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On the Singular Solutions of Nonlinear Singular Partial Differential Equations (Asymptotic Analysis and Microlocal Analysis of PDE)

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On the Singular Solutions of Nonlinear Singular Partial Differential Equations

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Abstract

Let us consider the following nonlinear singular partial differential equation:

\[(t \partial_t)^m u = F(t, x, \{(t \partial_t)^j \partial_x^\alpha u\}_{j+|\alpha| \leq m, j < m})\]

in the complex domain. Denote by \(S_+\) [resp. \(S_{\log}\)] the set of all the solutions \(u(t, x)\) with asymptotics \(u(t, x) = O(|t|^a)\) [resp. \(u(t, x) = O(1/|\log t|^a)\)] (as \(t \to 0\) uniformly in \(x\)) for some \(a > 0\). Clearly \(S_{\log} \supset S_+\). The paper gives a sufficient condition for \(S_{\log} = S_+\) to be valid.

The paper deals with nonlinear singular partial differential equations of the form

\[(t \partial/\partial t)^m u = F(t, x, \{(t \partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j+|\alpha| \leq m, j < m})\]

in the complex domain. In Gérard-Tahara [1] the author has determined all the singular solutions \(u(t, x)\) of (E) under the condition that \(u(t, x) = O(|t|^a)\) (as \(t \to 0\) uniformly in \(x\)) for some \(a > 0\).

The present paper investigates singular solutions \(u(t, x)\) of (E) under a weaker condition that \(u(t, x) = O(1/|\log t|^a)\) (as \(t \to 0\) uniformly in \(x\)) for some \(a > 0\).

§1. Equations.

Notations: \(t \in C, x = (x_1, \ldots, x_n) \in C^n, N = \{0, 1, 2, \ldots\}, \) and \(N^* = \{1, 2, \ldots\}\).

For \(\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n\) we write \(|\alpha| = \alpha_1 + \cdots + \alpha_n\) and

\[\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.\]

Let \(m \in N^*, N = \#\{(j, \alpha) \in N \times N^n; j + |\alpha| \leq m, j < m\}, \) and write the variable \(Z\) as

\[Z = \{Z_{j, \alpha}\}_{j + |\alpha| \leq m, j < m} \in C^N.\]
Let $F(t, x, Z)$ be a function in the variables $(t, x, Z)$ defined in a neighborhood of the origin $(0, 0, 0) \in C_t \times C_x^m \times C_Z^N$, and assume the following:

(A1) $F(t, x, Z)$ is holomorphic near $(0, 0, 0)$;
(A2) $F(0, x, 0) \equiv 0$ near $x = 0$;
(A3) $\frac{\partial F}{\partial Z_{j,\alpha}}(0, x, 0) \equiv 0$ near $x = 0$, if $|\alpha| > 0$.

In this paper we always assume the conditions (A1), (A2), (A3), and we will consider the following nonlinear partial differential equation

\[(E) \quad \left(t \frac{\partial}{\partial t}\right)^m u = F(t, x, \left\{ \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u \right\}_{j<m}^{j+|\alpha| \leq m})\]

with $u = u(t, x)$ as an unknown function.

For (E) we set

\[C(\lambda, x) = \lambda^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0, x, 0)\lambda^j\]

and denote by $\lambda_1(x), \ldots, \lambda_m(x)$ the roots of the equation $C(\lambda, x) = 0$ in $\lambda$. These $\lambda_1(x), \ldots, \lambda_m(x)$ are called the characteristic exponents of (E).

The following is our basic problem:

**Problem.** Determine all kinds of local singularities which appear in the solutions of (E).

§2. Gérard-Tahara (1993)

Let us recall the result in Gérard-Tahara [1]. Denote:

- $R(C \setminus \{0\})$ denotes the universal covering space of $C \setminus \{0\}$;
- $S_\theta = \{t \in R(C \setminus \{0\}); \ |\arg t| < \theta\}$;
- $S(\epsilon(s)) = \{t \in R(C \setminus \{0\}); \ 0 < |t| < \epsilon(\arg t)\}$, where $\epsilon(s)$ is a positive-valued continuous function on $R_s$;
- $D_r = \{x \in C^n; \ |x| \leq r\}$;
- $C\{x\}$ denotes the ring of convergent power series in $x$, or equivalently the ring of germs of holomorphic functions at the origin of $C^n$. 
Definition 1. We denote by $\tilde{\mathcal{O}}_+$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):

i) $u(t,x)$ is a holomorphic function on $S(\epsilon(s)) \times D_r$ for some positive-valued continuous function $\epsilon(s)$ and some $r > 0$;

ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x| \leq r} |u(t,x)| = O(|t|^a) \quad (\text{as } t \to 0 \text{ in } S_\theta).$$

