A Sharp Existence and Uniqueness Theorem for Linear Fuchsian Partial Differential Equations

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Abstract

This paper considers the equation $Pu = f$, where $u$ and $f$ are continuous with respect to $t$ and holomorphic with respect to $z$, and $P$ is the linear Fuchsian partial differential operator

$$P = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, z)(\mu(t)D_z)^\alpha(tD_t)^j.$$ 

We will give a sharp form of unique solvability in the following sense: we can find a domain $\Omega$ such that if $f$ is defined on $\Omega$, then we can find a unique solution $u$ also defined on $\Omega$.

1 Introduction and Result

Denote by $N$ the set of nonnegative integers, and let $(t, z) = (t, z_1, \ldots, z_n) \in \mathbb{R} \times \mathbb{C}^n$. Let $R > 0$ be sufficiently small, and for $\rho \in (0, R]$, let $B_\rho$ be the polydisk $\{z \in \mathbb{C}^n; |z_i| < \rho \text{ for } i = 1, 2, \ldots, n\}$.

Given any bounded, open subset $D$ of $\mathbb{C}^n$, we define by $A(D)$ the Banach space of all functions $g(z)$ holomorphic in $D$ and continuous up to $\overline{D}$; the norm in this space is given by $\|g\|_D = \max_{z \in \overline{D}} |g(z)|$. Let $T > 0$. Then we denote by $C^0([0, T], A(D))$ the set of functions continuous on the interval $[0, T]$ and valued in the space $A(D)$.

We say that a continuous, positive-valued function $\mu(t)$ on the interval $(0, T)$ is a weight function if $\mu(t)$ is increasing and the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s} ds$$

is well-defined on $(0, T)$, i.e., the integral on the right is finite. (See Tahara [7].)

Consider now the linear partial differential operator

$$P = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, z)(\mu(t)D_z)^\alpha(tD_t)^j.$$ 

(1.2)

Here, $D_t = \partial/\partial t$ and $D_z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$; $\mu(t)$ is a weight function; and the coefficients $a_{j,\alpha}(t, z)$ belong in the space $C^0([0, T], A(B_R))$, i.e., for any
$s \in [0, T]$, each of the functions $a_{j, \alpha}(s, z)$, when viewed as a function of $z$, is holomorphic in $B_R$ and continuous up to $\overline{B_R}$. We associate a polynomial with this operator, called the \textit{characteristic polynomial} of $\mathcal{P}$, and we define it by

$$\mathcal{C}(\lambda, z) = \lambda^m + a_{m-1,0}(0,z)\lambda^{m-1} + \cdots + a_{0,0}(0,z). \quad (1.3)$$

Its roots $\lambda_1(z), \ldots, \lambda_m(z)$ will be referred to as \textit{characteristic exponents}. In what follows, we will assume that there exists a positive number $L$ such that

$$\Re \lambda_j(z) \leq -L < 0 \quad \text{for all } z \in B_R \text{ and } 1 \leq j \leq m. \quad (1.4)$$

Baouendi and Goulaouic [1] studied the above operator in the case when $\mu(t) = t^a \ (a > 0)$. They called such operator a Fuchsian partial differential operator, which for them is the "natural" generalization of a Fuchsian ordinary differential operator. In their paper, they gave some generalizations of the classical Cauchy-Kowalewski and Holmgren theorems for this type of operators. Their method has been applied and extended to various cases as can be seen, for example, in Tahara [6], Mandai [5] and Yamane [8].

In a previous paper [4], the author proved existence and uniqueness theorems similar to those given in [1], but for general $\mu(t)$. Essentially, he proved the following unique solvability result.

\textbf{Theorem 1.} Let $\mathcal{P}$ be as in (1.2). Then given any $\rho \in (0, R)$, there exists an $\varepsilon \in (0, T]$ such that for any $f(t, z) \in C^0([0, T], \mathcal{A}(B_R))$, the equation $\mathcal{P}u = f$ has a unique solution $u(t, z) \in C^0([0, \varepsilon], \mathcal{A}(B_\rho))$ satisfying for $1 \leq p \leq m$ the relation $(tD_t)^pu \in C^0([0, \varepsilon], \mathcal{A}(B_\rho))$.

