Existence of singular solutions with bounds of linear partial differential equations in the complex domain (Asymptotic Analysis and Microlocal Analysis of PDE)

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Existence of singular solutions with bounds of linear partial differential equations in the complex domain

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§0 Introduction

In this paper we consider a linear partial differential equation in the complex domain \( \mathbb{C}^{d+1} \), \( L(z, \partial)u(z) = f(z) \). \( L(z, \partial_z) \) is an \( m \)-th linear partial differential operator with coefficients are holomorphic in a neighborhood \( U \) of \( z = 0 \) in \( \mathbb{C}^{d+1} \). The inhomogeneous term \( f(z) \) has singularities on a complex hypersurface \( K \). The author reported the results concerning the growth properties and the asymptotic behaviors of solutions, and those concerning the existence of solutions with asymptotic expansion, when \( f(z) \) has an asymptotic expansion, at the conference held here, RIMS of Kyoto Univ. (see [5], [6], [7] and [8]). In the present paper our concern is the existence of solutions, when \( f(z) \) has not necessary asymptotic expansion, but the singularities are tempered, that is, singularities are of the fractional order. The details will be given elsewhere.

§1 Notations and Definitions

In order to state our problem and results more precisely, let us introduce notations, function spaces and characteristic polygon.

1.1. Notations. \( z = (z_0, z_1, \cdots, z_d) = (z_0, z') \in \mathbb{C} \times \mathbb{C}^d \). \( |z| = \max\{|z_i|; 0 \leq i \leq d\} \) and \( |z'| = \max\{|z_i|; 1 \leq i \leq d\} \). Its dual variables are \( \xi = (\xi_0, \xi') = (\xi_0, \xi_1, \cdots, \xi_d) \). \( \partial_i = \partial/\partial z_i \), and \( \partial = (\partial_0, \partial_1, \cdots, \partial_d) = (\partial_0, \partial') \). \( \mathbb{Z} \) is the set of all integers and \( \mathbb{N} \) is the set of all nonnegative integers. For a multi-index \( \alpha = (\alpha_0, \alpha') \in \mathbb{N} \times \mathbb{N}^d \), \( |\alpha| = \alpha_0 + |\alpha'| = \sum_{i=0}^{d} \alpha_i \). For a polydisk \( U = U_0 \times U' \) in \( \mathbb{C}^{d+1} \), where \( U_0 = \{z_0 \in \mathbb{C}; |z_0| < R_0\} \) and \( U' = \{z \in \mathbb{C}; |z'| < R\} \), set \( U_0(\theta) = \{z_0 \in U_0 - \{0\}; |\arg z_0| < \theta\} \) and \( U(\theta) = U_0(\theta) \times U' \). \( U(\theta) \) is a sectorial region with respect to \( z_0 \). \( K \) is a complex hypersurface through \( z = 0 \) in \( U \). We choose the coordinate so that \( \{z \in U; z_0 = 0\} \)

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1.2. Function spaces. For a region $U$ in $\mathbb{C}^n$, $\mathcal{O}(U)$ is the set of all holomorphic functions on $U$.

**Definition 1.1.** (i). $\mathcal{O}_{\text{temp},c}(U(\theta))$ is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any $\theta'$ with $0 < \theta' < \theta$

\[ |u(z)| \leq C|z_0|^c \quad z \in \Omega(\theta') \]

holds for a constant $C = C(\theta')$.

(ii). $\mathcal{O}_{\text{temp}}(U(\theta)) = \bigcup_{c \in \mathbb{R}} \mathcal{O}_{\text{temp},c}(U(\theta))$. We say that $u(z) \in \mathcal{O}_{\text{temp}}(U(\theta))$ is tempered singular, or regular singular, on $K$ in $U(\theta)$.

**Definition 1.2.** $A_{\{\kappa\}}(U(\theta))(0 < \kappa \leq +\infty)$ is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any $\theta'$ with $0 < \theta' < \theta$

\[ |\partial_0^N u(z)| \leq AB^N \Gamma(N(1 + \frac{1}{\kappa}) + 1) \quad \text{for } z \in U(\theta') \]

holds for all $n \in \mathbb{N}$ and for some constants $A = A(\theta')$ and $B = B(\theta')$.

