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A Sharp Existence and Uniqueness Theorem for Linear Fuchsian Partial Differential Equations

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Abstract

This paper considers the equation $Pu = f$, where $u$ and $f$ are continuous with respect to $t$ and holomorphic with respect to $z$, and $P$ is the linear Fuchsian partial differential operator

$$P = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, z)(\mu(t)D_z)^\alpha(tD_t)^j.$$  

We will give a sharp form of unique solvability in the following sense: we can find a domain $\Omega$ such that if $f$ is defined on $\Omega$, then we can find a unique solution $u$ also defined on $\Omega$.

1 Introduction and Result

Denote by $\mathbb{N}$ the set of nonnegative integers, and let $(t, z) = (t, z_1, \ldots, z_n) \in \mathbb{R} \times \mathbb{C}^n$. Let $R > 0$ be sufficiently small, and for $\rho \in (0, R]$, let $B_\rho$ be the polydisk $\{z \in \mathbb{C}^n; |z_i| < \rho \text{ for } i = 1, 2, \ldots, n\}$.

Given any bounded, open subset $D$ of $\mathbb{C}^n$, we define by $A(D)$ the Banach space of all functions $g(z)$ holomorphic in $D$ and continuous up to $\overline{D}$; the norm in this space is given by $||g||_D = \max_{z \in \overline{D}} |g(z)|$. Let $T > 0$. Then we denote by $C^0([0, T], A(D))$ the set of functions continuous on the interval $[0, T]$ and valued in the space $A(D)$.

We say that a continuous, positive-valued function $\mu(t)$ on the interval $(0, T)$ is a weight function if $\mu(t)$ is increasing and the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s} \, ds$$

(1.1)

is well-defined on $(0, T)$, i.e., the integral on the right is finite. (See Tahara [7].)

Consider now the linear partial differential operator

$$P = (tD_t)^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t, z)(\mu(t)D_z)^\alpha(tD_t)^j.$$  

(1.2)

Here, $D_t = \partial/\partial t$ and $D_z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$; $\mu(t)$ is a weight function; and the coefficients $a_{j,\alpha}(t, z)$ belong in the space $C^0([0, T], A(B_R))$, i.e., for any
$s \in [0, T]$, each of the functions \(a_{j, \alpha}(s, z)\), when viewed as a function of \(z\), is holomorphic in \(B_R\) and continuous up to \(\overline{B_R}\). We associate a polynomial with this operator, called the \textit{characteristic polynomial} of \(\mathcal{P}\), and we define it by

\[
\mathcal{C}(\lambda, z) = \lambda^m + a_{m-1,0}(0, z)\lambda^{m-1} + \cdots + a_{0,0}(0, z).
\]  
(1.3)

Its roots \(\lambda_1(z), \ldots, \lambda_m(z)\) will be referred to as \textit{characteristic exponents}. In what follows, we will assume that there exists a positive number \(L\) such that

\[
\Re \lambda_j(z) \leq -L < 0 \quad \text{for all } z \in B_R \text{ and } 1 \leq j \leq m.
\]  
(1.4)

Baouendi and Goulaouic [1] studied the above operator in the case when \(\mu(t) = t^a\) \((a > 0)\). They called such operator a Fuchsian partial differential operator, which for them is the "natural" generalization of a Fuchsian ordinary differential operator. In their paper, they gave some generalizations of the classical Cauchy-Kowalewski and Holmgren theorems for this type of operators. Their method has been applied and extended to various cases as can be seen, for example, in Tahara [6], Mandai [5] and Yamane [8].

In a previous paper [4], the author proved existence and uniqueness theorems similar to those given in [1], but for general \(\mu(t)\). Essentially, he proved the following unique solvability result.

**Theorem 1.** Let \(\mathcal{P}\) be as in (1.2). Then given any \(\rho \in (0, R)\), there exists an \(\epsilon \in (0, T]\) such that for any \(f(t, z) \in C^0([0, T], A(B_R))\), the equation \(\mathcal{P}u = f\) has a unique solution \(u(t, z) \in C^0([0, \epsilon], A(B_{\rho}))\) satisfying for \(1 \leq p \leq m\) the relation \((t D_t)^p u \in C^0([0, \epsilon], A(B_{\rho}))\).

