplet \( (N, M, L) \) is locally isomorphic to \( (\mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R}) \) by a local coordinate system \((z, \tau) = (x + \sqrt{-1} y, t + \sqrt{-1} s)\) of \( X \) around each point of \( N \). We say such a coordinate system admissible. We shall mainly follow the notation in Kashiwara-Schapira [K-S]; we denote the normal deformations of \( N \) and \( Y \) in \( M \) and \( L \) by \( \overline{M}_N \) and \( \tilde{L}_Y \) respectively and regard \( \overline{M}_N \) as a closed submanifold of \( \tilde{L}_Y \). We have the following commutative diagram:

![Diagram](https://example.com/diagram.png)

and by admissible coordinates we have locally the following relation:

\[
\begin{align*}
\mathbb{R}^n_x \times \{0\} &\to M = \mathbb{R}^n_x \times \mathbb{R}_t \\
\mathbb{C}^n_z \times \{0\} &\to L = \mathbb{C}^n_z \times \mathbb{R}_t \\
X &\to \mathbb{C}^n_z \times \mathbb{C}_\tau
\end{align*}
\]

With these coordinates, we often identify \( T_Y X \) and \( T_Y L \) with \( X \) and \( L \) respectively.

The projection \( \tau_Y: T_Y L \to Y \) and \( s_L: T_Y L \to \tilde{L}_Y \) induce natural mappings:

\[
T^*_N Y \leftarrow T_N M \times T^*_N Y \xrightarrow{i_Y^*} T^*_N M \times T^*_M \tilde{L}_Y \xrightarrow{i_M^*} T^*_M \tilde{L}_Y
\]

and by these mappings, we identify \( T^*_N M T_Y L \) with \( T_N M \times T^*_N Y \) and \( T_N M \times T^*_M \tilde{L}_Y \).
$T_{Y}L \setminus T_{Y}Y$ has two components with respect to its fiber. We denote one of them by $T_{Y}L^{+}$ and represent (at least locally) by fixing an admissible coordinate system

$$T_{Y}L^{+} = \{(z, t) \in T_{Y}L; \, t > 0\}.$$ Moreover set $T_{N}M^{+} := T_{Y}L^{+} \cap T_{N}M$. Set an open embedding $f : T_{Y}L^{+} \hookrightarrow T_{Y}L$ and $f_{N} := f|_{T_{N}M^{+}} : T_{N}M^{+} \hookrightarrow T_{N}M$. We regard $T_{N}M^{+} \times T_{N}Y$ as an open set of $T_{T_{N}M}^{*}T_{Y}L$. Moreover $f$ induces mappings:

$$T_{T_{N}M}^{*}T_{Y}L^{+} \hookrightarrow \cup \rightarrow T_{N}M^{+} \times T_{T_{N}M}^{*}T_{Y}L.$$ Hence we identify $T_{T_{N}M}^{*}T_{Y}L^{+}$ with $T_{N}M^{+} \times T_{N}^{*}Y$ and $f_{N}$ with $f_{N} \times \text{id}$.

Let $\pi_{N,M} : T_{\frac{*}{M}N}L_{\mathrm{Y}} \rightarrow \overline{M}_{N}$ and $\pi_{N|M} : T_{T_{N}M}^{*}T_{Y}L \rightarrow T_{N}M$, be the natural projections. We denote as usual by $\nu$ and $\mu$ the Sato specialization and microlocalization functors respectively.

## General Boundary Values

By using an admissible coordinate system we define a continuous section $\sigma : Y \rightarrow \dot{T}_{Y}X$ by $z \mapsto (z, 1)$. Similarly we define $\tau : Y \rightarrow \dot{T}_{Y}X$ by $z \mapsto (z, 1)$. In general, let $Z$ be a complex manifold, $\tau : E \rightarrow Z$ a complex vector bundle. Then, denote by $D^{b}_{C^{\times}}(E)$ the subcategory of $D^{b}(E)$ consisting of $C^{\times}$-conic objects.

