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Semiclassical behavior of the scattering phase near a critical value of the potential

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0 Introduction

We study the asymptotic behavior in the semiclassical limit of the scattering phase and the time delay for the one-dimensional Schrödinger equation

$$-h^2\frac{d^2u}{dx^2}+V(x)u=Eu \tag{0.1}$$

when the energy E is near a critical value of the potential $V(x)$. We suppose here that the critical value is the maximum V_0 of the potential and consider two cases where $V(x)$ attains the maximum at one point (case I) and two points (case II). We assume moreover that these maxima are non-degenerate, i.e. the radii of curvature are finite.

We will show the following facts. Near the critical value V_0, the leading term of the asymptotic expansion as h tends to 0 of the time delay function (the derivative with respect to the energy of the scattering phase) is logarithmic in both cases. Moreover its coefficient is, in case I, the radius of curvature of the barrier top and in case II, it is the average of the radii of curvature of the two barrier tops, plus a contribution from the potential well. This contribution from the potential well is an oscillating function with wavelength $O(h/\log h^{-1})$ near the barrier top and it can be continued analytically into the well, where its behavior is like in the Breit- Wigner formula.

This physical problem is closely related to the purely mathematical problem originated by H.Weyl [We] on the eigenvalue asymptotics for the Laplace operator in a bounded domain $\Omega\in\mathbf{R}^n$ with regular boundary. If $\mathcal{N}(E)$ is the number of eigenvalues not excedding

E (counting multiplicity), then

$$\mathcal{N}(E) \sim \frac{\tau_n}{(2\pi)^n}\mathrm{vol}(\Omega)E^{n/2}, \quad E \to +\infty,$$

where $\tau_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball in $\mathbf{R}^n$. L.Hörmander [Hö] generalized this result to compact Riemannian manifolds without boundary with the best estimate of the error.

On the other hand, the asymptotic behavior of the scattering phase $\theta(E)$ for the exterior problem with an obstacle Ω was studied, say, by A.Jensen and T.Kato [Je-Ka], V.Petkov and G.Popov [Pe-Po] and R.Melrose [Me]. This is defined to be the half of the argument of the determinant of the scattering matrix which is unitary on $L^2(S^{n-1})$. They showed that $\theta(E)$ also has the Weyl type asymptotic formula when $E \to +\infty$:

$$\theta(E) = -\frac{\tau_n}{(2\pi)^n}\mathrm{vol}(\Omega)E^{n/2} + O(E^{(n-1)/2}).$$

In the case of Schrödinger operator on $\mathbf{R}^n$ with a scalar potential $V(x)$, the semiclassical problem is also important. Let $V(x) \in C_0^\infty(\mathbf{R}^n)$ for simplicity. Then the spectrum of the Schrödinger operator $P = -h^2\Delta + V(x)$ on $L^2(\mathbf{R}^n)$ consists of negative eigenvalues and the essential spectrum lying on $\mathbf{R}_+$. We can define the eigenvalue counting function $\mathcal{N}(E,h)$ for negative E and the scattering phase $\theta(E,h)$ for positive E both depending on the small parameter h. The semiclassical problem is to describe the asymptotic behavior of $\mathcal{N}(E,h)$ or $\theta(E,h)$ as h tends to 0 while E is restricted near a fixed energy E_0. Remark that the high energy regime ($E \to +\infty$) is a particular case of the semiclassical regime ($h \to 0$) with $h = E^{-1/2}$ (see [Ro]).

The spectral shift function $s(E,h)$ combines these two notions. It is defined as a Schwartz distribution on $\mathbf{R}$ modulo a real constant by the following equation:

$$\mathrm{Tr}(f(P) - f(P_0)) = -\int_{-\infty}^{+\infty} f'(E)s(E,h)dE,$$

where $P_0 = -h^2\Delta$ and f is an arbitrary function in the Schwartz space. $s(E,h)$ is uniquely determined by asking $s(E,h) = 0$ for $E << 0$. Under this condition, it is easy to see that $s(E,h) = \mathcal{N}(E,h)$ for negative E. The remarkable fact discovered by M.S.Birman and M.G.Krein [Bi-Kr] is the relation between the spectral shift function and the scattering phase for positive E:

$$\theta(E,h) = \pi s(E,h) \mod(\pi\mathbf{Z}).$$

The semiclassical asymptotics of the spectral shift function is closely related to its classical mechanics counterpart. Let $s^{cl}(E,h)$ be the classical analogue of the spectral shift function defined by

$$\int\int_{\mathbf{R}^{2n}} \{f(p(x,\xi)) - f(p_0(\xi))\}dxd\xi = -\int_{-\infty}^{+\infty} f'(E)s^{cl}(E)dE,$$

where $p(x,\xi) = \xi^2 + V(x)$ and $p_0(\xi) = \xi^2$. We see immediately that

$$s^{cl}(E) = \tau_n \int_{\mathbf{R}^n} \{(E - V(x))_+^{n/2} - E_+^{n/2}\} dx,$$

where $a_+ = \max(a, 0)$. An energy level E is said to be *non-trapping* for the classical Hamiltonian $p(x,\xi)$ if every Hamiltonian flow on $p^{-1}(E)$ generated by p escapes to infinity as time goes to both $+$ and $-$ infinity.

D.Robert and H.Tamura [Ro-Ta] proved that if E is non-trapping in an interval, $s(E,h)$ has a uniform complete asymptotic expansioin as $h \to 0$ and

$$s(E,h) = (2\pi h)^{-n} s^{cl}(E) + O(h^{1-n}) \quad \text{as} \quad h \to 0. \tag{0.2}$$

If the energy is trapping, however, it is believed that the scattering phase varies very rapidly because of the presence of poles of the scattering matrix called *resonances* near the real axis.

Let us mention here two works on the scattering phase associated with a trapped trajectory in a potential well. A resonance generated by such a closed trajectory is called *shape resonance* and is exponentially close to the real axis with respect to h. C.Gérard, A.Martinez and D.Robert [Gé-Ma-Ro] proved that the scattering phase increases by π at the real part of a shape resonance. In other words, they showed the Breit-Wigner formula at the bottom of the potential well.

S.Nakamura [Na] considered two Hamiltonians P_i and P_e corresponding to the bounded and unbounded component of $p^{-1}(E)$ respectively and showed that the spectral shift function is approximated by the sum of that for P_e, the asymptotic behavior of which we know from the result of [Ro-Ta], and the eigenvalue counting function for P_i. These eigenvalues are near the shape resonances and cause the rapid variation of the scattering phase.

The barrier top energy $E = V_0$ is trapping because it takes infinite time for classical particles to arrive at the barrier top. Remark also that in case I, E is non-trapping above and below V_0 and in case II, non-trapping above V_0 and trapping below V_0 because of the potential well. This change of classical structure at the barrier top is represented as the logarithmic singularity of $s^{cl}(E)$ at $E = V_0$ (see Lemma 2.3).

We will see that there is a correction term for $s^{cl}(E)$ in a complex neighborhood of V_0 of radius $O(h)$ and that it cancels the singularity of $s^{cl}(E)$. This term is written in terms of the Fredholm determinant of the harmonic oscillator. This comes from the fact that the operator P is reduced to the harmonic oscillator microlocally near the non-degenerate critical point (see [He-Sj]). Out of this neighborhood, that is where $|V_0 - E|/h$ tends to infinity, this term disappears from the leading term and we recover $s^{cl}(E)$.

Once we prove that the leading term is holomorphic in a complex domain, we can differentiate the asymptotic formula term by term. In fact, the scattering phase itself is also holomorphic in a complex neighborhood of V_0 and so is the remainder term in the intersection. We can then estimate the derivative of the remainder term by its supremum by Cauchy's formula. Thus we get the asymptotic formula of the time delay function.

The method is based on the *exact WKB analysis* (see [Gé-Gr]). In the one-dimesional case, the scattering matrix can be defined as a 2×2 matrix and each element is written by wronskians of Jost solutions (see Section 2). Therefore the problem is reduced to the connection problem of Sturm-Liouville equations with small parameter.

In this paper we start from the asymptotic formulas of the scattering matrix obtained by [Ra] in case I and by [Fu-Ra 1] in case II.