We remark that although $f(t, z)$, viewed as a function of $z$, is defined on $B_R$, the existence of the solution $u(t, z)$ is only guaranteed up to $B_\rho$, with $\rho < R$. Moreover, any two solutions of $\mathcal{P}u = f$ can only be shown to coincide in a neighborhood of the origin which is smaller than the neighborhood on which the two are defined.

In this paper, we shall present a formulation leading to an existence and uniqueness result sharper than the one given above. The result is sharper in the sense that the solution $u(t, z)$ of the equation $\mathcal{P}u = f$ will now have the same domain of definition as the inhomogeneous part $f(t, z)$.

To proceed, we will need the following definitions.

\textbf{Definition 1.} Let $\tau \in (0, T)$, $\gamma > 0$ and $\varphi(t)$ be the one in (1.1). We define

(i) $\omega_{\tau}[\gamma] = \{z \in \mathbb{C}^n; |z_i| < R - \gamma \varphi(\tau) \text{ for } i = 1, 2, \ldots, n\}$, and

(ii) $\Omega_T[\gamma] = \{(\tau, z) \in \mathbb{R} \times \mathbb{C}^n; 0 \leq \tau \leq T \text{ and } z \in \omega_{\tau}[\gamma]\}$.

\textbf{Definition 2.} Let $p \in \mathbb{N}$ and $\gamma > 0$.

(i) We say that $f(t, z)$ belongs in $\mathcal{K}_0(\Omega_T[\gamma])$ if for each $\tau \in [0, T]$, we have $f(t) \in C^0([0, \tau], \mathcal{A}(\omega_{\tau}[\gamma]))$. 

(ii) $\mathcal{K}_0(\Omega_T[\gamma])$ is a set of functions $f(t, z)$ which are defined for all $t \in [0, T]$ and for all $z$ in a neighborhood of the origin with radius $\gamma$, such that $f(t, z)$ is holomorphic in $\mathbb{C}^n$ and bounded for all $t$. The set $\mathcal{K}_0(\Omega_T[\gamma])$ is a Banach space with the norm $\|f\|_{C^0([0, T], \mathcal{A}(\omega_{\tau}[\gamma]))}$.
(ii) We say that \( w(t, z) \) belongs in \( C^0_p([0, \tau], \mathcal{A}(\omega_\tau[\gamma])) \) if for all \( 0 \leq j \leq p, \) we have \( (tD_t)^j w(t) \in C^0([0, \tau], \mathcal{A}(\omega_\tau[\gamma])). \)

(iii) We say that \( u(t, z) \) belongs in \( \mathcal{K}_p(\Omega_T[\gamma]) \) if for each \( \tau \in [0, T] \), we have \( u(t) \in C^0_p([0, \tau], \mathcal{A}(\omega_\tau[\gamma])). \)

Under the above assumptions, we now state the following main result.

**Theorem 2.** Let \( P \) be the operator given in (1.2). Then there exist constants \( T_0 > 0 \) and \( \gamma_0 > 0 \) depending on \( P \) such that for any \( f(t, z) \in \mathcal{K}_0(\Omega_{T_0}[\gamma_0]), \) the equation

\[
P u = f \quad \text{in} \quad \Omega_{T_0}[\gamma_0]
\]

has a unique solution \( u(t, z) \) in \( \mathcal{K}_m(\Omega_{T_0}[\gamma_0]). \)

Moreover, the solution satisfies the a priori estimate

\[
\sum_{p=0}^{m} \max_\Delta |(tD_t)^p u| \leq C \max_\Delta |f|,
\]

where \( \Delta \) is the closure of \( \Omega_{T_0}[\gamma_0] \) and \( C > 0 \) is some constant dependent on the above equation and on the domain \( \Omega_{T_0}[\gamma_0]. \)

