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A Sharp Existence and Uniqueness Theorem for Linear Fuchsian Partial Differential Equations

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Abstract

This paper considers the equation $Pu = f$, where $u$ and $f$ are continuous with respect to $t$ and holomorphic with respect to $z$, and $P$ is the linear Fuchsian partial differential operator

$$P = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, z)(\mu(t)D_z)^\alpha(tD_t)^j.$$ 

We will give a sharp form of unique solvability in the following sense: we can find a domain $\Omega$ such that if $f$ is defined on $\Omega$, then we can find a unique solution $u$ also defined on $\Omega$.

1 Introduction and Result

Denote by $\mathbb{N}$ the set of nonnegative integers, and let $(t, z) = (t, z_1, \ldots, z_n) \in \mathbb{R} \times \mathbb{C}^n$. Let $R > 0$ be sufficiently small, and for $\rho \in (0, R]$, let $B_{\rho}$ be the polydisk $\{z \in \mathbb{C}^n; |z_i| < \rho \text{ for } i = 1, 2, \ldots, n\}$.

Given any bounded, open subset $D$ of $\mathbb{C}^n$, we define by $A(D)$ the Banach space of all functions $g(z)$ holomorphic in $D$ and continuous up to $\overline{D}$; the norm in this space is given by $\|g\|_D = \max_{z \in \overline{D}} |g(z)|$. Let $T > 0$. Then we denote by $C^0([0, T], A(D))$ the set of functions continuous on the interval $[0, T]$ and valued in the space $A(D)$.

We say that a continuous, positive-valued function $\mu(t)$ on the interval $(0, T)$ is a weight function if $\mu(t)$ is increasing and the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s} \, ds \quad (1.1)$$

is well-defined on $(0, T)$, i.e., the integral on the right is finite. (See Tahara [7].)

Consider now the linear partial differential operator

$$P = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, z)(\mu(t)D_z)^\alpha(tD_t)^j. \quad (1.2)$$

Here, $D_t = \partial/\partial t$ and $D_z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$; $\mu(t)$ is a weight function; and the coefficients $a_{j,\alpha}(t, z)$ belong in the space $C^0([0, T], A(B_R))$, i.e., for any
$s \in [0, T]$, each of the functions $a_{j, \alpha}(s, z)$, when viewed as a function of $z$, is holomorphic in $B_R$ and continuous up to $\overline{B_R}$. We associate a polynomial with this operator, called the characteristic polynomial of $\mathcal{P}$, and we define it by

$$C(\lambda, z) = \lambda^m + a_{m-1,0}(0, z)\lambda^{m-1} + \cdots + a_{0,0}(0, z).$$

(1.3)

Its roots $\lambda_1(z), \ldots, \lambda_m(z)$ will be referred to as characteristic exponents. In what follows, we will assume that there exists a positive number $L$ such that

$$\Re \lambda_j(z) \leq -L < 0 \quad \text{for all } z \in B_R \text{ and } 1 \leq j \leq m.$$  

(1.4)

Baouendi and Goulaouic [1] studied the above operator in the case when $\mu(t) = t^a$ ($a > 0$). They called such operator a Fuchsian partial differential operator, which for them is the "natural" generalization of a Fuchsian ordinary differential operator. In their paper, they gave some generalizations of the classical Cauchy-Kowalewski and Holmgren theorems for this type of operators. Their method has been applied and extended to various cases as can be seen, for example, in Tahara [6], Mandai [5] and Yamane [8].

In a previous paper [4], the author proved existence and uniqueness theorems similar to those given in [1], but for general $\mu(t)$. Essentially, he proved the following unique solvability result.

**Theorem 1.** Let $\mathcal{P}$ be as in (1.2). Then given any $\rho \in (0, R)$, there exists an $\varepsilon \in (0, T]$ such that for any $f(t, z) \in C^0([0, T], A(B_R))$, the equation $\mathcal{P}u = f$ has a unique solution $u(t, z) \in C^0([0, \varepsilon], A(B_\rho))$ satisfying for $1 \leq p \leq m$ the relation $(tD_t)^p u \in C^0([0, \varepsilon], A(B_\rho))$.

We remark that although $f(t, z)$, viewed as a function of $z$, is defined on $B_R$, the existence of the solution $u(t, z)$ is only guaranteed up to $B_\rho$, with $\rho < R$. Moreover, any two solutions of $\mathcal{P}u = f$ can only be shown to coincide in a neighborhood of the origin which is smaller than the neighborhood on which the two are defined.