We remark that although \(f(t, z)\), viewed as a function of \(z\), is defined on \(B_R\), the existence of the solution \(u(t, z)\) is only guaranteed up to \(B_{\rho}\), with \(\rho < R\). Moreover, any two solutions of \(\mathcal{P}u = f\) can only be shown to coincide in a neighborhood of the origin which is smaller than the neighborhood on which the two are defined.

In this paper, we shall present a formulation leading to an existence and uniqueness result sharper than the one given above. The result is sharper in the sense that the solution \(u(t, z)\) of the equation \(\mathcal{P}u = f\) will now have the same domain of definition as the inhomogeneous part \(f(t, z)\).

To proceed, we will need the following definitions.

**Definition 1.** Let \(\tau \in (0, T), \gamma > 0\) and \(\varphi(t)\) be the one in (1.1). We define

(i) \(\omega_{\tau}[\gamma] = \{z \in \mathbb{C}^n; |z_i| < R - \gamma \varphi(\tau) \text{ for } i = 1, 2, \ldots, n\}\), and

(ii) \(\Omega_T[\gamma] = \{(\tau, z) \in \mathbb{R} \times \mathbb{C}^n; 0 \leq \tau \leq T \text{ and } z \in \omega_{\tau}[\gamma]\}\).

**Definition 2.** Let \(p \in \mathbb{N}\) and \(\gamma > 0\).

(i) We say that \(f(t, z)\) belongs in \(K_0(\Omega_T[\gamma])\) if for each \(\tau \in [0, T]\), we have \(f(t) \in C^0([0, \tau], A(\omega_{\tau}[\gamma]))\).
(ii) We say that \( w(t, z) \) belongs in \( C^0_p([0, \tau], \mathcal{A}(\omega_\tau[\gamma])) \) if for all \( 0 \leq j \leq p \), we have \( (tD_t)^j w(t) \in C^0([0, \tau], \mathcal{A}(\omega_\tau[\gamma])) \).

(iii) We say that \( u(t, z) \) belongs in \( \mathcal{K}_p(\Omega_T[\gamma]) \) if for each \( \tau \in [0, T] \), we have \( u(t) \in C^0_p([0, \tau], \mathcal{A}(\omega_\tau[\gamma])) \).

Under the above assumptions, we now state the following main result.

**Theorem 2.** Let \( \mathcal{P} \) be the operator given in (1.2). Then there exist constants \( T_0 > 0 \) and \( \gamma_0 > 0 \) depending on \( \mathcal{P} \) such that for any \( f(t, z) \in \mathcal{K}_0(\Omega_{T_0}[\gamma_0]) \), the equation
\[
\mathcal{P}u = f \quad \text{in} \quad \Omega_{T_0}[\gamma_0]
\]
has a unique solution \( u(t, z) \) in \( \mathcal{K}_m(\Omega_{T_0}[\gamma_0]) \).

Moreover, the solution satisfies the a priori estimate
\[
\sum_{p=0}^{m} \max_{\Delta} |(tD_t)^p u| \leq C \max_{\Delta} |f|,
\]
where \( \Delta \) is the closure of \( \Omega_{T_0}[\gamma_0] \) and \( C > 0 \) is some constant dependent on the above equation and on the domain \( \Omega_{T_0}[\gamma_0] \).

Note that \( f(t, z) \) and \( u(t, z) \) both have \( \Omega_{T_0}[\gamma_0] \) as their domain of definition. This fact allows us to restate our theorem in the following manner: for any \( T, \gamma > 0 \), let \( X_{T, \gamma} \) and \( Y_{T, \gamma} \) be the spaces \( \mathcal{K}_m(\Omega_T[\gamma]) \) and \( \mathcal{K}_0(\Omega_T[\gamma]) \), respectively. Let \( \mathcal{W}_{T, \gamma} \) be the subspace of \( X_{T, \gamma} \) consisting of functions \( u \in X_{T, \gamma} \) such that \( \mathcal{P}u \) belongs in \( Y_{T, \gamma} \). Define a linear operator \( \Psi \) from \( X_{T, \gamma} \) to \( Y_{T, \gamma} \) with domain \( \mathcal{W}_{T, \gamma} \) by \( \Psi u = \mathcal{P}u \). Let \( \| \cdot \|_{T, \gamma} \) denote the maximum norm in the closure of \( \Omega_T[\gamma] \). Then \( X_{T, \gamma} \) and \( Y_{T, \gamma} \) are Banach spaces; given \( u \in X_{T, \gamma} \) and \( f \in Y_{T, \gamma} \), we define their norms by \( \sum_{p=0}^{m} \| (tD_t)^p u \|_{T, \gamma} \) and \( \| f \|_{T, \gamma} \), respectively. Note further that the operator \( \Psi \) is a closed linear operator from \( X_{T, \gamma} \) to \( Y_{T, \gamma} \). The above theorem can now be stated as

**Theorem 2'.** There exist \( T_0, \gamma_0 > 0 \) depending on \( \mathcal{P} \) such that the operator \( \Psi \) is a one-one, closed linear operator from \( X_{T_0, \gamma_0} \) onto \( Y_{T_0, \gamma_0} \).