1 Results

We consider the one-dimensional Schrödinger equation (0.1) where the potential $V(x)$ satisfies the following conditions:

(H1) $V(x)$ is real on $\mathbf{R}$ and dilation analytic, that is, $V(x)$ is holomorphic in a sector $\mathcal{S} = \{x \in \mathbf{C}; |\operatorname{Im} x| < \tan\theta_0 |\operatorname{Re} x|\} \cup \{|\operatorname{Im} x| < \delta\}$ for some $0 < \theta_0 < \pi/2$ and $\delta > 0$.

(H2) $V(x)$ is short range, that is, there exists a positive constant $\epsilon > 0$ such that $|V(x)| \leq (1 + |x|)^{-1-\epsilon}$ in $\mathcal{S}$.

Let V_0 be the maximum of the potential on the real axis which we assume to be positive. We consider the two cases:

(Case I) $V^{-1}(V_0) = \{o_1\}$

(Case II) $V^{-1}(V_0) = \{o_1, o_2\} \quad (o_1 < o_2)$

In both cases, we assume furthermore that the radius of curvature ρ_j is finite at each critical points:

(H3) $V''(o_j) = -\rho_j^{-1} < 0 \quad (j = 1, 2)$

Put $\lambda = V_0 - E$. The energy is under the maximum of the potential when λ is positive. If λ is positive and sufficiently small, the equation $V(x) - E = 0$ has 2 real roots $\alpha_1(\lambda), \beta_1(\lambda)$ near o_1 ($\alpha_1 < o_1 < \beta_1$) in both cases and 2 other real roots $\alpha_2(\lambda), \beta_2(\lambda)$ near o_2 ($\alpha_2 < o_2 < \beta_2$) in case II. We then define the action integrals between these turning points and $\pm\infty$ as follows. For the simplicity of notations, we use the convention that $*$ stands for 1 in case I and 2 in case II.

Definition 1.1 For $\lambda = V_0 - E$ positive and sufficiently small, we set

$$S_j(\lambda) = \int_{\alpha_j}^{\beta_j} \sqrt{V(x) - E}\,dx, \quad (j = 1, 2),$$

$$S_{\rm e}(\lambda) = \left(\int_{-\infty}^{\alpha_1} + \int_{\beta_*}^{\infty}\right) \{\sqrt{E - V(x)} - \sqrt{E}\}dx - \sqrt{E}(\beta_* - \alpha_1),$$
$$S_{\rm i}(\lambda) = \int_{\beta_1}^{\alpha_2} \sqrt{E - V(x)}dx \quad \text{(in case II)}.$$

Let us remark here that the classical analogue of the spectral shift function $s^{cl}(E)$ is related with these actions by

$$s^{cl}(E) = \begin{cases} 2S_{\rm e}(\lambda) & (\text{case } I) \\ 2(S_{\rm e}(\lambda) + S_{\rm i}(\lambda)) & (\text{case } II) \end{cases}$$

A very important fact is that $S_{\rm e}$ and $S_{\rm i}$ have a logarithmic singularity at $\lambda = 0$ (see Lemma 2.3). We will see that there is a correction term which affects the leading term of the scattering phase and that it regularizes $s^{cl}(E)$ by canceling the logarithmic singularity. This correction term is written by the following interesting function $N(z)$, which is the Jost function of the harmonic oscillator (see Remark 2.5). For the properties of this function, see Lemma 2.4.

Definition 1.2 We define the function $N(z)$ on $\{z \in \mathbb{C}\backslash; |\arg z| < \pi\}$ by

$$N(z) = \frac{\sqrt{2\pi}}{\Gamma(z + 1/2)} e^{z \log(z/e)}.$$

Let us define, using this function, the *regularized* actions at the barrier top:

Definition 1.3 We define the real functions $\sigma_{\rm e}(\lambda, h)$ and $\sigma_{\rm i}(\lambda, h)$ for positive and small λ by

$$\sigma_{\rm e,i}(\lambda, h) = S_{\rm e,i}(\lambda) - \frac{h}{2}\left\{\arg N(i\frac{S_1(\lambda)}{\pi h}) + \arg N(i\frac{S_*(\lambda)}{\pi h})\right\},$$

where $\arg(iS_j(\lambda)/(\pi h)) = \pi/2$ for $\lambda > 0$. $\sigma_{\rm e,i}(\lambda, h)$ can be extended as holomorphic functions to a complex neighborhood of $\lambda = 0$ (see Proposition 2.6).