In this paper, we shall present a formulation leading to an existence and uniqueness result sharper than the one given above. The result is sharper in the sense that the solution $u(t, z)$ of the equation $\mathcal{P}u = f$ will now have the same domain of definition as the inhomogeneous part $f(t, z)$.

To proceed, we will need the following definitions.

**Definition 1.** Let $\tau \in (0, T)$, $\gamma > 0$ and $\varphi(t)$ be the one in (1.1). We define

(i) $\omega_\tau[\gamma] = \{z \in \mathbb{C}^n; |z_i| < R - \gamma \varphi(\tau) \text{ for } i = 1, 2, \ldots, n\}$, and

(ii) $\Omega_T[\gamma] = \{(r, z) \in \mathbb{R} \times \mathbb{C}^n; 0 \leq \tau \leq T \text{ and } z \in \omega_\tau[\gamma]\}$.

**Definition 2.** Let $p \in \mathbb{N}$ and $\gamma > 0$.

(i) We say that $f(t, z)$ belongs in $K_0(\Omega_T[\gamma])$ if for each $\tau \in [0, T)$, we have $f(t) \in C^0([0, \tau], A(\omega_\tau[\gamma]))$. 


(ii) We say that $w(t, z)$ belongs in $C^0_p([0, \tau], \mathcal{A}(\omega_{\tau}[\gamma]))$ if for all $0 \leq j \leq p$, we have $(tD_t)^j w(t) \in C^0([0, \tau], \mathcal{A}(\omega_{\tau}[\gamma]))$.

(iii) We say that $u(t, z)$ belongs in $\mathcal{K}_p(\Omega_T[\gamma])$ if for each $\tau \in [0, T]$, we have $u(t) \in C^0_p([0, \tau], \mathcal{A}(\omega_{\tau}[\gamma]))$.

Under the above assumptions, we now state the following main result.

**Theorem 2.** Let $\mathcal{P}$ be the operator given in (1.2). Then there exist constants $T_0 > 0$ and $\gamma_0 > 0$ depending on $\mathcal{P}$ such that for any $f(t, z) \in \mathcal{K}_0(\Omega_{T_0}[\gamma_0])$, the equation

$$\mathcal{P} u = f \quad \text{in} \quad \Omega_{T_0}[\gamma_0]$$

(1.5)

has a unique solution $u(t, z)$ in $\mathcal{K}_m(\Omega_{T_0}[\gamma_0])$.

Moreover, the solution satisfies the a priori estimate

$$\sum_{p=0}^{m} \max_{\Delta} |(tD_t)^p u| \leq C \max_{\Delta} |f|,$$

(1.6)

where $\Delta$ is the closure of $\Omega_{T_0}[\gamma_0]$ and $C > 0$ is some constant dependent on the above equation and on the domain $\Omega_{T_0}[\gamma_0]$.

Note that $f(t, z)$ and $u(t, z)$ both have $\Omega_{T_0}[\gamma_0]$ as their domain of definition. This fact allows us to restate our theorem in the following manner: for any $T, \gamma > 0$, let $X_{T, \gamma}$ and $Y_{T, \gamma}$ be the spaces $\mathcal{K}_m(\Omega_T[\gamma])$ and $\mathcal{K}_0(\Omega_T[\gamma])$, respectively. Let $W_{T, \gamma}$ be the subspace of $X_{T, \gamma}$ consisting of functions $u \in X_{T, \gamma}$ such that $\mathcal{P} u$ belongs to $Y_{T, \gamma}$. Define a linear operator $\Psi$ from $X_{T, \gamma}$ to $Y_{T, \gamma}$ with domain $W_{T, \gamma}$ by $\Psi u = \mathcal{P} u$. Let $\|\cdot\|_{T, \gamma}$ denote the maximum norm in the closure of $\Omega_T[\gamma]$. Then $X_{T, \gamma}$ and $Y_{T, \gamma}$ are Banach spaces; given $u \in X_{T, \gamma}$ and $f \in Y_{T, \gamma}$, we define their norms by $\sum_{p=0}^{m} \|(tD_t)^p u\|_{T, \gamma}$ and $\|f\|_{T, \gamma}$, respectively. Note further that the operator $\Psi$ is a closed linear operator from $X_{T, \gamma}$ to $Y_{T, \gamma}$. The above theorem can now be stated as

**Theorem 2'.** There exist $T_0, \gamma_0 > 0$ depending on $\mathcal{P}$ such that the operator $\Psi$ is a one-one, closed linear operator from $X_{T_0, \gamma_0}$ onto $Y_{T_0, \gamma_0}$.