### 2.1 Theorem. For any object $\mathcal{F}$ of $D^{b}(X)$ such that $\nu_{Y}(\mathcal{F}) \in \text{Ob}(D^{b}_{C^{\times}}(T_{Y}X))$, there exists the following natural isomorphism:

$$f_{\pi}^{-1} \mu_{T_{N}M}(\nu_{Y}(i_{i'}^{L}\mathcal{F})) \Rightarrow f_{\pi}^{-1} \tau_{\pi}^{-1} \mu_{N}(\nu_{Y}(\mathcal{F})) \otimes \omega_{L/X}.$$ 

### 2.2 Definition. For any object $\mathcal{F}$ of $D^{b}(X)$ such that $\nu_{Y}(\mathcal{F}) \in \text{Ob}(D^{b}_{C^{\times}}(T_{Y}X))$, we define by virtue of Kashiwara-Schapira [K-S] and Theorem 2.1:

$$\beta : f_{\pi}^{-1} s_{L^{*}}^{-1} \mu_{M}(Rj_{L*} \overline{f}_{L}^{-1} i_{L}^{*}\mathcal{F}) \rightarrow f_{\pi}^{-1} \mu_{T_{N}M}(\nu_{Y}(i_{i'}^{L}\mathcal{F})) \Rightarrow f_{\pi}^{-1} \tau_{\pi}^{-1} \mu_{N}(\nu_{Y}(\mathcal{F})) \otimes \omega_{L/X}.$$ 

### 2.3 Definition (Laurent-Monteiro Fernandes [L-MF 2]). We say an object $\mathcal{F}$ of $D^{b}(X)$ is near-hyperbolic at $x_{0} \in N$ (in $dt$-codirection) if there exist positive constants $C$ and $\epsilon_{1}$ such that

$$\text{SS}(\mathcal{F}) \cap \{(z, \tau; z^{*}, \tau^{*}) \in T^{*}X; |z - x_{0}|, |\tau| < \epsilon_{1}, 0 < \text{Re} \tau\}$$

$$\subset \{(z, \tau; z^{*}, \tau^{*}) \in T^{*}X; |\text{Re} \tau^{*}| < C(|\text{Im} z^{*}|(|\text{Im} z| + |\text{Im} \tau|) + |\text{Re} z^{*}|)\}$$ holds by an admissible coordinate system. Here $\text{SS}(\mathcal{F})$ denotes the microsupport of $\mathcal{F}$. 

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2.4 Theorem. Let \( F \) be a object of \( \mathcal{D}^b(X) \). Assume that \( \nu_Y(F) \in \text{Ob}(\mathcal{D}^b_{\mathbb{C}^\times}(T_Y X)) \) and \( F \) is near-hyperbolic at \( x_0 \in N \). Then, for any \( p^* \in T^*_{T_N M^+} T_Y L^+ \)

\[
\beta: s_{L^+}^{-1} \mu_{M_N}^{\mathcal{D}_{X}}(Rj_{L^*} \tilde{p}^{-1}_L i_L^* \mathcal{F})_{p^*} \to \mu_N(\sigma^{-1} \nu_Y(F))_{\tau_{Y\pi}(p^*)} \otimes \omega_{L/X}
\]

is an isomorphism.

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable \( \mathcal{D} \)-Module and its nearby-cycle.

As usual, we denote by \( \mathcal{D}_X \) the sheaf on \( X \) of holomorphic differential operators, and by \( \{\mathcal{D}^{(m)}_X\}_{m \in \mathbb{N}_0} \) the usual order filtration on \( \mathcal{D}_X \).

3.1 Definition. Denote by \( \mathcal{I}_Y \) the defining Ideal of \( Y \) in \( \mathcal{O}_X \) with a convention that \( \mathcal{I}^j_Y = \mathcal{O}_X \) for \( j \leq 0 \). The \( V \)-filtration \( \{V^k_Y(\mathcal{D}_X)\}_{k \in \mathbb{Z}} \) (along \( Y \)) is a filtration on \( \mathcal{D}_X|_Y \) defined by

\[
V^k_Y(\mathcal{D}_X) := \bigcap_{j \in \mathbb{Z}} \{P \in \mathcal{D}_X|_Y; P \mathcal{I}^j_Y \subset \mathcal{I}^{j-k}_Y\}.
\]

Let us denote by \( \theta \) the Euler operator. Note that \( \theta \in V^0_Y(\mathcal{D}_X) \setminus V^1_Y(\mathcal{D}_X) \) and that \( \theta \) can be represented by \( \tau \partial_r \) by admissible coordinates.