$u(z) \in A_{\{+\infty\}}(U(\theta))$ means that $u(z)$ is holomorphic at $z = 0$. $A_{\{\kappa\}}(U(\theta))$ is coincident with $\text{Asy}_{\{\kappa\}}(U(\theta))$ in the preceding papers [5],[7] and [8], which consists of all $u(z) \in \mathcal{O}(U(\theta))$ with asymptotic expansion of Gevrey type, that is, for any $\theta'$ with $0 < \theta' < \theta$

\[ |u(z) - \sum_{n=0}^{N-1} u_n(z') z_0^n| \leq AB^N|z_0|^N \Gamma\left(\frac{N}{\kappa} + 1\right) \quad \text{for } z \in U(\theta'), \]

where $u_n(z') \in \mathcal{O}(U')$ ($n \in \mathbb{N}$), $A = A(\theta')$ and $B = B(\theta')$. The notation $u(z) \sim 0$ in $A_{\{\kappa\}}(U(\theta))$ means that $u_n(z') \equiv 0$ for all $n$ in (1.3).

1.3. Characteristic polygon. Let $L(z, \partial)$ be an $m$-th order linear partial differential operator with holomorphic coefficients in a neighborhood of $z = 0$,

\[ L(z, \partial) = \sum_{|\alpha| \leq m} a_\alpha(z) \partial^\alpha. \]

We introduce the characteristic polygon of $L(z, \partial)$ with respect to hypersurface $K = \{z_0 = 0\}$, which is indispensable for our purpose, to study the existence of solutions with bounds. Let us introduce a notation $\underline{J}(a, b) := \{(x, y) \in \mathbb{R}^2; x \leq a, y \geq b\}$, which means an infinite rectangle. Let $j_\alpha$ be the
valuation of \( a_\alpha(z) \) with respect to \( z_0 \), that is, if \( a_\alpha(z) \neq 0 \), \( a_\alpha(z) = z_0^{i_\alpha} b_\alpha(z) \) with \( b_\alpha(0, z') \neq 0 \) and set \( j_\alpha = \infty \) for \( a_\alpha(z) \equiv 0 \). Define

\[
e_{L, \alpha} = j_\alpha - \alpha_0,
\]

where \( e_{L, \alpha} = +\infty \) if \( a_\alpha(z) \equiv 0 \).

The characteristic polygon of \( \Sigma \) is defined by

\[
\Sigma := \text{the convex hull of } \bigcup_{\alpha} \{1, e(\alpha) \in \mathbb{R}^2 \mid 0 \leq i \leq p^* - 1\}.
\]

The boundary of \( \Sigma \) consists of a vertical half line \( \Sigma(0) \) and a horizontal half line \( \Sigma(p^*) \) and \( p^* - 1 \) segments \( \Sigma(i) \) \((1 \leq i \leq p^* - 1)\) with slope \( \gamma_i \), \( 0 = \gamma_{p^*} < \gamma_{p^* - 1} < \cdots < \gamma_1 < \gamma_0 = +\infty \).

Let \( \{ (m_i, e(i)) \in \mathbb{R}^2 \mid 0 \leq i \leq p^* - 1\} \) be vertices of \( \Sigma \), where \( 0 \leq m_{p^* - 1} < \cdots < m_i < m_{i-1} < \cdots < m_0 = m \). So the endpoints of \( \Sigma(i) \) \((1 \leq i \leq p^* - 1)\) are \((m_i-1, e(i-1))\) and \((m_i, e(i))\). We call the slope \( \gamma_i \) of \( \Sigma(i) \) the \( i \)-th characteristic index of \( L(z, \partial) \) with respect to \( K = \{ z_0 = 0 \} \).

![Figure 1: Characteristic polygon](image-url)
Let \( \Delta(i) \) be a subset of multi-indices and \( l_i \in \mathbb{N} \) \((0 \leq i \leq p^* - 1)\) defined by

\[
\begin{align*}
\Delta(i) := & \{ \alpha \in \mathbb{N}^{d+1}; |\alpha| = m_i, e_{L,\alpha} = e(i) \}, \\
l_i := & \max\{|\alpha'| : \alpha \in \Delta(i)\}.
\end{align*}
\]

(1.6)

Define a subset \( \Delta_0(i) \) of \( \Delta(i) \) and a polynomial \( \chi_{L,i}(z', \xi') \) in \( \xi' \) \((0 \leq i \leq p^* - 1)\) by

\[
\begin{align*}
\Delta_0(i) = & \{ \alpha \in \Delta(i); |\alpha'| = l_i \}, \\
\chi_{L,i}(z', \xi') = & \sum_{\alpha \in \Delta_0(i)} b_{\alpha}(0, z') \xi'^{\alpha}.
\end{align*}
\]

(1.7)

\( \chi_{L,i}(z', \xi') \) is homogeneous in \( \xi' \) with degree \( l_i \).