Note that \( f(t, z) \) and \( u(t, z) \) both have \( \Omega_{T_0}[\gamma_0] \) as their domain of definition. This fact allows us to restate our theorem in the following manner: for any \( T, \gamma > 0, \) let \( X_{T, \gamma} \) and \( Y_{T, \gamma} \) be the spaces \( \mathcal{K}_m(\Omega_T[\gamma]) \) and \( \mathcal{K}_0(\Omega_T[\gamma]), \) respectively. Let \( W_{T, \gamma} \) be the subspace of \( X_{T, \gamma} \) consisting of functions \( u \in X_{T, \gamma} \) such that \( Pu \) belongs in \( Y_{T, \gamma}. \) Define a linear operator \( \Psi \) from \( X_{T, \gamma} \) to \( Y_{T, \gamma} \) with domain \( W_{T, \gamma} \) by \( \Psi u = Pu. \) Let \( \| \cdot \|_{T, \gamma} \) denote the maximum norm in the closure of \( \Omega_T[\gamma]. \) Then \( X_{T, \gamma} \) and \( Y_{T, \gamma} \) are Banach spaces; given \( u \in X_{T, \gamma} \) and \( f \in Y_{T, \gamma}, \) we define their norms by \( \sum_{p=0}^{m} \| (tD_t)^p u \|_{T, \gamma} \) and \( \| f \|_{T, \gamma}, \) respectively. Note further that the operator \( \Psi \) is a closed linear operator from \( X_{T, \gamma} \) to \( Y_{T, \gamma}. \) The above theorem can now be stated as

**Theorem 2’.** There exist \( T_0, \gamma_0 > 0 \) depending on \( P \) such that the operator \( \Psi \) is a one-one, closed linear operator from \( X_{T_0, \gamma_0} \) onto \( Y_{T_0, \gamma_0}. \)

Since \( \Psi \) is an injection, \( \Psi^{-1} \) exists and is also closed. The Closed Graph Theorem further implies that \( \Psi^{-1} \) is continuous. The estimate given in (1.6) is just a consequence of the continuity of \( \Psi^{-1}. \)

2 Preliminary Discussion

We can rewrite the operator \( P \) as

\[
P = Q + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j, \alpha}(t, z)(\mu(t)D_z)^{\alpha}(tD_t)^j,
\]
where the operator $Q$ is defined by

$$Q = (tD_t)^m + a_{m-1,0}(0,z)(tD_t)^{m-1} + \cdots + a_{0,0}(0,z)$$

(2.1)

and

$$c_{j,\alpha}(t, z) = \begin{cases} a_{j,\alpha}(t, z) & \text{if } |\alpha| \neq 0, \\ a_{j,\alpha}(t, z) - a_{j,\alpha}(0,z) & \text{if } |\alpha| = 0. \end{cases}$$

Note that the coefficients of $Q$ are holomorphic functions of $z$ in $B_R$. Note further that the characteristic exponents of $Q$ are the same as that of $P$, and hence satisfy (1.4).

**Lemma 1.** Fix $\tau > 0$ and let $g(t) \in C^0([0, \tau], A(\omega_{\tau}[\gamma]))$. Then the equation $Qu = g$ has a unique solution $u(t) \in C^0_m([0, \tau], A(\omega_{\tau}[\gamma]))$ given by

$$u(t) = \frac{1}{m!} \sum_{\sigma \in S_m} \int_0^t \cdots \int_0^{s_2} \left( \frac{s_m}{t} \right)^{-\lambda_{\sigma(m)}} \left( \frac{s_{m-1}}{s_m} \right)^{\lambda_{\sigma(m-1)}} \cdots \left( \frac{\theta}{s_2} \right)^{-\lambda_{\sigma(1)}} g(s_1) \frac{ds_1}{s_1} \frac{ds_2}{s_2} \cdots \frac{ds_m}{s_m}.$$  

(2.2)

Here, $S_m$ is the group of permutations of $\{1, 2, \ldots, m\}$.