3.2 Definition. A coherent \( \mathcal{D}_X|_Y \)-Module \( M \) is said to be regular-specializable (along \( Y \)) if there exist locally a coherent \( \mathcal{O}_X \)-sub-Module \( M_0 \) of \( M \) and a non-zero polynomial \( b(\alpha) \in \mathbb{C}[\alpha] \) such that the following conditions are satisfied:

1. \( M_0 \) generates \( M \) over \( \mathcal{D}_X \); that is, \( M = \mathcal{D}_X M_0 \);
2. \( b(\theta) M_0 \subset (\mathcal{D}^{(m)}_X \cap V^{k-1}_Y(\mathcal{D}_X)) M_0 \), where \( m \) is the degree of \( b(\alpha) \).

In what follows, we shall omit the phrase "along \( Y \)" since \( Y \) is fixed.

3.3 Remark. (1) Let \( M \) be a coherent \( \mathcal{D}_X|_Y \)-Module for which \( Y \) is non-characteristic. Then, it is easy to see that \( M \) is regular-specializable.

(2) Kashiwara-Kawai [K-K] proved that every regular-holonomic \( \mathcal{D}_X|_Y \)-Module is regular-specializable.

3.4 Proposition. If \( M \) is a regular-specializable \( \mathcal{D}_X|_Y \)-Module, \( R\text{Hom}_{\mathcal{D}_X}(M, \nu_Y(\mathcal{O}_X)) \) and \( R\text{Hom}_{\mathcal{D}_X}(M, \nu_Y(\mathcal{O}_X)) \) are objects of \( \mathcal{D}^b_{\mathbb{C}^\times}(T_Y X) \) and \( \mathcal{D}^b_{\mathbb{C}^\times}(T_Y X) \) respectively.

Let \( \iota: Y \to X \) be the natural inclusion. Then the induced system, or the inverse image in the sense of \( \mathcal{D} \)-Modules is defined by \( D\iota^* M := \mathcal{O}_Y \otimes_{\iota^{-1} \mathcal{O}_X} \mathcal{O}_X \).

For any regular-specializable \( \mathcal{D}_X \)-Module \( M \), the nearby-cycle \( \Psi_Y(M) \) of \( M \) and the vanishing-cycle \( \Phi_Y(M) \) of \( M \) in the theory of \( \mathcal{D} \)-Modules can be defined. For the definitions of \( \Psi_Y(M) \) and \( \Phi_Y(M) \), we refer to Laurent [L], Mebkhout [Me]. We shall recall the following two results:
3.5 Proposition (Laurent [L], Mebkhout [Me]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, $\Psi_Y(\mathcal{M})$, $\Phi_Y(\mathcal{M})$ and each cohomology of $\mathcal{D}_Y^* \mathcal{M}$ are coherent $\mathcal{D}_Y$-Modules. Moreover, there exists the following distinguished triangle:

$$\Phi_Y(\mathcal{M}) \rightarrow \Psi_Y(\mathcal{M}) \rightarrow \mathcal{D}_Y^* \mathcal{M} \rightarrow +1.$$

Here, $\text{Var} := \varphi(\partial)\tau$ with $\varphi(\zeta) := (e^{2\pi \sqrt{-1} \zeta} - 1)/\zeta$.

3.6 Theorem (Laurent [L]). Let $\mathcal{E}_{Y|X}^\mathbb{R}$ be the sheaf of real holomorphic microfunctions on $T_Y^* X$ as usual. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$-Module. Then, there exists the following isomorphism of distinguished triangles:

$$\mathbf{R}\mathcal{H}\text{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \rightarrow \mathbf{R}\mathcal{H}\text{om}_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{O}_Y) \rightarrow \mathbf{R}\mathcal{H}\text{om}_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{O}_Y) \rightarrow +1.$$

3.7 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$\mathbf{R}\mathcal{H}\text{om}_{\mathcal{D}_Y}(\mathcal{D}_Y^* \mathcal{M}, \mathcal{O}_Y) \simeq \mathbf{R}\mathcal{H}\text{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

4 Boundary Values for Regular-Specializable System

We denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaf of holomorphic functions on $X$, of hyperfunctions on $M$ and of microfunctions on $T^*_M X$ respectively.