For the characteristic exponents $\lambda_1(x), \ldots, \lambda_m(x)$, we set

$$\mu = \# \{ i ; \Re \lambda_i(0) > 0 \}.$$ 

When $\mu = 0$, this is equivalent to the fact that $\Re \lambda_i(0) \leq 0$ for all $i = 1, \ldots, m$. When $\mu \geq 1$, by a renumeration we may assume

\begin{equation}
\begin{cases}
\Re \lambda_i(0) > 0 & \text{for } 1 \leq i \leq \mu, \\
\Re \lambda_i(0) \leq 0 & \text{for } \mu + 1 \leq i \leq m.
\end{cases}
\end{equation}

Then we already have:

Theorem 1 (Gérard-Tahara [1]). Denote by $S_+$ the set of all $\tilde{\mathcal{O}}_+$-solutions of (E). Then we have:

I) When $\mu = 0$, we have $S_+ = \{u_0\}$ where $u_0 = u_0(t,x)$ is the unique holomorphic solution of (E) satisfying $u_0(0,x) \equiv 0$.

II) When $\mu \geq 1$, under (1.1) and the following additional conditions

1) $\lambda_i(0) \neq \lambda_j(0)$ for $1 \leq i \neq j \leq \mu$ ,

2) $C(1,0) \neq 0$ ,

3) $C(i + j_1\lambda_1(0) + \cdots + j_\mu\lambda_\mu(0), 0) \neq 0$ for any $(i, j) \in N \times N^\mu$ satisfying $i + |j| \geq 2$ (where $j = (j_1, \ldots, j_\mu)$),

we have

$$S_+ = \{U(\phi_1, \ldots, \phi_\mu) ; (\phi_1, \ldots, \phi_\mu) \in (C\{x\})^\mu \},$$

where $U(\phi_1, \ldots, \phi_\mu)$ is an $\tilde{\mathcal{O}}_+$-solution of (E) determined by $(\phi_1, \ldots, \phi_\mu) \in (C\{x\})^\mu$ and having the expansion of the following form:

$$U(\phi_1, \ldots, \phi_\mu) = \sum_{i \geq 1} u_i(x) t^i$$

$$+ \phi_1(x) t^{\lambda_1(x)} + \cdots + \phi_\mu(x) t^{\lambda_\mu(x)}$$

$$+ \sum_{i+2m|j| \geq k+2m} \phi_{i,j,k}(x) t^{i+j_1\lambda_1(x)+\cdots+j_\mu\lambda_\mu(x)} (\log t)^k.$$
§3. Problems.

In Theorem 1 we have restricted ourselves to the study of singular solutions in $\tilde{O}_+$. But, there seems to be a possibility that (E) has singular solutions which do not belong in the class $\tilde{O}_+$, as is seen in the following example.

Example 1. The equation

$$t \frac{\partial u}{\partial t} = u \left( \frac{\partial u}{\partial x} \right)^k$$

(where $(t, x) \in C^2$ and $k \in N^*$) has a family of singular solutions

$$u(t, x) = \left( \frac{1}{k} \right)^{1/k} \frac{x + \alpha}{(c - \log t)^{1/k}}, \quad \alpha, c \in C,$$

which do not belong in the class $\tilde{O}_+$.

In order to include this kind of singular solutions in our framework, we introduce the following new class of singular solutions:

Definition 2. We denote by $\tilde{O}_{\log}$ the set of all $u(t, x)$ satisfying the following conditions i) and ii):

i) $u(t, x)$ is a holomorphic function on $S(\epsilon(s)) \times D_r$ for some positive-valued continuous function $\epsilon(s)$ and some $r > 0$;

ii) there is an $\alpha > 0$ such that for any $\theta > 0$ we have

$$\max_{|x| \leq r} |u(t, x)| = O \left( \frac{1}{|\log t|^\alpha} \right) \quad \text{(as } t \to 0 \text{ in } S_\theta \text{).}$$

Clearly we have $\tilde{O}_{\log} \supset \tilde{O}_+$. Therefore, if we denote by $S_{\log}$ the set of all $\tilde{O}_{\log}$-solutions of (E), we have $S_{\log} \supset S_+$.

We will say that $u(t, x)$ is a solution with temperate singularities if $u(t, x) \in S_+$, and that $u(t, x)$ is a solution with logarithmic singularities if $u(t, x) \in S_{\log} \setminus S_+$.

Our next problems can be set up as follows:

Problem 1. When does $S_{\log} = S_+$ hold?

Problem 2. When does $S_{\log} \neq S_+$ hold?

Note that the problem 1 asserts that new singular solutions do not appear and that the problem 2 asserts that new singular solutions really appear in the solutions of (E).

In this paper we will give a partial answer and a conjecture on the problem 1. The problem 2 will be discussed in the forthcoming paper.
§4. A result and a conjecture.