A result in symmetric entire functions asserts that $u(t, z)$ is holomorphic with respect to $z$. The fact that it belongs in $C^0_m([0, \gamma], A(\omega_{\tau}[\gamma]))$ is seen in the integral expression, but may actually be obtained a priori. (See [1].)

To facilitate computation, we define for $\lambda = (\lambda_1, \ldots, \lambda_m)$ the function

$$G^t_{\theta}(\lambda) \overset{\text{def}}{=} \frac{1}{m!} \sum_{\sigma \in S_m} \left( \frac{s_m}{t} \right)^{-\lambda_{\sigma(m)}} \left( \frac{s_{m-1}}{s_m} \right)^{-\lambda_{\sigma(m-1)}} \cdots \left( \frac{\theta}{s_2} \right)^{-\lambda_{\sigma(1)}},$$

(2.3)

for some dummy variables $s_2, \ldots, s_m$. Define, too, the integral operator

$$\int_{[t, \theta]}^{(m)} g \overset{\text{def}}{=} \int_0^t \cdots \int_0^{s_2} g(\theta) \frac{d\theta}{\theta} \frac{ds_2}{s_2} \cdots \frac{ds_m}{s_m}.$$  

(2.4)

Using the above, we can now write the solution $u(t)$ of the equation $Qu = g$ as

$$u(t) = \int_{[t, \theta]}^{(m)} G^t_{\theta}(\lambda) g.$$

In our proof of the main theorem, it will be necessary to consider the action of the differential operator $(tD_t)^p$ on integral expressions similar to the one in (2.2). One can easily verify the following

**Lemma 2.** Let $u(t)$ be the solution of $Qu = g$. Then for a natural number $p$ less than $m$, we have

$$(tD_t)^p u = \sum_{i=m-p}^m \int_{[t, s_1]}^{(i)} g \times \left\{ \frac{1}{m!} \sum_{\sigma \in S_m} h_i(\sigma, \lambda) \left( \frac{s_i}{t} \right)^{-\lambda_{\sigma(i)}} \times \left( \frac{s_{i-1}}{s_i} \right)^{-\lambda_{\sigma(i-1)}} \cdots \left( \frac{s_1}{s_2} \right)^{-\lambda_{\sigma(1)}} \right\},$$

(2.5)
where the functions \( h_i(\sigma, \lambda) \) are suitable polynomial functions of the characteristic exponents \( \lambda_1(z), \ldots, \lambda_m(z) \).

For brevity, let us set, for a natural number \( k \),

\[
H^t_\varnothing(k, \lambda) = \frac{1}{m!} \sum_{\sigma \in S_m} h_k(\sigma, \lambda) \left( \frac{s_k}{s_{k-1}} \right)^{-\lambda_{\sigma(k)}} \left( \frac{s_{k-1}}{s_{k-2}} \right)^{-\lambda_{\sigma(k-1)}} \cdots \left( \frac{s_2}{s_1} \right)^{-\lambda_{\sigma(1)}}.
\]  

(2.6)

By symmetry, the functions \( H^t_\varnothing(k, \lambda) \) are holomorphic with respect to \( z \) and thus belong in \( A(B_R) \).

The next lemma is useful in evaluating some integral expressions in the proof.

**Lemma 3.** Let \( k \) be natural number. Then the following equalities hold:

(a) \[
\int_0^s \int_0^{s_k} \int_0^{s_k-1} \cdots \int_0^{s_1} \frac{ds_0}{s_0} \cdots \frac{ds_{k-1}}{s_{k-1}} = \frac{1}{L^k}
\]

(b) \[
\int^{t \epsilon_k \epsilon_1} \cdots \int_0^{s_1} \frac{\mu(s_k) \cdots \mu(s_1)}{s_k} ds_k \cdots ds_1 = \frac{1}{L^k k!}
\]

The first equality is obvious. The second can be proved by reversing the order of integration and recalling that \( t \varphi'(t) = \mu(t) \).