4.1 Definition (Oaku [Oa 2], Oaku-Yamazaki [O-Y]). We set:

$$\mathcal{C}_{N|M} := s_{L*}^{-1} \mu_{\overline{M}_N}(Rj_L^* \nu_{Y}\mathcal{O}_X) \otimes \mathcal{O}_{M/X}[n + 1].$$

We can regard $\mathcal{C}_{N|M}$ as a microlocalization of $\nu_N(\mathcal{B}_M)$.

4.2 Proposition. (1) $\mathcal{C}_{N|M}$ is concentrated in degree zero; that is, $\mathcal{C}_{N|M}$ is regarded as a sheaf on $T^*_N M \otimes L$. Further $\mathcal{C}_{N|M}|_{T^*_N M} = \nu_N(\mathcal{B}_M)$ holds.

(2) There exists the following exact sequence on $T^*_N M$:

$$0 \rightarrow \nu_Y(\mathcal{O}_L)|_{T^*_N M} \rightarrow \nu_N(\mathcal{B}_M) \rightarrow \hat{\pi}_{N|M*} \mathcal{C}_{N|M} \rightarrow 0.$$

Here $\mathcal{B}_L := \mathcal{O}_{L/X} \otimes \mathcal{O}_{L/X}$ is the sheaf of hyperfunctions with holomorphic parameters on $L$. Note that $\nu_Y(\mathcal{B}_L)$ is concentrated in degree zero.
4.3 Definition. Let $M$ be a regular-specializable $D_X|_Y$-Module. By Proposition 3.4, $\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_X}\left(M, \mathcal{O}_X\right)$ satisfies the assumption of Theorem 2.1. Thus, by Definition 2.2 and Theorem 3.6, we define:

$$\beta: f_\pi^{-1}\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_X}(M, C_{N|M}) \to f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_Y}(\Psi_Y(M), C_N).$$

4.4 Theorem. (1) The morphism $\beta$ gives a monomorphism

$$\beta^0: f_\pi^{-1}\mathcal{H}om_{D_X}(M, C_{N|M}) \to f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{D_Y}(\Psi_Y(M), C_N).$$

(2) The restriction of $\beta^0$ to the zero-section $T_N M^+$ coincides with the boundary value morphism in the sense of Monteiro Fernandes [MF1].

4.5 Definition. Let $M$ be a coherent $D_X|_Y$-Module. Then we say $M$ is near-hyperbolic at $x_0 \in N$ (in $dt$-codirection) if $\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_X}(M, \mathcal{O}_X)$ is near-hyperbolic in the sense of Definition 2.3. Here, we remark that $SS(\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_X}(M, \mathcal{O}_X)) = \text{char}(M)$.

The following theorem is a direct consequence of Theorem 2.4:

4.6 Theorem. Let $M$ be a regular-specializable $D_X|_Y$-Module. Assume that $M$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* \in T^*_T M^+T_Y L^+$

$$\beta: \mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_X}(M, C_{N|M})_{p^*} \to \mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_Y}(\Psi_Y(M), C_N)_{\tau_{Y\pi}(p^*)}$$

is an isomorphism.

4.7 Remark. Let $C^F_{N|M}$ be the sheaf of $F$-mild microfunctions on $T^*_T M^+T_Y L$, and set $\mathcal{C}^A_{N|M} := \mathcal{H}^n(\mu_N(\mathcal{O}_X|_Y)) \otimes \mathcal{O}_{N/Y}$ (see Oaku [Oa1], [Oa2], and Oaku-Yamazaki [O-Y]). Let $M$ be a regular-specializable $D_X|_Y$-Module. Set $M_Y := \mathcal{H}^0(D\iota^*M) = \mathcal{O}_Y \otimes \iota^{-1}M$.

By the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

$$
\begin{align*}
\mathcal{H}om_{D_X}(M, \mathcal{C}^F_{N|M}) & \to \mathcal{H}om_{D_X}(M, \mathcal{C}^A_{N|M}) & \to \mathcal{H}om_{D_Y}(M_Y, C_N) \\
\mathcal{H}om_{D_X}(M, \mathcal{C}^A_{N|M}) & \to \mathcal{H}om_{D_Y}(M_Y, C_N)
\end{align*}
$$

that is, the boundary value morphism

$$\gamma^F: f_\pi^{-1}\mathcal{H}om_{D_X}(M, C^F_{N|M}) \to f_\pi^{-1}\tau_{Y\pi}^{-1}\mathcal{H}om_{D_Y}(\Psi_Y(M), C_N)$$