Since \( \Psi \) is an injection, \( \Psi^{-1} \) exists and is also closed. The Closed Graph Theorem further implies that \( \Psi^{-1} \) is continuous. The estimate given in (1.6) is just a consequence of the continuity of \( \Psi^{-1} \).

2 Preliminary Discussion

We can rewrite the operator \( \mathcal{P} \) as
\[
\mathcal{P} = Q + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j, \alpha}(t, z)(\mu(t)D_z)^\alpha(tD_t)^j,
\]
where the operator $Q$ is defined by
\[ Q = (tD_t)^m + a_{m-1,0}(0,z)(tD_t)^{m-1} + \cdots + a_{0,0}(0,z) \] (2.1)
and
\[ c_{j,\alpha}(t,z) = \begin{cases} \alpha_{j,\alpha}(t,z) & \text{if } |\alpha| \neq 0, \\ \alpha_{j,\alpha}(t,z) - \alpha_{j,\alpha}(0,z) & \text{if } |\alpha| = 0. \end{cases} \]

Note that the coefficients of $Q$ are holomorphic functions of $z$ in $B_R$. Note further that the characteristic exponents of $Q$ are the same as that of $P$, and hence satisfy (1.4).

**Lemma 1.** Fix $\tau > 0$ and let $g(t) \in C^0([0, \tau], A(\omega_{\tau}[\gamma]))$. Then the equation $Qu = g$ has a unique solution $u(t) \in C^0_m([0, \tau], A(\omega_{\tau}[\gamma]))$ given by
\[ u(t) = \sum_{\sigma \in S_m} \frac{1}{m!} \int_0^t \int_0^{s_{m-1}} \ldots \int_0^{s_1} g(s_{m}) \frac{d\theta}{\theta} \frac{ds_{m}}{s_{m}} \ldots \frac{ds_{1}}{s_{1}}. \] (2.2)

Here, $S_m$ is the group of permutations of $\{1, 2, \ldots, m\}$.

A result in symmetric entire functions asserts that $u(t,z)$ is holomorphic with respect to $z$. The fact that it belongs in $C^0_m([0, \gamma], A(\omega_{\tau}[\gamma]))$ is seen in the integral expression, but may actually be obtained a priori. (See [1].)

To facilitate computation, we define for $\lambda = (\lambda_1, \ldots, \lambda_m)$ the function
\[ G^t_\theta(\lambda) = \frac{1}{m!} \sum_{\sigma \in S_m} \left( \frac{s_{m}}{t} \right)^{-\lambda_{\sigma(m)}} \left( \frac{s_{m-1}}{s_{m}} \right)^{-\lambda_{\sigma(m-1)}} \ldots \left( \frac{\theta}{s_{2}} \right)^{-\lambda_{\sigma(1)}}, \] (2.3)
for some dummy variables $s_2, \ldots, s_m$. Define, too, the integral operator
\[ \int_{[t;\theta]}^{(m)} g = \frac{1}{m!} \sum_{\sigma \in S_m} \left( \frac{s_{m}}{t} \right)^{-\lambda_{\sigma(m)}} \left( \frac{s_{m-1}}{s_{m}} \right)^{-\lambda_{\sigma(m-1)}} \ldots \left( \frac{\theta}{s_{2}} \right)^{-\lambda_{\sigma(1)}} \] (2.4)

Using the above, we can now write the solution $u(t)$ of the equation $Qu = g$ as
\[ u(t) = \int_{[t;\theta]}^{(m)} G^t_\theta(\lambda) g. \]

In our proof of the main theorem, it will be necessary to consider the action of the differential operator $(tD_t)^p$ on integral expressions similar to the one in (2.2). One can easily verify the following