We will see (see (2.6)) that far from the barrier top $\lambda = 0$, $\sigma_{\rm e,i}(\lambda, h)$ coincide with $S_{\rm e,i}(\lambda)$. More precisely,

$$\sigma_{\rm e,i}(\lambda, h) \to S_{\rm e,i}(\lambda) \quad \text{as} \quad |\lambda|/h \to +\infty \tag{1.1}$$

We also define a holomorphic function which will express the width of resonances near the real axis in case II.

Definition 1.4 We define the positive function $\gamma(\lambda, h)$ for positive and small λ by

$$\gamma(\lambda, h) = \frac{|N(i\frac{S_1(\lambda)}{\pi h})N(i\frac{S_2(\lambda)}{\pi h})| - 1}{|N(i\frac{S_1(\lambda)}{\pi h})N(i\frac{S_2(\lambda)}{\pi h})| + 1}.$$

As function of λ, $\gamma(\lambda, h)$ extends holomorphicaly to a complex neighborhood of $\lambda = 0$ (see Lemma 2.7).

Theorem 1.5 *There exists $C > 0$ such that if λ is real and $|\lambda| \leq Ch$, then we have in case I*

$$h\theta(E,h) = \sigma_e(\lambda, h) + O(h^2 \log(1/h)), \tag{1.2}$$

and in case II

$$h\theta(E,h) = \sigma_e(\lambda, h) + h\tan^{-1}\left\{\gamma(\lambda, h)\tan\frac{\sigma_i(\lambda, h)}{h}\right\} + O(h^2\log(1/h)). \tag{1.3}$$

The asymptotic formula (1.2) and (1.3) are just the analogues of the results of [Ro-Ta] and [Na] respectively. The second term in the right hand side of (1.3) is related to the presence of the potential well and causes rapid variations caused by the potential well. It will be seen more clearly in the next corollary as a rapid oscillation of the *time delay*, which is the derivative of the scattering phase with respect ot the energy E.

Corollary 1.6 *There exists $C > 0$ such that if λ is real and $|\lambda| \leq Ch$, then we have in case I*

$$h\frac{d\theta}{dE} = \rho_1 \log\frac{1}{h} + O(1)$$

and if λ is real and $|\lambda| \leq Ch/\log(1/h)$, then we have in case II

$$h\frac{d\theta}{dE} = \frac{\rho_1 + \rho_2}{2}\left\{1 + \frac{\gamma}{(1-\gamma^2)\cos^2(\sigma_i/h) + \gamma^2}\right\}\log\frac{1}{h} + O(1)$$

The leading term is logarithmic with respect to h hence the Weyl law fails in such small neighborhoods of the potential maximum. The function

$$B(E,h) = \frac{\gamma}{(1-\gamma^2)\cos^2(\sigma_i/h) + \gamma^2}$$

is the contribution from the potentail well. The function $\gamma(E,h)$ tends to 0 for $E < V_0$, to 1 for $E > V_0$ and equals 1/3 for $E = V_0$. When γ is small, B presents spikes at each zero of $\cos(\sigma_i/h)$ of height $1/\gamma$ and width γ (see Lemma 2.7). Hence this is an extension of the *Breit-Wigner formula* to the potential maximum. The zeros of $\cos(\sigma_i/h)$ are given by the Bohr-Sommerfeld type quantization condition

$$\sigma_i(E,h) = (n + \frac{1}{2})\pi h.$$

It follows from Proposition (2.6) that the distance between two such successive zeros when $|\lambda| \leq Ch/\log(1/h)$ is $2\pi(\rho_1+\rho_2)^{-1}h\log(1/h)$.

2 Proofs

2.1 Proof of Theorem 1.6

We calculate the asymptotic formula of the scattering phase starting from the results of [Ra] and [Fu-Ra 1] for the scattering matrix.

Let $f_l^{\pm}(x)$ and $f_r^{\pm}(x)$ be Jost solutions such that

$$\begin{aligned} f_r^{\pm}(x) &\sim e^{\pm i\sqrt{E}x/h} \quad \text{as} \quad \operatorname{Re} x \to +\infty \quad \text{in} \quad \mathcal{S}, \\ f_l^{\pm}(x) &\sim e^{\pm i\sqrt{E}x/h} \quad \text{as} \quad \operatorname{Re} x \to -\infty \quad \text{in} \quad \mathcal{S}. \end{aligned}$$

The existence (and uniqueness) of such solutions is guaranteed by the short range assumption (H.2) for E in $\Pi_{\theta_0} = \{E \in \mathbb{C}\backslash\{0\}; |\arg E| < 2\theta_0\}$. These solutions are holomorphic in $(x, E) \in \mathcal{S} \times \Pi_{\theta_0}$.