and $\beta^0$ are compatible. In particular, if $Y$ is non-characteristic for $M$, then it is known that $\Psi_Y(M) \cong D\iota^*M \cong M_Y$ and by Oaku [Oa2] (cf. Oaku-Yamazaki [O-Y]) we have:

$$\gamma_{N|M}:\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_X}(M, \mathcal{C}^A_{N|M}) \to \tau_{Y\pi}^{-1}\mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}_{D_Y}(M_Y, C_N).$$
In this case we see that $\beta^0$ is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa 2]). In particular, the restriction of $\beta^0$ to the zero-section $T_N^* M^+$ is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Further, if $Y$ is non-characteristic for $M$ and $\pm dt \in T_Y^* M$ is hyperbolic for $M$, then the nearly-hyperbolic condition is satisfied and $\beta$ is an isomorphism.

4.8 Example. Assume that $X = \mathbb{C}^{n+1}$ and so on by an admissible coordinate system.

(1) Let $b(\alpha)$ be a non-zero polynomial with degree $m$, and $Q \in \mathcal{D}_X^{(m)} \cap \mathcal{V}_Y^{-1} (\mathcal{D}_X)$. Set $M := \mathcal{D}_X / \mathcal{D}_X (b(\theta) + Q)$. Then $M$ is regular-specializable. Assume that $b(\alpha) = \prod_{j=1}^{\mu} (\alpha - \alpha_j)^{\nu_j}$ $(\alpha_i - \alpha_j \notin \mathbb{Z}$ for $1 \leq i \neq j \leq \mu$, note that $\sum_{j=1}^{\mu} \nu_j = m)$. Then a direct calculation shows that $\Psi_Y (M) \simeq \mathcal{D}_Y^{\oplus m}$, and $\beta^0$ is equivalent to $\gamma$ in Oaku [Oa 2]: Let $p^* = (x_0, f_0; \sqrt{-1} \langle \xi_0, dx \rangle)$ be a point of $T_{N,M}^* \mathcal{T}_Y^* L^+$, and $f(x,t)$ a germ of $\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \mathcal{C}_{N|M})$ at $p^*$. Then, we can see that $f(x,t)$ has a defining function

$$F(z, \tau) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_j} F_{jk}(z, \tau) \tau^{\alpha_j} (\log \tau)^{k-1}.$$ 

Here each $F_{jk}(z, \tau)$ is holomorphic on a neighborhood of $\{(z, 0) \in X; |x_0 - z| < \epsilon, \text{Im } z \in \Gamma \}$ with a positive constant $\epsilon$ and an open convex cone $\Gamma$ such that $\xi_0 \in \text{Int}(\Gamma^\circ)$ (the interior of the dual cone $\Gamma^\circ$ of $\Gamma$). Then, $\beta^0(f)$ is equivalent to $\{ \text{sp}_N (F_{jk}(x + \sqrt{-1} \Gamma 0,0)); 1 \leq k \leq \nu_j, 1 \leq j \leq \mu \}$. Moreover, if the principal symbol of $b(\theta) + Q$ is written as $\tau^m P(z, \tau; z^*, \tau^*)$ for a hyperbolic polynomial $P$ at $dt$-codirection, the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [T].

(2) Take an operator $A(z; \partial_z) \in \mathcal{D}_Y^{(1)}$ at the origin and set $A^0 := \text{id}$ and $A^{(j)} := \frac{1}{j!} A \circ A^{(j-1)} \in \mathcal{D}_Y^{(j)}$ for $j \geq 1$. Let $p^* = (0,1; \sqrt{-1} \langle \xi, dx \rangle)$ be a point of $T_{N,M}^* \mathcal{T}_Y^* L^+$ and set $p_0 := (0; \sqrt{-1} \langle \xi, dx \rangle) \in T_{N,Y}^*$. Set $P := (\theta - \alpha_1)(\theta - \alpha_2) - \tau A(z; \partial_z) \theta \in \mathcal{D}_X|_Y$, where $(\alpha_1, \alpha_2) \in \mathbb{C}^{\oplus 2}$. Consider $M := \mathcal{D}_X / \mathcal{D}_X P = \mathcal{D}_X u$, where $u := 1 \mod P$. Let $f(x,t)$ be a germ of $\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \mathcal{C}_{N|M})$ at $p^*$. Then:

(i) If $(\alpha_1, \alpha_2) = (-1,0)$, then

$$\Phi_Y (M) = \frac{V_Y^0 (\mathcal{D}_X) u + V_Y^1 (\mathcal{D}_X) (\theta + 1) u}{V_Y^{-1}(\mathcal{D}_X) u + V_Y^0 (\mathcal{D}_X) (\theta + 1) u} = \mathcal{D}_Y [u] + \mathcal{D}_Y [\partial_x (\theta + 1) u] \simeq \mathcal{D}_Y^{\oplus 2},$$

$$\Psi_Y (M) = \frac{V_Y^{-1}(\mathcal{D}_X) u + V_Y^0 (\mathcal{D}_X) (\theta + 1) u}{V_Y^{-2}(\mathcal{D}_X) u + V_Y^{-1}(\mathcal{D}_X) (\theta + 1) u} = \mathcal{D}_Y [\tau u] + \mathcal{D}_Y [\tau (\theta + 1) u] \simeq \mathcal{D}_Y^{\oplus 2},$$

and $\text{Var}: ([u], [\partial_x (\theta - 1) u]) \mapsto ([\tau u], 0)$. Hence $M_Y \simeq \mathcal{D}_Y [\tau (\theta + 1) u] \simeq \mathcal{D}_Y$. In this case $f(x,t)$ has the following defining function:

$$F(z, \tau) = U_0(z) + \frac{U_{-1}(z)}{\tau} - \sum_{j=1}^{\infty} \frac{A^{(j)} U_{-1}(z)}{j-1} \tau^{j-1} - AU_{-1}(z) \log \tau.$$
and $\beta^{0}(f(x, t))$ is given by $\{\text{sp}_{N}(U_{i})(x)\}_{i=-1,0}$ at $p_{0}$. If $f(x, t)$ is $F$-mild at $p_{0}$, then $U_{-1}(z) = 0$ and $\gamma^{F}(f(x, t)) = \{f(x, +0)\} = \{\text{sp}_{N}(U_{0})(x)\}$.

(ii) If $(\alpha_{1}, \alpha_{2}) = (0, 1)$, then:

$$
\Phi_{Y}(M) = \frac{V_{Y}^{1}(D_{X}u) + V_{Y}^{2}(D_{X})u}{V_{Y}^{0}(D_{X})u + V_{Y}^{1}(D_{X})u} \rho_{\tau} A_{\tau} \rho_{\tau} - 1 d_{\tau} u \simeq D_{Y}^{\oplus 2},
$$

$$
\Psi_{Y}(M) = \frac{V_{Y}^{0}(D_{X})u + V_{Y}^{1}(D_{X})u}{V_{Y}^{1}(D_{X})u + V_{Y}^{2}(D_{X})u} \rho_{\tau} A_{\tau} \rho_{\tau} - 1 d_{\tau} u \simeq D_{Y}^{\oplus 2},
$$

and $\text{Var} [\partial_{\tau} u] = \text{Var} [\partial_{\tau}^{2} u] = 0$. Hence $M_{Y} \simeq D_{Y} [\partial_{\tau} u] + D_{Y} [\partial_{\tau}^{2} u] \simeq D_{Y}^{\oplus 2}$. In this case $f(x, t)$ has the following defining function:

$$
F(z, \tau) = U_{0}(z) + \sum_{j=0}^{\infty} A^{(j)} U_{1}(z) \tau^{j+1},
$$

and $f(x, t)$ is always $F$-mild. Hence $\beta^{0}(f(x, t))$ at $p_{0}$ coincides with $\gamma^{F}(f(x, t)) = \{\partial_{\tau}^{2} f(x, +0)\}_{i=0, 1} = \{\text{sp}_{N}(U_{i})(x)\}_{i=0, 1}$ (if $\tau \not= 0$, $M$ is isomorphic to $D_{X}/D_{X}(\partial_{\tau}^{2} A(z; \partial_{z}) \partial_{\tau})$ for which $Y$ is non-characteristic).