Since $\Psi$ is an injection, $\Psi^{-1}$ exists and is also closed. The Closed Graph Theorem further implies that $\Psi^{-1}$ is continuous. The estimate given in (1.6) is just a consequence of the continuity of $\Psi^{-1}$.

## 2 Preliminary Discussion

We can rewrite the operator $\mathcal{P}$ as

$$\mathcal{P} = Q + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j, \alpha}(t, z)(\mu(t)D_z)^{\alpha}(tD_t)^j,$$
where the operator $Q$ is defined by

$$Q = (tD_t)^m + a_{m-1,0}(0,z)(tD_t)^{m-1} + \cdots + a_{0,0}(0,z)$$  \hspace{1cm} (2.1)

and

$$c_{j,\alpha}(t,z) = \begin{cases} a_{j,\alpha}(t,z) & \text{if } |\alpha| \neq 0, \\ a_{j,\alpha}(t,z) - a_{j,\alpha}(0,z) & \text{if } |\alpha| = 0. \end{cases}$$

Note that the coefficients of $Q$ are holomorphic functions of $z$ in $B_R$. Note further that the characteristic exponents of $Q$ are the same as that of $P$, and hence satisfy (1.4).

**Lemma 1.** Fix $\tau > 0$ and let $g(t) \in C^0([0,\tau],A(\omega_\tau[\gamma]))$. Then the equation $Qu = g$ has a unique solution $u(t) \in C_m^0([0,\tau],A(\omega_\tau[\gamma]))$ given by

$$u(t) = \frac{1}{m!} \sum_{\sigma \in S_m} \int_0^t \int_0^{s_2} \cdots \int_0^{s_m} \frac{(s_m)}{t}^{-\lambda_{\sigma(m)}} \left( \frac{s_{m-1}}{s_m} \right)^{-\lambda_{\sigma(m-1)}} \cdots \left( \frac{s_1}{s_2} \right)^{-\lambda_{\sigma(1)}} g(s_1) \frac{ds_1}{s_1} \frac{ds_2}{s_2} \cdots \frac{ds_m}{s_m}.$$  \hspace{1cm} (2.2)

Here, $S_m$ is the group of permutations of $\{1,2,\ldots,m\}$.

A result in symmetric entire functions asserts that $u(t,z)$ is holomorphic with respect to $z$. The fact that it belongs in $C_m^0([0,\gamma],A(\omega_\tau[\gamma]))$ is seen in the integral expression, but may actually be obtained a priori. (See [1].)

To facilitate computation, we define for $\lambda = (\lambda_1,\ldots,\lambda_m)$ the function

$$G^t_\theta(\lambda) \overset{\text{def}}{=} \frac{1}{m!} \sum_{\sigma \in S_m} \left( \frac{s_m}{t} \right)^{-\lambda_{\sigma(m)}} \left( \frac{s_{m-1}}{s_m} \right)^{-\lambda_{\sigma(m-1)}} \cdots \left( \frac{\theta}{s_2} \right)^{-\lambda_{\sigma(1)}},$$  \hspace{1cm} (2.3)

for some dummy variables $s_2,\ldots,s_m$. Define, too, the integral operator

$$\int_{[t,\theta]}^{(m)} g \overset{\text{def}}{=} \int_0^t \int_0^{s_2} \cdots \int_0^{s_m} g(\theta) \frac{d\theta}{\theta} \frac{ds_2}{s_2} \cdots \frac{ds_m}{s_m}.$$  \hspace{1cm} (2.4)

Using the above, we can now write the solution $u(t)$ of the equation $Qu = g$ as

$$u(t) = \int_{[t,\theta]}^{(m)} G^t_\theta(\lambda) g.$$  \hspace{1cm} (2.5)

In our proof of the main theorem, it will be necessary to consider the action of the differential operator $(tD_t)^p$ on integral expressions similar to the one in (2.2). One can easily verify the following