The two pairs (f_l^+, f_l^-) and (f_r^+, f_r^-) are basis of solutions to (0.1) hence they are related with each other by a constant 2×2 matrix $\mathbf{T}(E, h)$:

$$\begin{pmatrix} f_l^+ \\ f_l^- \end{pmatrix} = \mathbf{T}(E,h) \begin{pmatrix} f_r^+ \\ f_r^- \end{pmatrix}.$$

The determinant of this matrix is 1 since $[f_l^+, f_l^-] = \det \mathbf{T}\, [f_r^+, f_r^-]$ and the wronskians $[f_l^+, f_l^-]$ and $[f_r^+, f_r^-]$ are both $-2i\sqrt{E}/h$. Moreover $f_{l,r}^- = (f_{l,r}^+)^*$ where $f^*(x, E) = \overline{f(\bar{x}, \bar{E})}$ by definition, hence $\mathbf{T}$ is of the form

$$\mathbf{T} = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \quad aa^* - bb^* = 1. \tag{2.1}$$

The elements a and b are written in terms of Jost solutions:

$$a(E,h) = \frac{ih}{2\sqrt{E}}[f_l^+, f_r^-], \quad b(E,h) = -\frac{ih}{2\sqrt{E}}[f_l^+, f_r^+],$$

and we see that they, as well as $a^*(E, h)$ and $b^*(E, h)$, are all holomorphic in $E \in \Pi_{\theta_0}$.

On the other hand, the scattering matrix is defined as the matrix associated with the change of basis between the outgoing pair of solutions (f_r^+, f_l^-) and the incoming pair of solutions (f_l^+, f_r^-); if

$$p_+ f_r^+ + p_- f_l^- = q_+ f_l^+ + q_- f_r^-,$$

then

$$\begin{pmatrix} p_+ \\ p_- \end{pmatrix} = \mathbf{S}(E,h) \begin{pmatrix} q_+ \\ q_- \end{pmatrix}.$$

We see by an elementary computation that in terms of a and b, $\mathbf{S}$ is written by

$$\mathbf{S} = \frac{1}{a^*} \begin{pmatrix} 1 & -b^* \\ b & 1 \end{pmatrix}.$$

Suppose now that E is positive. Then $\mathbf{S}$ is unitary by (2.1) and hence its determinant is a complex number with module 1. The scattering phase $\theta(E,h)$ is defined as half of the argument of $\det \mathbf{S}$:

$$\det \mathbf{S}(E,h) = e^{2i\theta(E,h)}. \tag{2.2}$$

It is real and given in terms of the element a of the matrix $\mathbf{T}$ by

$$\theta(E,h) = \arg a(E,h) = -\arg \overline{a(E,h)}. \tag{2.3}$$

Remark that $\theta(E,h)$ can also be defined for complex $E \in \Pi_{\theta_0}$ as complex valued function by

$$\theta(E,h) = \frac{1}{2i} \log \frac{a(E,h)}{a^*(E,h)}.$$

More precisely we have

Lemma 2.1 *There exists $C > 0$ such that $\theta(E,h)$ extends holomorphically to the disk $|E - V_0| < Ch$ in case I and $|E - V_0| < Ch\log(1/h)$ in case II.*

Proof: Since a and a^* are holomorphic in Π_{θ_0}, $\theta(E,h)$ is singular only at zeros of a and a^*. The zeros of a are complex conjugates of those of a^* and hence it is enough to study the asymptotic distribution of zeros of a^*, that is, resonances. This was done in [Ra] and [Fu-Ra 1] in cases I and II respectively. They calculated the leading term of the asymptotic expansion of a^* (see Theorem 2.2) and majorated by Rouché's theorem the distance between its zeros and those of a^*. □

Recall here the asymptotic formula of $a^*(E,h)$ obtained in [Ra] and [Fu-Ra 1]. See also [Fu] where $S_l + S_r = -S_e$ and $S_{12} = S_i$.