To estimate the derivatives with respect to \( z \), we have the following lemma. (For a proof, see Hörmander [3], Lemma 5.1.3.)

**Lemma 4.** Let the function \( v(z) \) be holomorphic in \( B_R \), and suppose there are positive constants \( K \) and \( c \) such that

\[
\| v \|_\rho \leq \frac{K}{(R-\rho)^c} \quad \text{for every } \rho \in (0, R).
\]  

(2.7)

Then we have

\[
\| D_z^\alpha v \|_\rho \leq \frac{Ke^{1|\alpha|(c+1)|\alpha|}}{(R-\rho)^{c+|\alpha|}} \quad \text{for every } \rho \in (0, R).
\]  

(2.8)

In the above, we define \((c)_p = (c)(c+1) \cdots (c+p-1)\).

### 3 Proof of Main Theorem

Let \( f \) be any element of \( \mathcal{K}_0(\Omega, \gamma_0) \). Here, the constants \( T_0 > 0 \) and \( \gamma_0 > 0 \) satisfy some conditions which will later be specified. For convenience, we will drop the subscript in both and instead use \( T \) and \( \gamma \); we will again use the subscript upon stating the conditions that these constants need to satisfy.
We will use the method of successive approximations to solve the equation \( Pu = f \). Define the approximate solutions as follows:

\[
    u_0(t) = \int_{[t; s]}^{(m)} G_{s}^{t} (\lambda) f
\]  

(3.1)

and for \( k \geq 1 \),

\[
    u_k(t) = \int_{[t; s]}^{(m)} G_{s}^{t} (\lambda) [f - S(s)u_{k-1}].
\]  

(3.2)

Here, \( t \in [0, T] \), and for brevity, we have set \( S(t) = \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j, \alpha}(t, z) \cdot (\mu(t)D_{z})^{\alpha}(tD_{t})^{j} \). Note that for all \( k \), the approximate solutions \( u_k(t, z) \) are defined on \( \Omega_{T_0}[\gamma_0] \). Furthermore, they are continuous with respect to \( t \) and holomorphic with respect to \( z \) on this region.

For each \( k \), we also define the sequence of functions \( v_k(t) = u_k(t) - u_{k-1}(t) \), with \( u_1 \equiv 0 \). Then the \( v_k(t, z) \)'s are also defined on the same region as the \( u_k(t, z) \)'s, and are also continuous with respect to \( t \) and holomorphic with respect to \( z \). Using the expression for \( u_k(t) \), we have

\[
    v_0(t) = \int_{[t; s]}^{(m)} G_{s}^{t} (\lambda) f
\]

and for \( k \geq 1 \),

\[
    v_k(t) = - \int_{[t; s]}^{(m)} G_{s}^{t} (\lambda) S(s)v_{k-1}.
\]  

(3.3)

To prove that the approximate solutions converge to the real solution, we will henceforth fix one \( t \in [0, T] \), and estimate the functions \( v_k(t) \). Let \( C \) be the bound on \([0, T] \times \overline{B}_R \) of all \( c_{j, \alpha}(t, z) \), and \( K \) be the bound in \( \Omega_T[\gamma] \) of \( f(t, z) \). As \( G_{s}^{t} (\lambda) \) and \( H_{s}^{t}(k, \lambda) \), we have for \( 1 \leq k \leq m \) and for some \( D > 0 \):

\[
    \sup_{z \in \overline{B}_R} |G_{s}^{t} (\lambda)| \leq \left( \frac{s}{t} \right)^L \quad \text{and} \quad \sup_{z \in \overline{B}_R} |H_{s}^{t}(k, \lambda)| \leq D \left( \frac{s}{t} \right)^L.
\]  

(3.4)