**Lemma 2.** Let $u(t)$ be the solution of $Qu = g$. Then for a natural number $p$ less than $m$, we have

$$(tD_t)^p u = \sum_{i=m-p}^m \int_{[t,s_i]}^{(i)} g \times \left\{ \frac{1}{m!} \sum_{\sigma \in S_m} h_i(\sigma,\lambda) \left( \frac{s_i}{t} \right)^{-\lambda_{\sigma(i)}} \right\} \overset{\lambda_{\sigma(i)}}{\cdots} \left( \frac{s_1}{s_2} \right)^{-\lambda_{\sigma(1)}},$$  \hspace{1cm} (2.5)
where the functions \( h_i(\sigma, \lambda) \) are suitable polynomial functions of the characteristic exponents \( \lambda_1(z), \ldots, \lambda_m(z) \).

For brevity, let us set, for a natural number \( k \),

\[
H^t_\theta(k, \lambda) = \frac{1}{m!} \sum_{\sigma \in S_m} h_k(\sigma, \lambda) \left( \frac{s_k}{t} \right)^{-\lambda_{\sigma(k)}} \left( \frac{s_{k-1}}{s_k} \right)^{-\lambda_{\sigma(k-1)}} \cdots \left( \frac{\theta}{s_2} \right)^{-\lambda_{\sigma(1)}}. \tag{2.6}
\]

By symmetry, the functions \( H^t_\theta(k, \lambda) \) are holomorphic with respect to \( z \) and thus belong in \( \mathcal{A}(B_R) \).

The next lemma is useful in evaluating some integral expressions in the proof.

**Lemma 3.** Let \( k \) be a natural number. Then the following equalities hold:

(a) \[
\int_0^t \int_0^{s_k-1} \cdots \int_0^{s_1} \frac{ds_0}{s_0} \cdots \frac{ds_{k-1}}{s_{k-1}} \frac{\partial^L}{\partial t^L} = \frac{1}{L^k}
\]

(b) \[
\int_0^t \int_0^{s_k} \cdots \int_0^{s_1} \mu(s_k) \frac{\mu(s_{k-1})}{s_{k-1}} \cdots \frac{\mu(s_1)}{s_1} \frac{ds_0}{s_0} \cdots \frac{ds_k}{s_k} = \frac{1}{L!}
\]

The first equality is obvious. The second can be proved by reversing the order of integration and recalling that \( t\varphi'(t) = \mu(t) \).

To estimate the derivatives with respect to \( z \), we have the following lemma. (For a proof, see Hörmander [3], Lemma 5.1.3.)

**Lemma 4.** Let the function \( v(z) \) be holomorphic in \( B_R \), and suppose there are positive constants \( K \) and \( c \) such that

\[
\|v\|_\rho \leq \frac{K}{(R-\rho)^c} \quad \text{for every } \rho \in (0,R). \tag{2.7}
\]

Then we have

\[
\|D_z^\alpha v\|_\rho \leq \frac{K e^{1\alpha}(c+1)\alpha}{(R-\rho)^c+\alpha} \quad \text{for every } \rho \in (0,R). \tag{2.8}
\]

In the above, we define \( (c)_p = (c)(c+1) \cdots (c+p-1) \).

## 3 Proof of Main Theorem

Let \( f \) be any element of \( \mathcal{K}_0(\Omega_{T_0}[\gamma_0]) \). Here, the constants \( T_0 > 0 \) and \( \gamma_0 > 0 \) satisfy some conditions which will later be specified. For convenience, we will drop the subscript in both and instead use \( T \) and \( \gamma \); we will again use the subscript upon stating the conditions that these constants need to satisfy.
We will use the method of successive approximations to solve the equation $Pu = f$. Define the approximate solutions as follows:

$$u_0(t) = \int_{[t:s]}^{(m)} G_{s}^t(\lambda) f$$

and for $k \geq 1$,

$$u_k(t) = \int_{[t:s]}^{(m)} G_{s}^t(\lambda) [f - S(s)u_{k-1}].$$

Here, $t \in [0, T]$, and for brevity, we have set $S(t) = \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j,\alpha}(t, z) \cdot (\mu(t)D_{z})^{\alpha}(tD_{t})^{j}$. Note that for all $k$, the approximate solutions $u_k(t, z)$ are defined on $\Omega_{T_0}[\gamma_0]$. Furthermore, they are continuous with respect to $t$ and holomorphic with respect to $z$ on this region.