Theorem 2.2 *The asymptotic formula of $a^*(E,h)$ is given by the following formulas:*

(Case I)

$$a^*(E,h) = e^{(S_1 - iS_e)/h} N(i\frac{S_1}{\pi h})(1 + O(h\log h)), \tag{2.4}$$

(Case II)

$$a^*(E,h) = e^{(S_1+S_2-iS_e)/h} \left(e^{iS_i/h} + N(i\frac{S_1}{\pi h})N(i\frac{S_2}{\pi h})e^{-iS_i/h}\right) \times(1 + O(h\log h)) \tag{2.5}$$

Let us calculate the argument of a^* through (2.4) and (2.5). For simplicity, put

$$r_j(\lambda,h) = |N(i\frac{S_j(\lambda)}{\pi h})|, \quad \phi_j(\lambda,h) = \arg N(i\frac{S_j(\lambda)}{\pi h}).$$

In case I, we get immediately

$$\pi h\theta(E,h) = S_{\mathrm{e}} - h\phi_1 + O(h^2\log h) = \sigma_{\mathrm{e}} + O(h^2\log h).$$

In case II, we have

$$\begin{aligned}\pi h\theta(E,h) &= S_{\mathrm{e}} - h\ \arg\left(e^{iS_{\mathrm{i}}/h} + r_1r_2e^{i(\phi_1+\phi_2)}e^{-iS_{\mathrm{i}}/h}\right) + O(h^2\log h)\\ &= \sigma_{\mathrm{e}} - h\ \arg\left(e^{i\sigma_{\mathrm{i}}/h} + r_1r_2e^{-i\sigma_{\mathrm{i}}/h}\right) + O(h^2\log h),\end{aligned}$$

and since

$$\begin{aligned}\arg\left(e^{i\sigma_{\mathrm{i}}/h} + r_1r_2e^{-i\sigma_{\mathrm{i}}/h}\right) &= \arg\left\{(1+r_1r_2)\cos\frac{\sigma_{\mathrm{i}}}{h} + i(1-r_1r_2)\sin\frac{\sigma_{\mathrm{i}}}{h}\right\}\\ &= \tan^{-1}\left(\frac{1-r_1r_2}{1+r_1r_2}\tan\frac{\sigma_{\mathrm{i}}}{h}\right),\end{aligned}$$

we get (1.3).

2.2 Regularized actions

We first recall the analytic property in a complex neighborhood of $\lambda = 0$ of the action integrals $S_j(\lambda)$, $S_{\mathrm{e}}(\lambda)$ and $S_{\mathrm{i}}(\lambda)$ which were defined for small positive λ in Definition 1.1. Let $D(R)$ denote the disk $\{\lambda \in \mathbf{C}; |\lambda| < R\}$. See [Fu-Ra 1] for the proof of the following lemma.

Lemma 2.3 *There exist a positive constant R and functions $g_j(\lambda)$ $(j = 1, 2)$, $g_{\mathrm{e}}(\lambda)$ and $g_{\mathrm{i}}(\lambda)$ holomorphic in $D(R)$ such that $S_j(\lambda)$, $S_{\mathrm{e}}(\lambda)$ and $S_{\mathrm{i}}(\lambda)$ are all real for $0 < \lambda < R$ and*

$$S_j(\lambda) = \pi\rho_j\lambda(1+\lambda g_j(\lambda)) \quad (j = 1,2).$$

$$S_{\mathrm{e,i}}(\lambda) = S_{\mathrm{e,i}}(0) + \frac{1}{2\pi}(S_1(\lambda) + S_*(\lambda))\log\lambda + \lambda g_{\mathrm{e,i}}(\lambda),$$

where $\log\lambda > 0$ when $\arg\lambda = 0$.

We also recall some properties of the function $N(z)$.

Lemma 2.4 *$N(z)$ is holomorphic in $\{z \in \mathbf{C}\backslash\{0\}; |\arg z| < \pi\}$ and in this domain,*

$$\lim_{|z|\to\infty} N(z) = 1. \tag{2.6}$$

In particular, on the positive imaginary axis $z = it, t > 0$, we have

$$|N(it)|^2 = 1 + e^{-2\pi t}, \tag{2.7}$$

$$\arg N(it) = t \log t + t g(t), \tag{2.8}$$

where g is a real and analytic function and extends holomorphically to a complex neighborhood of the origin.

