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SINGULARITIES OF THE BERGMAN KERNEL AND NEWTON POLYHEDRA

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1. Introduction

In this note, we announce a result about the singularity of the Bergman kernel for pseudoconvex domains of finite type. In the case of some class of pseudoconvex domains, we show that the growth order of the Bergman kernel at the boundary is determined by the shape of the *Newton polyhedron* of the defining function of the domain and that the boundary limit of the Bergman kernel takes the same value as that in the case of *local model.*

2. Background and our results

2.1. **Background.** Let Ω be a domain in $\mathbb{C}^n$ and $H^2(\Omega)$ the set of the L^2-holomorphic functions on Ω. The Bergman kernel $B(z)$ of Ω (on the diagonal) is defined by

$$B(z) = \sum |\phi_\alpha(z)|,$$

where $\{\phi_\alpha\}$ is a complete orthonomal basis of $H^2(\Omega)$.

There are many studies about the singularity of the Bergman kernel at the boundary of pseudoconvex domains. Let us recall important results which are deeply connected with our study.

2.1.1. *Strictly pseudoconvex case.* Assume that Ω is a C^∞-smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Hörmander [9] and Diederich [3],[4] computed the *boundary limit* of the Bergman kernel as follows.

$$\lim_{z\to p} B(z)\cdot d(z-p)^{n+1} = \frac{n!}{4\pi^n} \times (\text{Levi determinant at } p), \tag{2.1}$$

where d means the distance. Later C. Fefferman [7] obtained the following strong result of the asymptotic expansion of the Bergman kernel:

$$B(z) = \frac{\varphi(z)}{\rho(z)^{n+1}} + \psi(z)\log\rho(z), \tag{2.2}$$

where $\rho \in C^\infty(\bar{\Omega})$ is the defining function (i.e. $\Omega = \{z; \rho(z) > 0\}$ and $|d\rho| > 0$ on $\partial\Omega$) and $\varphi(z)$ and $\psi(z)$ are C^∞-smooth on $\bar{\Omega}$ and $\varphi(z)$ is positive on the boundary.

2.1.2. *Semiregular (h-extendible) case.* In the weakly pseudoconvex and of finite type case, although there is not so strong result like asymptotic expansion (2.2), many precise results have been obtained. In particular the following result due to Boas-Straube-Yu [2] (see also Diederich-Herbort [6]) is very important. Assume that Ω is a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ and $p \in \partial\Omega$ is a boundary point of semiregular (h-extendible) with the multi-type $(1, 2m_1, \dots, 2m_n)$ (see [5],[12] for precise definition). In [2],[6],

$$\lim_{\substack{z \to p \\ z \in \Lambda}} B(z) \cdot d(z-p)^{2+\sum_{j=1}^{n} 1/m_j} = B_0(\varpi), \tag{2.3}$$

where Λ is a nontangential cone, B_0 is the Bergman kernel of *local model* of Ω at p and ϖ is some point in Ω.

Recently the author [10] computed an asymptotic expansion of the Bergman kernel in the case of tube domains of finite type. Let f be a convex C^∞-smooth function on $\mathbb{R}^n$ such that $f(0) = df(0) = 0$ and ω_f the set in $\mathbb{R}^{n+1}$ defined by $y_0 + f(y_1, \dots, y_n) = y_0 + f(y') < 0$. Let Ω_f be a tube domain defined by $\Omega_f = \mathbb{R}^{n+1} + \omega_f$. Assume that the origin is a point of finite type. This assumption implies that $f(y')$ has an expression near the origin: $f(y') = P(y')[1 + h(y')]$ where $P(y')$ is a convex polynomial such that $P(t^{1/2m_1}y_1, \dots, t^{1/2m_n}y_n) = tP(y_1, \dots, y_n)$ $(m_j \in \mathbb{N})$ and $|h(y')| \le C\sigma(\tau)^\gamma$ where $\sigma(\tau) = \sum_{j=1}^{n} y_j^{2m_j}$ and $C > 0$, $\gamma \in (0, 1]$ are some numbers. Here we set

$$\begin{aligned} \Delta_P &= \{\tau \in \mathbb{R}^n; P(\tau) < 1\}, \\ \Gamma_\delta &= \{(\tau, \rho^{1/m}) \in \Delta_P \times [0, \delta); P(\tau)[1 + C\rho^\gamma \sigma(\tau)^\gamma] < 1\}. \end{aligned}$$

Let $\sigma : \omega_f \to \Delta_P \times (0, \infty)$ be a mapping defined by $\sigma(y_0, y_1, \dots, y_n) = (\tau_1, \dots, \tau_n, \rho)$ where $\tau_j = -y_j \cdot y_0^{-1/2m_j}$ and $\rho = -y_0$.

Theorem 2.1 ([10]). *The Bergman kernel $B(z)$ of Ω_f has the form in some small neighborhood of the origin:*

$$B(z) = \frac{\varphi(\tau, \rho^{1/m})}{\rho^{\sum_{j=1}^{n} 1/m_j + 2}} + \psi(\tau, \rho^{1/m}) \log \rho, \tag{2.4}$$

with $\varphi(\tau, \rho^{1/m}) \in C^\infty(\Gamma_\delta)$ and $\psi(\tau, \rho^{1/m}) \in C^\infty(\overline{\Delta_P} \times [0, \delta))$, where $\delta > 0$ is a small number, m is the least common multiple of $\{m_1, \dots, m_p\}$ and $\varphi(\tau, 0) = \varphi(\tau) > 0$.