For each $k$, we also define the sequence of functions $v_k(t) = u_k(t) - u_{k-1}(t)$, with $u_{-1} \equiv 0$. Then the $v_k(t, z)$'s are also defined on the same region as the $u_k(t, z)$'s, and are also continuous with respect to $t$ and holomorphic with respect to $z$. Using the expression for $u_k(t)$, we have $v_0(t) = \int_{[t:s]}^{(m)} G_{s}^t(\lambda) f$ and for $k \geq 1$,

$$v_k(t) = - \int_{[t:s]}^{(m)} G_{s}^t(\lambda) S(s) v_{k-1}.$$

To prove that the approximate solutions converge to the real solution, we will henceforth fix one $t \in [0, T]$, and estimate the functions $v_k(t)$. Let $C$ be the bound on $[0, T] \times \overline{B}_R$ of all $c_{j,\alpha}(t, z)$, and $K$ be the bound in $\overline{\Omega}_T[\gamma]$ of $f(t, z)$. As $G_{s}^t(\lambda)$ and $H_{s}^t(k, \lambda)$, we have for $1 \leq k \leq m$ and for some $D > 0$:

$$\sup_{s \in \overline{B}_R} |G_{s}^t(\lambda)| \leq (\frac{s}{t})^{L} \quad \text{and} \quad \sup_{s \in \overline{B}_R} |H_{s}^t(k, \lambda)| \leq D\left(\frac{s}{t}\right)^{L}. $$

We can easily see that $\|v_0(t)\|_{\omega_t}$ is bounded by $KL^{-m}$ for any $0 \leq t \leq T$. Here, we have written for convenience $\|\cdot\|_{\omega_t}$ in place of $\|\cdot\|_{\omega_{t}[\gamma]}$. For general $k$, we note that $v_k(t)$ is given by the iterated integral

$$v_k(t) = (-1)^k \int_{[t:s_k]}^{(m)} G_{s_k}^t(\lambda) S(s_k) \int_{[s_k:s_{k-1}]}^{(m)} G_{s_{k-1}}^{s_k} (\lambda) S(s_{k-1}) \cdots \int_{[s_2:s_1]}^{(m)} G_{s_1}^{s_2} (\lambda) S(s_1) \int_{[s_1:s_0]}^{(m)} G_{s_0}^{s_1} (\lambda) f(s_0).$$

The expression above can be expanded using Lemma 2, and thus obtain a finite sum whose number of terms is less than $(mJ)^k$, where $J$ is the cardinality of the set $\{(j, \alpha); 0 \leq j \leq m-1 \text{ and } |\alpha| \leq m-j\}$. Each term of the finite sum
has the form
\[ I = (-1)^k \int_{[t;\sigma_k]}^{(m)} G_{s_k}(\lambda) c_{j_k,\alpha_k}(\mu D_z)^{\alpha_k} \int_{[s_k;\sigma_{k-1}]}^{(i_k)} \cdots \int_{[\sigma_{k-1};\sigma_1]}^{(i_2)} H_{s_1}^{s_2}(i_2, \lambda) c_{j_1,\alpha_1}(\mu D_z)^{\alpha_1} \int_{[\sigma_1;\sigma_0]}^{(i_1)} H_{\sigma_0}(i_1, \lambda) f(s_0), \]  
\[ (3.6) \]

where for each \( p \), the relations \( m - j_p \leq i_p \leq m \) and \( |\alpha_p| \leq m - j_p \) hold. (Here, \( \alpha_p \) is a multi-index and should not be confused with the \( p \)th component of \( \alpha \).) The above is further equal to
\[ I = (-1)^k \int_{[t;\sigma_k]}^{(m)} \int_{[s_k;\sigma_{k-1}]}^{(i_k)} \cdots \int_{[\sigma_1;\sigma_0]}^{(i_1)} G_{s_k}^{t} c_{j_k,\alpha_k}(\mu D_z)^{\alpha_k} \times H_{s_{k-1}}^{s_k} c_{j_{k-1},\alpha_{k-1}}(\mu D_z)^{\alpha_{k-1}} \]  
\[ \times H_{\sigma_0}(i_1, \lambda) f(s_0), \]  
\[ (3.7) \]

Let \( F_k(s) \) denote the integrand of the above integral. Let \( R_{\sigma_0} = R - \gamma \varphi(s_0) \). Then all the functions above, when viewed as a function of \( z \), belong in \( A(\omega_{\sigma_0}[]\gamma] \). (This explains the necessity of the assumption that the coefficients be defined up to \( B_R \), for all \( t \) in the interval \([0, T]\).)