We can easily see that \( \|v_0(t)\|_{\omega_t} \) is bounded by \( KL^{-m} \) for any \( 0 \leq t \leq T \). Here, we have written for convenience \( \|\cdot\|_{\omega_t} \) in place of \( \|\cdot\|_{\omega_t[\gamma]} \). For general \( k \), we note that \( v_k(t) \) is given by the iterated integral

\[
    v_k(t) = (-1)^k \int_{[t; s_k]}^{(m)} G_{s_k}^{t} (\lambda) S(s_k) \int_{[s_k; s_{k-1}]}^{(m)} G_{s_{k-1}}^{s_k} (\lambda) S(s_{k-1}) \cdots
\]

\[
    \cdots \int_{[s_2; s_1]}^{(m)} G_{s_1}^{s_2} (\lambda) S(s_1) \int_{[s_1; s_0]}^{(m)} G_{s_0}^{s_1} (\lambda) f(s_0).
\]  

(3.5)

The expression above can be expanded using Lemma 2, and thus obtain a finite sum whose number of terms is less than \((mJ)^k\), where \( J \) is the cardinality of the set \( \{(j, \alpha); 0 \leq j \leq m - 1 \text{ and } |\alpha| \leq m - j \} \). Each term of the finite sum
has the form

\[ I = (-1)^k \int_{[t; s_k]}^{(m)} G^t_{s_k}(\lambda) c_{j_k, \alpha_k}(\mu D_z)^{\alpha_k} \int_{[s_k; s_{k-1}]}^{(i_k)} H^2_{s_k}(i_k, \lambda) c_{j_{k-1}, \alpha_{k-1}}(\mu D_z)^{\alpha_{k-1}} \cdots \int_{[s_1; s_0]}^{(i_1)} H^2_{s_1}(i_1, \lambda) f(s_0), \]  

(3.6)

where for each \( p \), the relations \( m - j_p \leq i_p \leq m \) and \( |\alpha_p| \leq m - j_p \) hold. (Here, \( \alpha_p \) is a multi-index and should not be confused with the \( p \)th component of \( \alpha \).)

The above is further equal to

\[ I = (-1)^k \int_{[t; s_k]}^{(m)} \cdots \int_{[s_1; s_0]}^{(i_1)} G^t_{s_k} c_{j_k, \alpha_k}(s_k)(\mu(s_k) D_z)^{\alpha_k} \times H^2_{s_k-1} c_{j_{k-1}, \alpha_{k-1}}(s_{k-1})(\mu(s_{k-1}) D_z)^{\alpha_{k-1}} \cdots \times H^2_{s_1} c_{j_1, \alpha_1}(s_1)(\mu(s_1) D_z)^{\alpha_1} \times H^2_{s_0} f(s_0). \]  

(3.7)

Let \( F_k(s) \) denote the integrand of the above integral. Let \( R(s_0) = R - \gamma \varphi(s_0) \). Then all the functions above, when viewed as a function of \( z \), belong in \( A(\omega_{s_0}[\gamma]) \).

This explains the necessity of the assumption that the coefficients be defined up to \( B_R \), for all \( t \) in the interval \([0, T]\).)

We can therefore apply Lemma 4 repeatedly, starting from the rightmost expression, to obtain the following estimate: for any \( \rho \in (0, R(s_0)) \), we have

\[ \|F_k(s)\|_{B_{\rho}} \leq K(CD)^k \mu(s_1)|\alpha_1| \cdots \mu(s_k)|\alpha_k| \left(\frac{s_0}{t}\right)^L \times \left(\frac{e}{R(s_0) - \rho}\right)|\alpha_1 + \cdots + \alpha_k| \]  

(3.8)