**Lemma 2.** Let $u(t)$ be the solution of $Qu = g$. Then for a natural number $p$ less than $m$, we have
\[ (tD_t)^p u = \sum_{i=m-p}^m \int_{[t;s_i]}^{(i)} \left\{ \frac{1}{m!} \sum_{\sigma \in S_m} h_i(\sigma, \lambda) \left( \frac{s_{i}}{t} \right)^{-\lambda_{\sigma(i)}} \times \left( \frac{s_{i-1}}{s_{i}} \right)^{-\lambda_{\sigma(i-1)}} \ldots \left( \frac{s_{1}}{s_{2}} \right)^{-\lambda_{\sigma(1)}} \right\}, \] (2.5)
where the functions $h_i(\sigma, \lambda)$ are suitable polynomial functions of the characteristic exponents $\lambda_1(z), \ldots, \lambda_m(z)$.

For brevity, let us set, for a natural number $k$,

$$H^t_\sigma(k, \lambda) = \frac{1}{m!} \sum_{\sigma \in S_m} h_k(\sigma, \lambda) \left( \frac{s_k}{t} \right)^{-\lambda_{\sigma(k)}} \left( \frac{s_{k-1}}{s_k} \right)^{-\lambda_{\sigma(k-1)}} \ldots \left( \frac{s_2}{s_1} \right)^{-\lambda_1}. \quad (2.6)$$

By symmetry, the functions $H^t_\sigma(k, \lambda)$ are holomorphic with respect to $z$ and thus belong in $A(B_R)$.

The next lemma is useful in evaluating some integral expressions in the proof.

**Lemma 3.** Let $k$ be natural number. Then the following equalities hold:

(a) $$\int_0^s \int_0^{s_1} \int_0^{s_k-1} \int_0^{s_k-2} \ldots \frac{ds_0}{s_0} \frac{ds_1}{s_1} \ldots \frac{ds_{k-1}}{s_{k-1}} = \frac{1}{L^k}$$

(b) $$\int_0^t \int_0^{s_k} \int_0^{s_1} \int_0^{s_k-1} \frac{\mu(s_k)}{s_k} \frac{\mu(s_{k-1})}{s_{k-1}} \ldots \frac{\mu(s_1)}{s_1} \times \left( \frac{s_0}{t} \right)^L \frac{s_0^{-1}}{[\varphi(t) - \varphi(s_0)]^k} ds_0 \ldots ds_k = \frac{1}{Lk!}$$

The first equality is obvious. The second can be proved by reversing the order of integration and recalling that $t \varphi'(t) = \mu(t)$.

To estimate the derivatives with respect to $z$, we have the following lemma. (For a proof, see Hörmander [3], Lemma 5.1.3.)

**Lemma 4.** Let the function $v(z)$ be holomorphic in $B_R$, and suppose there are positive constants $K$ and $c$ such that

$$\|v\|_{\rho} \leq \frac{K}{(R-\rho)^c} \quad \text{for every } \rho \in (0,R). \quad (2.7)$$

Then we have

$$\|D_z^\alpha v\|_\rho \leq \frac{Ke^{\tau \alpha}(c+1)^{\alpha}}{(R-\rho)^{c+\alpha}} \quad \text{for every } \rho \in (0,R). \quad (2.8)$$

In the above, we define $(c)_p = (c)(c+1) \ldots (c+p-1)$.

## 3 Proof of Main Theorem

Let $f$ be any element of $K_0(\Omega_{T_0}[\gamma_0])$. Here, the constants $T_0 > 0$ and $\gamma_0 > 0$ satisfy some conditions which will later be specified. For convenience, we will drop the subscript in both and instead use $T$ and $\gamma$; we will again use the subscript upon stating the conditions that these constants need to satisfy.
We will use the method of successive approximations to solve the equation \( Pu = f \). Define the approximate solutions as follows:

\[
    u_0(t) = \int_{[t:s]}^{(m)} G^t_s(\lambda) f
\]

and for \( k \geq 1 \),

\[
    u_k(t) = \int_{[t:s]}^{(m)} G^t_s(\lambda) \left[ f - S(s)u_{k-1} \right].
\]

Here, \( t \in [0, T] \), and for brevity, we have set \( S(t) = \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} c_{j,\alpha}(t, z) \). Note that for all \( k \), the approximate solutions \( u_k(t, z) \) are defined on \( \Omega_{T_0}[\gamma_0] \). Furthermore, they are continuous with respect to \( t \) and holomorphic with respect to \( z \) on this region.