We can therefore apply Lemma 4 repeatedly, starting from the rightmost expression, to obtain the following estimate: for any \( \rho \in (0, R_{\sigma_0}) \), we have
\[ ||F_k(s)||_{B_{\rho}} \leq K(CD)^k \mu(s_1)^{(|\alpha_1| \cdots |\alpha_k|)} \left( \frac{s_0}{t} \right)^L \times \]  
\[ \left( \frac{e}{R_{\sigma_0} - \rho} \right)^{|\alpha_1 + \cdots + \alpha_k|} |\alpha_1 + \cdots + \alpha_k|!. \]  
\[ (3.8) \]

If \( |\alpha_1 + \cdots + \alpha_k| = 0 \), then for sufficiently small \( T = T_0 \), the bound for any \( c_{j,0}(t, z) = a_{j,0}(t, z) - a_{j,0}(0, z) \) is actually small, since \( a_{j,0}(t, z) \) is continuous with respect to \( t \). In other words, by choosing a small \( T = T_0 \), we could find a small constant \( \delta \) such that for any \( t \in [0, T_0] \) and \( 0 \leq s \leq t \), the following holds:
\[ ||F_k(s)||_{\omega_t} \leq K \delta^k \left( \frac{s_0}{t} \right)^L. \]  
\[ (3.9) \]

Going back to the integral, we have
\[ ||I||_{\omega_t} \leq \int_{[t;\sigma_k]}^{(m)} \int_{[s_k;\sigma_{k-1}]}^{(i_k)} \cdots \int_{[\sigma_1;\sigma_0]}^{(i_1)} K \delta^k \left( \frac{s_0}{t} \right)^L \]  
\[ = K \frac{\delta^k}{L^{m+i_1+\cdots+i_k}} \leq K \left( \frac{\delta}{L_0} \right)^k, \]  
\[ (3.10) \]

for some constant \( L_0 \) dependent on \( L \). This is possible since \( i_p \leq m \) for all \( p \).
If $|\alpha_1 + \cdots + \alpha_k| \neq 0$, set the $\rho$ in (3.8) to be equal to $R - \gamma \varphi(t)$. This gives

$$
||F_k(s)||_{\omega_t} \leq K(CD)^k \mu(s_1)^{a_1} \cdots \mu(s_k)^{a_k} \left( \frac{s_0}{t} \right)^L \times |\alpha_1 + \cdots + \alpha_k|! \left( \frac{e}{\gamma[\varphi(t) - \varphi(s_0)]} \right)^{|\alpha_1 + \cdots + \alpha_k|}.
$$

(3.11)

By renaming if necessary, assume that for $p = 1, \ldots, q$, we have $|\alpha_p| \neq 0$. Note that $q \geq 1$. We will again use the continuity of $a_{j,0}(t,z)$ to estimate those expressions which are not acted upon by $D_z$, i.e., the $k-q$ cases when $|\alpha_p| = 0$. Just like before, we can show that for small $\delta$,

$$
||F_k(s)||_{\omega_t} \leq K(CD)^q \delta^{k-q} \mu(s_1)^{a_1} \cdots \mu(s_q)^{a_q} \left( \frac{s_0}{t} \right)^L \times |\alpha_1 + \cdots + \alpha_q|! \left( \frac{e}{\gamma[\varphi(t) - \varphi(s_0)]} \right)^{|\alpha_1 + \cdots + \alpha_q|}.
$$

(3.12)

Thus, the integral $I$ can now be estimated as follows:

$$
||I||_{\omega_t} \leq K(CD)^q \delta^{k-q} \left( \frac{e}{\gamma} \right)^{|\alpha_1 + \cdots + \alpha_q|} |\alpha_1 + \cdots + \alpha_q|!
$$

$$
\times \int_{[t:s_1]}^{(m)} \int_{[s_1:s_0]}^{(i_1)} \cdots \int_{[s_k:s_{k-1}]}^{(i_k)} \left( \frac{s_0}{t} \right)^L \mu(s_1)^{a_1} \cdots \mu(s_q)^{a_q} \left( \frac{1}{[\varphi(t) - \varphi(s_0)]} \right)^{|\alpha_1 + \cdots + \alpha_q|} \frac{d\xi_0}{\xi_0} \cdots \frac{d\xi_k}{\xi_k}.
$$