If \( |\alpha_1 + \cdots + \alpha_k| = 0 \), then for sufficiently small \( T = T_0 \), the bound for any \( c_{j,0}(t, z) = a_{j,0}(t, z) - a_{j,0}(0, z) \) is actually small, since \( a_{j,0}(t, z) \) is continuous with respect to \( t \). In other words, by choosing a small \( T = T_0 \), we could find a small constant \( \delta \) such that for any \( t \in [0, T_0] \) and \( 0 \leq s \leq t \), the following holds:

\[ \|F_k(s)\|_{\omega_t} \leq K\delta^k \left(\frac{s_0}{t}\right)^L. \]  

(3.9)

Going back to the integral, we have

\[ \|I\|_{\omega_t} \leq \int_{[t; s_k]}^{(m)} \cdots \int_{[s_1; s_0]}^{(i_1)} K\delta^k \left(\frac{s_0}{t}\right)^L \]  

\[ = K \frac{\delta^k}{L^{m+i_1+\cdots+i_k}} \leq K \left(\frac{\delta}{L_0}\right)^k, \]  

(3.10)

for some constant \( L_0 \) dependent on \( L \). This is possible since \( i_p \leq m \) for all \( p \).
If \(|\alpha_{1} + \cdots + \alpha_{k}| \neq 0\), set the \(\rho\) in (3.8) to be equal to \(R - \gamma \varphi(t)\). This gives
\[
||F_{k}(s)||_{\omega_{t}} \leq K(CD)^{k} \mu(s_{1})^{|\alpha_{1}|} \cdots \mu(s_{k})^{|\alpha_{k}|} \left(\frac{s_{0}}{t}\right)^{L} \\
\times |\alpha_{1} + \cdots + \alpha_{k}|! \left(\frac{e}{\gamma[\varphi(t) - \varphi(s_{0})]}\right)^{|\alpha_{1} + \cdots + \alpha_{k}|}.
\] (3.11)

By renaming if necessary, assume that for \(p = 1, \ldots, q\), we have \(|\alpha_{p}| \neq 0\).
Note that \(q \geq 1\). We will again use the continuity of \(a_{j,0}(t, z)\) to estimate those expressions which are not acted upon by \(D_{z}\), i.e., the \(k-q\) cases when \(|\alpha_{p}| = 0\). Just like before, we can show that for small \(\delta\),
\[
||F_{k}(s)||_{\omega_{t}} \leq K(CD)^{q} \delta^{k-q} \mu(s_{1})^{|\alpha_{1}|} \cdots \mu(s_{q})^{|\alpha_{q}|} \left(\frac{s_{0}}{t}\right)^{L} \\
\times |\alpha_{1} + \cdots + \alpha_{q}|! \left(\frac{e}{\gamma[\varphi(t) - \varphi(s_{0})]}\right)^{|\alpha_{1} + \cdots + \alpha_{q}|}.
\] (3.12)

Thus, the integral \(I\) can now be estimated as follows:
\[
||I||_{\omega_{t}} \leq K(CD)^{q} \delta^{k-q} \left(\frac{e}{\gamma}\right)^{|\alpha_{1} + \cdots + \alpha_{q}|} |\alpha_{1} + \cdots + \alpha_{q}|! \\
\times \int_{[t:s_{k}]}^{(m)} \int_{[s_{k};s_{k-1}]}^{(i_{k})} \cdots \int_{[s_{1};s_{0}]}^{(i_{1})} \frac{(s_{0})}{t} \mu(s_{1})^{|\alpha_{1}|} \cdots \mu(s_{q})^{|\alpha_{q}|} \\
\times |\alpha_{1} + \cdots + \alpha_{q}|! \left(\frac{e}{\gamma[\varphi(t) - \varphi(s_{0})]}\right)^{|\alpha_{1} + \cdots + \alpha_{q}|}.
\] (3.13)