For each \( k \), we also define the sequence of functions \( v_k(t) = u_k(t) - u_{k-1}(t) \), with \( u_{-1} \equiv 0 \). Then the \( v_k(t, z) \)'s are also defined on the same region as the \( u_k(t, z) \)'s, and are also continuous with respect to \( t \) and holomorphic with respect to \( z \). Using the expression for \( u_k(t) \), we have

\[
    v_0(t) = \int_{[t:s]}^{(m)} G^t_s(\lambda) f
\]

and for \( k \geq 1 \),

\[
    v_k(t) = -\int_{[t:s]}^{(m)} G^t_s(\lambda) S(s)v_{k-1}.
\]

To prove that the approximate solutions converge to the real solution, we will henceforth fix one \( t \in [0, T] \), and estimate the functions \( v_k(t) \). Let \( C \) be the bound on \( [0, T] \times \overline{B}_R \) of all \( c_{j,\alpha}(t, z) \), and \( K \) be the bound in \( \overline{\Omega_T[\gamma]} \) of \( f(t, z) \). As \( G^t_s(\lambda) \) and \( H^t_s(k, \lambda) \), we have for \( 1 \leq k \leq m \) and for some \( D > 0 \):

\[
    \sup_{z \in \overline{B}_R} |G^t_s(\lambda)| \leq \left( \frac{s}{t} \right)^L \quad \text{and} \quad \sup_{z \in \overline{B}_R} |H^t_s(k, \lambda)| \leq D \left( \frac{s}{t} \right)^L.
\]

We can easily see that \( ||v_0(t)||_{\omega_t} \) is bounded by \( KL^{-m} \) for any \( 0 \leq t \leq T \). Here, we have written for convenience \( ||\cdot||_{\omega_t} \) in place of \( ||\cdot||_{\omega_t[\gamma]} \). For general \( k \), we note that \( u_k(t) \) is given by the iterated integral

\[
    v_k(t) = (-1)^k \int_{[t:s_k]}^{(m)} G^t_{s_k}(\lambda) S(s_k) \int_{[s_k:s_{k-1}]}^{(m)} G^t_{s_{k-1}}(\lambda) S(s_{k-1}) \cdots \int_{[s_2:s_1]}^{(m)} G^t_{s_1}(\lambda) S(s_1) \int_{[s_1:s_0]}^{(m)} G^t_{s_0}(\lambda) f(s_0).
\]

The expression above can be expanded using Lemma 2, and thus obtain a finite sum whose number of terms is less than \( (mJ)^k \), where \( J \) is the cardinality of the set \( \{(j, \alpha); 0 \leq j \leq m - 1 \text{ and } |\alpha| \leq m - j\} \). Each term of the finite sum...
has the form

\[
I = (-1)^k \int_{[t;s_k]}^{(m)} G_{s_k}^t (\lambda) c_{j_k,\alpha_k} (\mu D_z)_{\alpha_k} \int_{[s_k;\alpha_k^{-1}]}^{(i_k)} H_{s_k-1}^{s_k} (i_k, \lambda) c_{j_k-1,\alpha_k-1} (\mu D_z)_{\alpha_k-1} \\
\cdots \int_{[s_2;s_1]}^{(i_2)} H_{s_1}^{s_2} (i_2, \lambda) c_{j_1,\alpha_1} (\mu D_z)_{\alpha_1} \int_{[s_1;\epsilon_0]}^{(i_1)} H_{\epsilon_0}^{s_1} (i_1, \lambda) f(s_0),
\]

where for each \( p \), the relations \( m - j_p \leq i_p \leq m \) and \( |\alpha_p| \leq m - j_p \) hold. (Here, \( \alpha_p \) is a multi-index and should not be confused with the \( p \)th component of \( \alpha \).)

The above is further equal to

\[
I = (-1)^k \int_{[t;s_k]}^{(m)} \cdots \int_{[s_1;\epsilon_0]}^{(i_1)} G_{s_k}^t c_{j_k,\alpha_k} (\mu(s_k) D_z)_{\alpha_k} \\
\times H_{s_k-1}^{s_k} c_{j_k-1,\alpha_k-1} (s_k-1) (\mu(s_k-1) D_z)_{\alpha_k-1} \cdots \\
\times H_{\epsilon_0}^{s_1} c_{j_1,\alpha_1} (s_1) (\mu(s_1) D_z)_{\alpha_1} H_{\epsilon_0}^{s_1} f(s_0).
\]

Let \( F_k(s) \) denote the integrand of the above integral. Let \( R_{s_0} = R - \gamma \varphi(s_0) \). Then all the functions above, when viewed as a function of \( z \), belong in \( \mathcal{A}(\omega_{s_0}[\gamma]) \).