(3.13)

Let $d = m + i_1 + \cdots + i_k$ and $b = |\alpha_1 + \cdots + \alpha_q|$. Note that $b \geq q$. Since for each $p$, we have $|\alpha_p| \leq m - j_p \leq i_p$, and using the fact that both $\varphi(t)$ and $\mu(t)$ are increasing on $(0, T_0)$, we have

$$
||I||_{\omega_t} \leq K(CD)^q \delta^{k-q} \left( \frac{e}{\gamma} \right)^b |\alpha_1 + \cdots + \alpha_q|!
$$

$$
\times \int_0^t \int_0^{\xi_b} \cdots \int_0^{\xi_1} \frac{\mu(\xi_b)}{\xi_b} \cdots \frac{\mu(\xi_1)}{\xi_1} \left( \frac{1}{\xi_0} \right)^L \frac{d\xi_0}{\xi_0} \left( \frac{1}{[\varphi(t) - \varphi(\xi_0)]^b} \right) \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_b}{\xi_b}.
$$

(3.14)

By (a) of Lemma 3, the second integral is equal to $L^{-d+b+1}$. Thus, the above simplifies into

$$
||I||_{\omega_t} \leq K(CD)^q \delta^{k-q} \left( \frac{e}{\gamma} \right)^b L^{-d+b+1} |\alpha_1 + \cdots + \alpha_q|!
$$

$$
\times \int_0^t \int_0^{\xi_b} \cdots \int_0^{\xi_1} \frac{\mu(\xi_b)}{\xi_b} \cdots \frac{\mu(\xi_1)}{\xi_1} \left( \frac{1}{\xi_0} \right)^L \frac{d\xi_0}{\xi_0} \left( \frac{1}{\xi_0 - \varphi(\xi_0)^b} \right) \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_b}{\xi_b}.
$$

(3.15)

The last integral is equal to $(Lb!)^{-1}$, by (b) of Lemma 3. Meanwhile, since $d \leq m(k+1)$, we can find a constant $L_1$, depending on $L$, such that $L^{-d} \leq L_1^k$. 

Substituting these results into the above equation, we get

\[ \|I\|_{\omega_{t}} \leq K(CD)^{q}q^{-q} \left(\frac{cL}{\gamma}\right)^{b} L_{1}^{k} = K \left(\frac{CD}{\delta}\right)^{q} (\delta L_{1})^{k} \left(\frac{cL}{\gamma}\right)^{b}. \]  

(3.16)

By taking a sufficiently small \(T_{0}\), we can find a \(\delta\) small enough such that \(\delta L_{1}\) above and \(\delta L_{0}^{-1}\) in (3.10) are both less than \((mJ)^{-1}\). Now, since \(q \leq b\), we can make the remaining expression less than one by choosing a large \(\gamma = \gamma_{0}\).

To summarize, we have shown that if \(T_{0}\) is sufficiently small and \(\gamma_{0}\) is sufficiently large, some constants \(K > 0\) and \(\delta_{0} < 1\) exist such that for all \(k\), we have

\[ \|v_{k}(t)\|_{\omega_{t}[\gamma_{0}]} \leq K\delta_{0}^{k} \] 

for any \(t \in [0, T_{0}]\).  

(3.17)

It follows that the series \(\sum_{k=0}^{\infty} v_{k}(t, z)\) is majorized by a convergent geometric series, and hence is itself convergent in \(C^{0}([0, \tau], A(\omega_{t}[\gamma_{0}]))\) for all \(\tau \in [0, T_{0}]\). This means that \(v_{k}(t)\) converges uniformly to \(u(t)\) on \(\Omega_{T_{0}}[\gamma_{0}]\).

By following the steps above, we can also show that for \(1 \leq p \leq m - 1\), the sequence \((tD_{t})^{p}u_{k}(t)\) converges uniformly to \((tD_{t})^{p}u(t)\) on \(\Omega_{T_{0}}[\gamma_{0}]\). Thus, it follows that on a compact subset of \(\Omega_{T_{0}}[\gamma_{0}]\), the sequence \(D_{z}^{\alpha}(tD_{t})^{p}u_{k}(t)\) converges to \(D_{z}^{\alpha}(tD_{t})^{p}u(t)\). This implies the convergence of the approximate solutions to the true solution \(u(t)\).

Uniqueness may be proved in a similar manner.

References