Let \(d = m + i_{1} + \cdots + i_{k}\) and \(b = |\alpha_{1} + \cdots + \alpha_{q}|\). Note that \(b \geq q\). Since for each \(p\), we have \(|\alpha_{p}| \leq m - j_{p} \leq i_{p}\), and using the fact that both \(\varphi(t)\) and \(\mu(t)\) are increasing on \((0, T_{0})\), we have
\[
||I||_{\omega_{t}} \leq K(CD)^{q} \delta^{k-q} \left(\frac{e}{\gamma}\right)^{b} b! \\
\times \int_{[t:s_{k}]}^{(m)} \int_{[s_{k};s_{k-1}]}^{(i_{k})} \cdots \int_{[s_{1};s_{0}]}^{(i_{1})} \frac{(s_{0})}{t} \mu(s_{1})^{|\alpha_{1}|} \cdots \mu(s_{q})^{|\alpha_{q}|} \\
\times |\alpha_{1} + \cdots + \alpha_{q}|! \left(\frac{e}{\gamma[\varphi(t) - \varphi(s_{0})]}\right)^{b}.
\] (3.14)

By (a) of Lemma 3, the second integral is equal to \(L^{-d+b+1}\). Thus, the above simplifies into
\[
||I||_{\omega_{t}} \leq K(CD)^{q} \delta^{k-q} \left(\frac{e}{\gamma}\right)^{b} L^{-d+b+1} b! \\
\times \int_{[t:s_{k}]}^{(m)} \int_{[s_{k};s_{k-1}]}^{(i_{k})} \cdots \int_{[s_{1};s_{0}]}^{(i_{1})} \frac{(s_{0})}{t} \mu(s_{1})^{|\alpha_{1}|} \cdots \mu(s_{q})^{|\alpha_{q}|} \\
\times |\alpha_{1} + \cdots + \alpha_{q}|! \left(\frac{e}{\gamma[\varphi(t) - \varphi(s_{0})]}\right)^{b} d\xi_{0} d\xi_{1} \cdots d\xi_{b}.
\] (3.15)

The last integral is equal to \((Lb!)^{-1}\), by (b) of Lemma 3. Meanwhile, since \(d \leq m(k+1)\), we can find a constant \(L_{1}\), depending on \(L\), such that \(L^{-d} \leq L_{1}^{k} \).
Substituting these results into the above equation, we get

$$
||I||_{\omega_t} \leq K(CD)^q \delta^q \left(\frac{eL}{\gamma}\right)^b L_1^k = K\left(\frac{CD}{\delta}\right)^q (\delta L_1)^k \left(\frac{eL}{\gamma}\right)^b.
$$

(3.16)

By taking a sufficiently small $T_0$, we can find a $\delta$ small enough such that $\delta L_1$ above and $\delta L_0^{-1}$ in (3.10) are both less than $(mJ)^{-1}$. Now, since $q \leq b$, we can make the remaining expression less than one by choosing a large $\gamma = \gamma_0$.

To summarize, we have shown that if $T_0$ is sufficiently small and $\gamma_0$ is sufficiently large, some constants $K > 0$ and $\delta_0 < 1$ exist such that for all $k$, we have

$$
||v_k(t)||_{\omega_t[\gamma_0]} \leq K\delta_0^k \quad \text{for any } t \in [0, T_0].
$$

(3.17)

It follows that the series $\sum_{k=0}^{\infty} v_k(t, z)$ is majorized by a convergent geometric series, and hence is itself convergent in $C^0([0, \tau], \mathcal{A}(\omega_t[\gamma_0]))$ for all $\tau \in [0, T_0]$. This means that $v_k(t)$ converges uniformly to $u(t)$ on $\Omega_{T_0}[\gamma_0]$.

By following the steps above, we can also show that for $1 \leq p \leq m - 1$, the sequence $(tD_t)^p u_k(t)$ converges uniformly to $(tD_t)^p u(t)$ on $\Omega_{T_0}[\gamma_0]$. Thus, it follows that on a compact subset of $\Omega_{T_0}[\gamma_0]$, the sequence $D_z^p(tD_t)^p u_k(t)$ converges to $D_z^p(tD_t)^p u(t)$. This implies the convergence of the approximate solutions to the true solution $u(t)$.

Uniqueness may be proved in a similar manner.

References