In this section we will give a result on the problem 1 in a general form. A function $\mu(t)$ on $(0,T)$ is called a weight function if it satisfies the following conditions $\mu_1) \sim \mu_3)$:

1) $\mu(t) \in C^0((0,T))$,
2) $\mu(t) > 0$ on $(0,T)$ and $\mu(t)$ is increasing in $t$,
3) $\int_0^T \frac{\mu(s)}{s} ds < \infty$.

By $\mu_2)$ and $\mu_3)$ the condition $\mu(t) \rightarrow 0$ (as $t \rightarrow +0$) is clear. In this paper we impose the additional condition on $\mu(t)$:

\begin{equation}
\mu(t) \in C^1((0,T)) \quad \text{and} \quad \left( t \frac{d\mu}{dt} \right)(t) = o(\mu(t)) \quad \text{(as $t \rightarrow +0$)}.
\end{equation}

The following functions are typical examples:

$$\mu(t) = \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $b > 1$, $c > 1$. Note that the function $\mu(t) = t^d$ with $d > 0$ does not satisfy the condition (4.1).

**Definition 3.** Let $\mu(t)$ be a weight function.

1. For $a > 0$ we denote by $\tilde{O}_a(\mu(t))$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):
   i) $u(t,x)$ is a holomorphic function on $S(\epsilon(s)) \times D_r$ for some positive-valued continuous function $\epsilon(s)$ and some $r > 0$;
   ii) for any $\theta > 0$ we have
   $$\max_{|x| \leq r} |u(t,x)| = O(\mu(|t|)^a) \quad \text{(as $t \rightarrow 0$ in $S_\theta$)}.$$

2. We define $\tilde{O}_+(\mu(t))$ by
   $$\tilde{O}_+(\mu(t)) = \bigcup_{a > 0} \tilde{O}_a(\mu(t)),$$

**Lemma 1.** 1) $\tilde{O}_{\log} = \tilde{O}_+(\mu(t))$ if $\mu(t) = 1/(-\log t)^b$ with $b > 1$.
2) If $\mu(t)$ satisfies (4.1) we have $\tilde{O}_+ \subset \tilde{O}_1(\mu(t)) \subset \tilde{O}_+(\mu(t))$.

**Proof.** (1) is clear. (2) is verified as follows. By (4.1), for any $\epsilon > 0$ there is a $\delta > 0$ such that $t\mu(t) \leq \epsilon \mu(t)$ holds on $(0,\delta]$ and therefore we have

$$\frac{d}{dt}(t^{-\epsilon}\mu(t)) \leq 0 \quad \text{for} \quad 0 < t \leq \delta.$$
Integrating this from $t$ to $\delta$ we have

$$\delta^{-\epsilon}\mu(\delta) \leq t^{-\epsilon}\mu(t) \quad \text{for } 0 < t \leq \delta$$

and so

(4.2)  
$$\left(\frac{\mu(\delta)}{\delta^\epsilon}\right)t^{\epsilon} \leq \mu(t) \quad \text{for } 0 < t \leq \delta.$$  

Since $\epsilon > 0$ is arbitrary, (4.2) leads us to the conclusion of (2). \qed

Denote by $S_+ (\mu(t))$ (resp. $S_a (\mu(t))$) the set of all $\tilde{O}_+ (\mu(t))$-solutions of (E) (resp. $\tilde{O}_a (\mu(t))$-solutions of (E)). By (2) of Lemma 1 we have

$$S_+ \subset S_1 (\mu(t)) \subset S_+ (\mu(t)).$$

The following theorem gives a sufficient condition for $S_+ (\mu(t)) = S_+$ to be valid.

**Theorem 2.** Let $\mu(t)$ be a weight function satisfying (4.1). Then, $S_+ (\mu(t)) = S_+$ is valid if

(4.3)  
$$\mathrm{Re} \lambda_i(0) < 0 \quad \text{for all } i = 1, \ldots, m$$

or if

(4.4)  
$$\mathrm{Re} \lambda_i(0) > 0 \quad \text{for all } i = 1, \ldots, m.$$  

In the case (4.3), by Theorem 1 we have $S_+ = \{ u_0 \}$ and therefore the condition $S_+ (\mu(t)) = S_+$ is equivalent to the fact that the local uniqueness of the solution is valid in $S_+ (\mu(t))$ which is already proved in Tahara [4],[5].

In the case (4.4) the proof of Theorem 2 consists of the following two parts:

$C_1$) if $u \in S_+ (\mu(t))$ we have $u \in S_m (\mu(t))$;

$C_2$) if $u \in S_m (\mu(t))$ we have $u \in S_+$.

The proofs of these $C_1$) and $C_2$) will be published in Tahara [6].

**Corollary.** If (4.3) or (4.4) holds, we have $S_{\log} = S_+$.  

**Remark.** The author believes that the following conjecture is true, though at present he has no idea to prove this conjecture:

**Conjecture.** $S_{\log} = S_+$ is valid if

(4.5)  
$$\mathrm{Re} \lambda_i(0) \neq 0 \quad \text{for all } i = 1, \ldots, m.$$
References


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