We can therefore apply Lemma 4 repeatedly, starting from the rightmost expression, to obtain the following estimate: for any \( \rho \in (0, R_{s_0}) \), we have

\[
\|F_k(s)\|_{B_\rho} \leq K(CD)^k (\mu(s_1)^{|\alpha_1|} \cdots \mu(s_k)^{|\alpha_k|} \left( \frac{s_0}{t} \right)^L \times \\
\left( \frac{e}{R_{s_0} - \rho} \right)^{|\alpha_1 + \cdots + \alpha_k|} |\alpha_1 + \cdots + \alpha_k|!.
\]

If \( |\alpha_1 + \cdots + \alpha_k| = 0 \), then for sufficiently small \( T = T_0 \), the bound for any \( c_{j_0}(t, z) = a_{j,0}(t, z) - a_{j,0}(0, z) \) is actually small, since \( a_{j,0}(t, z) \) is continuous with respect to \( t \). In other words, by choosing a small \( T = T_0 \), we could find a small constant \( \delta \) such that for any \( t \in [0, T_0] \) and \( 0 \leq s \leq t \), the following holds:

\[
\|F_k(s)\|_{\omega_t} \leq K \delta^k \left( \frac{s_0}{t} \right)^L.
\]

Going back to the integral, we have

\[
\|I\|_{\omega_t} \leq \int_{[t;s_k]}^{(m)} \cdots \int_{[s_1;\epsilon_0]}^{(i_1)} K \delta^k \left( \frac{s_0}{t} \right)^L \\
= K \frac{\delta^k}{L^{m+i_1+\cdots+i_k}} \leq K \left( \frac{\delta}{L_0} \right)^k,
\]

for some constant \( L_0 \) dependent on \( L \). This is possible since \( i_p \leq m \) for all \( p \).
If $|\alpha_1 + \cdots + \alpha_k| \neq 0$, set the $\rho$ in (3.8) to be equal to $R - \gamma \varphi(t)$. This gives

$$\|F_k(s)\|_{\omega_t} \leq (CD)^k \mu(s_1)^{\alpha_1} \cdots \mu(s_k)^{\alpha_k} \left(\frac{s_0}{t}\right)^L \times |\alpha_1 + \cdots + \alpha_k| ! \left(\frac{e}{\gamma [\varphi(t) - \varphi(s_0)]}\right)^{|\alpha_1 + \cdots + \alpha_k|}. \quad (3.11)$$

By renaming if necessary, assume that for $p = 1, \ldots, q$, we have $|\alpha_p| \neq 0$. Note that $q \geq 1$. We will again use the continuity of $a_{j,0}(t, z)$ to estimate those expressions which are not acted upon by $D_z$, i.e., the $k - q$ cases when $|\alpha_p| = 0$. Just like before, we can show that for small $\delta$,

$$\|F_k(s)\|_{\omega_t} \leq (CD)^q \delta^{k-q} \mu(s_1)^{\alpha_1} \cdots \mu(s_q)^{\alpha_q} \left(\frac{s_0}{t}\right)^L \times |\alpha_1 + \cdots + \alpha_q| ! \left(\frac{e}{\gamma [\varphi(t) - \varphi(s_0)]}\right)^{|\alpha_1 + \cdots + \alpha_q|}. \quad (3.12)$$

Thus, the integral $I$ can now be estimated as follows:

$$\|I\|_{\omega_t} \leq (CD)^q \delta^{k-q} \left(\frac{e}{\gamma}\right)^{|\alpha_1 + \cdots + \alpha_q|} |\alpha_1 + \cdots + \alpha_q| ! \times \int_{[t:s_1]}^{(m)} \int_{[s_1:s_0]}^{(i_1)} \cdots \int_{[s_k:s_{k-1}]}^{(i_k)} \frac{\mu(s_1)^{\alpha_1} \cdots \mu(s_k)^{\alpha_k}}{[\varphi(t) - \varphi(s_0)]^{\alpha_1 + \cdots + \alpha_k}} \left(\frac{s_0}{t}\right)^L \frac{1}{\xi_0^b} d\xi_0 \cdots d\xi_k \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_k}{\xi_k} \left(\frac{\xi_0}{t}\right)^L \frac{d\xi_0}{\xi_0} \cdots \frac{d\eta_1}{\eta_1} d\xi_0 \cdots d\xi_k \cdot \quad (3.13)$$