The aymptotic expansions (2.2) and (2.4) have very similar forms and the essential difference between these asymptotic formulas only appears in the expansion variable, i.e. (2.2) takes the Taylor type while (2.4) takes the Puiseux type. But Herbort's example, below, asserts that an analogous asymptotic formula does not always hold in general case of finite type.

2.1.3. *Herbort's counterexample.* Herbort [8] showed the Bergman kernel of the domain:

$$\Omega_{HE} = \{z \in \mathbb{C}^3; \Re(z_0) + |z_1|^6 + |z_1|^2|z_2|^2 + |z_2|^6 < 0\}.$$

satisfies the inequalities near the origin on a nontangential cone:

$$\frac{c_1}{\rho^3 \log(1/\rho)} < B(z) < \frac{c_2}{\rho^3 \log(1/\rho)},$$

where c_1, c_2 are positive constants. Note that Ω_{HE} is not convex at the origin. Although Ω_{HE} is a pseudoconvex domain of finie type, the above inequarities imply that the logarithmic function appears in the first term of the asymptotic expansion of its Bergman kernel. From Herbort's exapmle, it seems difficult to imagin what kind of pattern of the singularity of the Bergman kernel for general domains of finite type.

2.2. **The Newton polyhedron.** Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ = [0, \infty)$. First let us recall the definition of the Newton polyhedron of functions on the real space. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞-smooth function in the neighborhood of the origin with $f(0) = 0$. Then $f(x)$ has the asymptotic expansion at the origin:

$$f(x) \sim \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha,$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The *Newton polyhedron* $\Gamma_+(f)$ is the convex hull of the union of $\{\alpha + \mathbb{R}_+^n\}$ for α such that $c_\alpha \neq 0$. The *Newton diagram* $\Gamma(f)$ is the union of the compact faces of the Newton polyhedron $\Gamma_+(f)$. The *pricipal part* $f_0(x)$ of $f(x)$ is defined by

$$f_0(x) = \sum_{\alpha \in \Gamma(f)} c_\alpha x^\alpha.$$

We generalize these concepts to the case of the function on the complex space. Let $F : \mathbb{C}^n \to \mathbb{R}$ be a C^∞-smooth function in the neighborhood of the origin with $F(0) = 0$. Then F has the asymptotic expansion at the origin:

$$F(z) \sim \sum_{\alpha, \beta \in \mathbb{N}_0^n} C_{\alpha\beta} z^\alpha \bar{z}^\beta,$$

where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\bar{z}^\beta = \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$. The *Newton polyhedron* $\tilde{\Gamma}_+(F)$ is the convex hull of the union of $\{\alpha + \beta + \mathbb{R}_+^n\}$ for α, β such that $C_{\alpha,\beta} \neq 0$. The *Newton diagram* $\tilde{\Gamma}(F)$ is the union of the compact faces of the Newton polyhedron $\tilde{\Gamma}_+(F)$. The *principal part* $F_0(z)$ of $F(z)$ is defined by

$$F_0(z) = \sum_{\alpha + \beta \in \tilde{\Gamma}(F)} C_{\alpha,\beta} z^\alpha \bar{z}^\beta.$$

The *Newton distance* d_F is defined by

$$d_F = \min\{d > 0; (d, \ldots, d) \in \tilde{\Gamma}_+(F)\}.$$

Set $P = \{(d_F, \ldots, d_F)\} \in \tilde{\Gamma}(F)$. Let $\tilde{l}_F$ be the number of the $(n-1)$-dimensional faces on $\tilde{\Gamma}(F)$ containing P. Then define $l_F = \min\{\tilde{l}_F, n\}$.

2.3. **Boundary limit of the Bergman kernel.** Let F be a C^∞-smooth plurisubharmonic function on $\mathbb{C}^n$ satisfying that $F(0) = |\partial\nabla F(0)| = 0$. We consider the domain:

$$\Omega_F = \{(z_0, z) \in \mathbb{C} \times \mathbb{C}^n; \rho = \Im(z_0) - F(z_1, \dots, z_n) > 0\}.$$

We assume that $0 \in \partial\Omega_F$ is a point of finite type and that

$$F(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n) = F(z_1, \dots, z_n)$$

for $\theta_j \in \mathbb{R}$.

Theorem 2.2. *There is some positive constant $C(F)$ such that the Bergman kernel B of the domain Ω_F satisfies*

$$\lim_{\substack{\Im(z_0)\to 0\\ (z_0,z)\in\Lambda}} B(z_0, z) \cdot \rho^{2+2/d_F}(\log(1/\rho))^{l_F-1} = C(F), \tag{2.5}$$

where Λ is a nontangential cone. Moreover if let F_0 be a princepal part of F, then $C(F) = C(F_0)$.

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