(iii) If $(\alpha_{1}, \alpha_{2}) = (1, 1)$, then

$$
\Phi_{Y}(M) = \frac{V_{Y}^{2}(D_{X})u + V_{Y}^{3}(D_{X})(\theta - 1)u}{V_{Y}^{1}(D_{X})u + V_{Y}^{2}(D_{X})(\theta - 1)u} \rho_{\tau} A_{\tau} \rho_{\tau} - 1 d_{\tau} u \simeq D_{Y}^{\oplus 2},
$$

$$
\Psi_{Y}(M) = \frac{V_{Y}^{1}(D_{X})u + V_{Y}^{2}(D_{X})(\theta - 1)u}{V_{Y}^{0}(D_{X})u + V_{Y}^{1}(D_{X})(\theta - 1)u} \rho_{\tau} A_{\tau} \rho_{\tau} - 1 d_{\tau} u \simeq D_{Y}^{\oplus 2},
$$

and $\text{Var} : ([\partial_{\tau}^{2} u], [\partial_{\tau}^{3} (\theta - 1)u]) \mapsto (2\pi \sqrt{-1} [\partial_{\tau} (\theta - 1)u], 0)$. Hence $M_{Y} \simeq D_{Y} [\partial_{\tau} u] \simeq D_{Y}$. In this case $f(x, t)$ has the following defining function:

$$
F(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_{0}(z) \tau^{j+1} - \sum_{j=1}^{\infty} \sum_{k=1}^{j} A^{(j)} U_{1}(z) \tau^{j+1} + \sum_{j=0}^{\infty} A^{(j)} U_{1}(z) \tau^{j+1} \log \tau,
$$

and $\beta^{0}(f(x, t))$ is given by $\{\text{sp}_{N}(U_{i})(x)\}_{i=0, 1}$ at $p_{0}$. If $f(x, t)$ is $F$-mild at $p_{0}$, then $U_{0}(z) = 0$ and $\gamma^{F}(f(x, t)) = \{\partial_{\tau} f(x, +0)\} = \{\text{sp}_{N}(U_{1})(x)\}$.

(iv) If $(\alpha_{1}, \alpha_{2}) = (1, 2)$, then:

$$
\Phi_{Y}(M) = \frac{V_{Y}^{2}(D_{X})u + V_{Y}^{3}(D_{X})\partial_{\tau} u}{V_{Y}^{1}(D_{X})u + V_{Y}^{2}(D_{X})\partial_{\tau} u} \rho_{\tau} A_{\tau} \rho_{\tau} - 1 d_{\tau} u \simeq D_{Y}^{\oplus 2},
$$

$$
\Psi_{Y}(M) = \frac{V_{Y}^{1}(D_{X})u + V_{Y}^{2}(D_{X})\partial_{\tau} u}{V_{Y}^{0}(D_{X})u + V_{Y}^{1}(D_{X})\partial_{\tau} u} \rho_{\tau} A_{\tau} \rho_{\tau} - 1 d_{\tau} u \simeq D_{Y}^{\oplus 2},
$$

and $\text{Var} : ([\partial_{\tau}^{2} u], [\partial_{\tau}^{3} (\theta - 1)u]) \mapsto (0, 2A [\partial_{\tau} u])$. Hence

$$
M_{Y} \simeq \frac{D_{Y} [\partial_{\tau} u] + D_{Y} [\partial_{\tau}^{2} (\theta - 1)u]}{D_{Y} A [\partial_{\tau} u]}.
$$
In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_2(z) \tau^{j+2} + U_1(z) \tau - \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \frac{j A^{(j)} U_1(z)}{k} \tau^{j+1}$$

$$+ \left( \sum_{j=0}^{\infty} (j+1) A^{(j+1)} U_1(z) \tau^{j} \right) \tau^2 \log \tau,$$

and $\beta^0(f(x, t))$ is given by $\{ \text{sp}_N(U_1)(x) \}_{i=1,2}$ at $p_0$. $f(x, t)$ is $F$-mild under the condition that $AU_1(z) = 0$, and in this case $\gamma^F(f(x, t))$ at $p_0$ is given by $\gamma^F(f_3(x, t)) = \{ \partial_t f(x, +0) \}_{i=1,2} = \{ \text{sp}_N(U_1)(x), 2 \text{sp}_N(U_2)(x) \}$ with $A\partial_t f(x, +0) = A \text{sp}_N(U_1)(x) = 0$.

References