Let $d = m + i_1 + \cdots + i_k$ and $b = |\alpha_1 + \cdots + \alpha_q|$. Note that $b \geq q$. Since for each $p$, we have $|\alpha_p| \leq m - j_p \leq i_p$, and using the fact that both $\varphi(t)$ and $\mu(t)$ are increasing on $(0, T_0)$, we have

$$\|I\|_{\omega_t} \leq (CD)^q \delta^{k-q} \left(\frac{e}{\gamma}\right)^b b! \times \int_0^t \int_0^{\xi_0} \cdots \int_0^{\eta_d-b-2} \left(\frac{s_0}{\xi_0}\right)^L \frac{d\xi_0}{\xi_0} \cdots \frac{d\eta_1}{\eta_1} \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_k}{\xi_k} \left(\frac{\xi_0}{s_0}\right)^{L-2} \frac{1}{b!} d\xi_0 \cdots d\xi_k \cdot \quad (3.14)$$

By (a) of Lemma 3, the second integral is equal to $L^{-d+b+1}$. Thus, the above simplifies into

$$\|I\|_{\omega_t} \leq (CD)^q \delta^{k-q} \left(\frac{e}{\gamma}\right)^b L^{-d+b+1} b! \times \int_0^t \int_0^{\xi_0} \cdots \int_0^{\eta_d-b-2} \left(\frac{s_0}{\xi_0}\right)^L \frac{d\xi_0}{\xi_0} \cdots \frac{d\eta_1}{\eta_1} \frac{d\xi_1}{\xi_1} \cdots \frac{d\xi_k}{\xi_k}. \quad (3.15)$$

The last integral is equal to $(Lb!)^{-1}$, by (b) of Lemma 3. Meanwhile, since $d \leq m(k+1)$, we can find a constant $L_1$, depending on $L$, such that $L^{-d} \leq L_1^d$. 


Substituting these results into the above equation, we get

\[ ||I||_{\omega_{t}} \leq K(CD)^{q} \delta^{k-q} \left( \frac{eL}{\gamma} \right)^{b} L_{t}^{k} = K \left( \frac{CD}{\delta} \right)^{q} \delta^{k-q} \left( \frac{eL}{\gamma} \right)^{b}. \]  

(3.16)

By taking a sufficiently small \( T_{0} \), we can find a \( \delta \) small enough such that \( \delta L_{1} \) above and \( \delta L_{0}^{-1} \) in (3.10) are both less than \( (mJ)^{-1} \). Now, since \( q \leq b \), we can make the remaining expression less than one by choosing a large \( \gamma = \gamma_{0} \).

To summarize, we have shown that if \( T_{0} \) is sufficiently small and \( \gamma_{0} \) is sufficiently large, some constants \( K > 0 \) and \( \delta_{0} < 1 \) exist such that for all \( k \), we have

\[ ||v_{k}(t)||_{\omega_{t}[\gamma_{0}]} \leq K\delta_{0}^{k} \quad \text{for any } t \in [0, T_{0}]. \]  

(3.17)

It follows that the series \( \sum_{k=0}^{\infty} v_{k}(t, z) \) is majorized by a convergent geometric series, and hence is itself convergent in \( C^{0}([0, \tau], \mathcal{A}(\omega_{t}[\gamma_{0}])) \) for all \( \tau \in [0, T_{0}] \). This means that \( v_{k}(t) \) converges uniformly to \( u(t) \) on \( \Omega_{T_{0}}[\gamma_{0}] \).

By following the steps above, we can also show that for \( 1 \leq p \leq m-1 \), the sequence \( (tD_{t})^{p}u_{k}(t) \) converges uniformly to \( (tD_{t})^{p}u(t) \) on \( \Omega_{T_{0}}[\gamma_{0}] \). Thus, it follows that on a compact subset of \( \Omega_{T_{0}}[\gamma_{0}] \), the sequence \( D_{z}^{\alpha}(tD_{t})^{p}u_{k}(t) \) converges to \( D_{z}^{\alpha}(tD_{t})^{p}u(t) \). This implies the convergence of the approximate solutions to the true solution \( u(t) \).

Uniqueness may be proved in a similar manner.

References