§2 Existence of singular solutions

Let us return to the equation

\[
(Eq) \quad L(z, \partial)u(z) = f(z) \in \mathcal{O}(U(\theta)).
\]

The existence of singular solutions are studied by [2], [4], [10] and other papers referred in these papers. More generally we have

**Theorem 2.1.** Suppose that \( \chi_{L,0}(0, \xi') \neq 0 \). Then there is a solution \( u(z) \in \mathcal{O}(V(\theta)) \) of \((Eq)\) for some \( V \subset U \).

In this paper we consider the case \( f(z) \) has tempered singularities on \( K \). We have a solution \( u(z) \) by Theorem 2.1, but \( u(z) \) has not always tempered singularities. We can generally show \( |u(z)| \leq A \exp(c|z_0|^{-\gamma_1}) \) for \( z \in V(\theta') \) \((0 < \theta' < \theta)\). So our interest is to find a solution \( u(z) \in \mathcal{O}_{\text{temp},c'}(V(\theta')) \) of the equation

\[
L(z, \partial)u(z) = f(z) \in \mathcal{O}_{\text{temp},c'}(U(\theta'))
\]

for some polydisc \( V \subset U \) and constants \( c' \) and \( 0 < \theta' < \theta \).
Let us give conditions $(C_i) (0 \leq i \leq p^* - 1)$. For fixed $i$, $0 \leq i \leq p^* - 2$
$(C_i) \quad j_\alpha = 0$ for $\alpha \in \Delta_0(i)$ and $\chi_{L,i}(0, \xi') \neq 0$.

For $i = p^* - 1$
$(C_{p^* - 1}) \quad |\alpha'| \leq l_{p^* - 1}$ for $\alpha \in \{\alpha \in \mathbb{N}^{d+1}; e_{L,\alpha} = e(p^* - 1)\}$ and $\chi_{L,p^* - 1}(0, \xi') \neq 0$.

Our main existence theorem is

**Theorem 2.2.** Suppose $p^* \geq 2$ and $(C_i)$ hold for all $0 \leq i \leq p^* - 1$. Let $f(z) \in \mathcal{O}_{\text{temp},c}(U(\theta))$ and $\theta'$ be a constant with $0 < \theta' < \min\{\theta, \pi/2\gamma_1\}$. Then there is a solution $u(z) \in \mathcal{O}_{\text{temp},c'}(V(\theta'))$ of (Eq) for some polydisc $V$ and a constant $c'$.

We note that the opening angle $\theta'$ of sectorial region is restricted by $\gamma_1$.

We need two theorems in order to show Theorem 2.2. One is

**Theorem 2.3.** Suppose $p^* \geq 2$ and $L(z, \partial)$ satisfies $(C_{p^* - 1})$. Let $f(z) \in \mathcal{O}_{\text{temp},c}(U(\theta))$ and $\theta'$ be a constant with $0 < \theta' < \min\{\theta, \pi/2\gamma_{p^* - 1}\}$. Then there is a $v(z) \in \mathcal{O}_{\text{temp},c'}(V(\theta'))$ for some polydisc $V$ and a constant $c'$ such that $(Rf)(z) := (L(z, \partial)v(z) - f(z)) \sim 0$ in $A_{\{\gamma_{p^* - 1}\}}(V(\theta'))$.

The other is

**Theorem 2.4.** Suppose $p^* \geq 2$ and $L(z, \partial)$ satisfies $(C_i)$ for $i = 0, 1, \cdots, p^* - 2$ and let $f(z) \in A_{\{\gamma_{p^* - 1}\}}(U(\theta))$. Then for any $0 < \theta' < \min\{\theta, \pi/2\gamma_{p^* - 1}\}$ there is $u(z) \in A_{\{\gamma_{p^* - 1}\}}(V(\theta'))$ satisfying $L(z, \partial)u(z) = f(z)$ in $V(\theta')$ for some polydisc $V$.

Theorem 2.4 is given in [8] and [9], where we considered the existence of solutions with asymptotic expansion under the condition that $f(z)$ in (Eq) has an asymptotic expansion. We exclude $p^* = 1$ in the preceding theorems, however, we have from results in [4]

**Theorem 2.5.** Suppose $p^* = 1$ and $(C_0)$ holds. Let $f(z) \in \mathcal{O}_{\text{temp},c}(U(\theta))$. Then there is a solution $u(z) \in \mathcal{O}_{\text{temp},c'}(V(\theta))$ of (Eq) for some polydisc $V$ and a constant $c'$.

The operators of Fuchsian type (see [1]) satisfy the conditions of Theorem 2.5.

**Example.** Let

\[
(2.1) \quad L(z, \partial) = \partial_1^5 + A_1(z)\partial_1^3\partial_0 + A_2(z)\partial_0^2, \quad z = (z_0, z_1) \in \mathbb{C}^2,
\]
where \( A_i(z) = z_0^{j_i} B_i(z) \), \( j_i \in \mathbb{N} \), \( B_i(0) \neq 0 \) for \( i = 1, 2 \). According to the values of \( j_1 \) and \( j_2 \), several cases occur. However the conditions in Theorem 2.2 or the conditions in Theorem 2.5 hold for any case. So \( L(z, \partial)u(z) = f(z) \) has always a solution \( u(z) \) with tempered singularities in a sectorial region for \( f(z) \) with tempered singularities.

\section*{§3 Outline of the proof of Theroem 2.3.}

In order to find \( v(z) \) in Theorem 2.3 we construct a parametrix of \( L(z, \partial_z) \). The method of construction of the parametrix is a modification of that in [6]. We may assume \( e(p^*-1) = 0 \) and \( \theta_0 \) be a constant with \( 0 < \theta_0 < \pi/2 \gamma_{p^*-1} \).

\( v(z) = (Gf)(z) \) is constructed of the form

\[
(Gf)(z) := \int_S G(z, w)f(w)dw, \quad w = (w_0, w_1, \ldots, w_d) = (w_0, w'),
\]

where \( S \) is a chain in \( V(\theta_0) \). The kernel \( G(z, w) \) has the form

\[
G(z, w) = \frac{1}{2\pi i} \int_{\lambda_0}^{\infty} z_0^{\lambda}w_0^{-\lambda-1}K(z, w', \lambda)d\lambda.
\]

We can find \( K(z, w', \lambda) \) with the following:

1. \( K(z, w', \lambda) \) is holomorphic \( \{z_0; 0 < |z_0| < r_0, |\arg z_0| < \theta_0\} \times \{(z', w'); |z_j| < r_1 < r_2 < |w_j| < r_3, 1 \leq j \leq d\} \) and holomorphic in \( \lambda \) in some infinite region.

2. \( K(z, w', \lambda) \) has an asymptotic expansion

\[
K(z, w', \lambda) \sim \hat{K}(z, w', \lambda) = \sum_{n=0}^{\infty} k_n(z, w', \lambda)z_0^n,
\]

where \( \hat{K}(z, w', \lambda) \) is a formal power series of \( z_0 \).

3. \( \hat{K}(z, w', \lambda) \) satisfies formally

\[
L(z, \partial)(z_0^{\lambda}\hat{K}(z, w', \lambda)) = \frac{z_0^{\lambda}}{(2\pi i)^d} \prod_{j=1}^{d} \frac{1}{(w_j - z_j)}.
\]
As for $G(z, w)$ we have

\begin{equation}
L(z, \partial)G(z, w) = \delta(z, w) + R(z, w),
\end{equation}

where

\begin{equation}
\delta(z, w) = \frac{1}{(2\pi i)^{d+1}} \left( \int_{\lambda_0}^\infty z_0^\lambda w_0^{-\lambda-1} d\lambda \right) \prod_{j=1}^d \frac{1}{(w_j - z_j)}
\end{equation}

\begin{equation}
|R(z, w)| \leq C \exp(-c|z_0|^{-\gamma_{p^{*}-1}}).
\end{equation}

It follows from (3.3) and (3.4) that $(Rf)(z) = L(z, \partial)v(z) - f(z)$ satisfies the conclusions of Theorem 2.3.

References