Proof: The formula (2.6) is nothing but the Stirling formula and (2.7) follows easily from the product formula of the Gamma function:

$$|\Gamma(\frac{1}{2} + it)|^2 = \Gamma(\frac{1}{2} + it)\Gamma(\frac{1}{2} - it) = \frac{\pi}{\cosh \pi t}. \tag{2.9}$$

For (2.8), we have

$$\arg N(it) = t \log t - t - \arg \Gamma(1/2 + it).$$

Using (2.9), we can rewrite the last term of the right hand side as

$$\arg \Gamma(\frac{1}{2} + it) = \frac{i}{2} \log \pi - i \log \Gamma(\frac{1}{2} + it) - \frac{i}{2} \log(\cosh \pi t).$$

This function can be extended analytically to $\mathbf{C} \backslash i(\mathbf{Z} + 1/2)$ and equals 0 when $t = 0$. Hence we can write $\arg N(it)$ in the form (2.8). □

Remark 2.5 The function $N(z)$ can be characterized as Jost function of the harmonic oscillator (see [Vo]). Let $\psi_\pm(x)$ be the solutions to (0.1) with $V(x) = x^2$ and $h = 1$ whose asymptotic behavior is

$$\psi_\pm(x) \sim (x^2 - E)^{-1/4} \exp\left(\pm \int_{x_0}^{x} (y^2 - E)^{1/2} dy\right) \quad \text{as} \quad x \to \mp\infty.$$

It is possible to define these solutions for arbitrary $x_0 \in \mathbb{R}$ when E is negative. Then the Jost function of the harmonic oscillator, which is defined as the wronskian of these solutions, is written by

$$\frac{1}{2}[\psi_+, \psi_-] = N(-\frac{E}{2}).$$

Proposition 2.6 *There exists $C > 0$ such that the functions $\sigma_{\mathrm{e}}(\lambda, h)$ and $\sigma_{\mathrm{i}}(\lambda, h)$ can be extended as holomorphic functions with respect to λ in $D(Ch)$ and the following asymptotic formula holds in this domain:*

$$\sigma_{\mathrm{e,i}}(\lambda, h) = S_{\mathrm{e,i}}(0) - \frac{\rho_1 + \rho_*}{2} \lambda \log \frac{1}{h} + O(\lambda) \quad \textit{as} \quad h \to 0.$$

Proof: From Lemmas 2.3 and 2.4, we get

$$\sigma_{\mathrm{e,i}}(\lambda, h) = S_{\mathrm{e,i}}(0) - \frac{\rho_1 + \rho_*}{2} \lambda \log \frac{1}{h} + G(\lambda, h),$$

with
$$G=\lambda g_{\mathrm{e,i}}-\sum_{j=1,*}\left[\frac{S_j}{2\pi}\left\{g(\frac{S_j}{\pi h})+\log\left(\rho_j(1+\lambda g_j)\right)\right\}+\frac{\rho_j}{2}\lambda^2 g_j\log\frac{1}{h}\right].$$

□

2.3 Proof of Corollary 1.7

Here we deduce Corollary 1.6 from Theorem 1.5 by making use of the analyticity of the remainder terms in the asymptotic formulas (1.2) and (1.3). Let $R_I(\lambda,h)$, $R_{II}(\lambda,h)$ be the the remainder terms of (1.2), (1.3) respectively;

$$R_I(\lambda,h)=\pi h\theta(E,h)-\sigma_{\mathrm{e}}(\lambda,h),$$

$$R_{II}(\lambda,h)=\pi h\theta(E,h)-\sigma_{\mathrm{e}}(\lambda,h)-h\tan^{-1}\left\{\gamma(\lambda,h)\tan\frac{\sigma_{\mathrm{i}}(\lambda,h)}{h}\right\}.$$

First let us observe some properties of the function $\gamma(\lambda,h)$.

Lemma 2.7 *There exist positive C and R such that the function $\gamma(\lambda,h)$ is holomorphic in $(-R,R)\times i(-Ch,Ch)$. Moreover, on $(-R,R)$ in particular, $0<\gamma<1$ and*

(i) if $\lambda=O(h)$, there exist $0<\gamma_0<\gamma_1<1$ independent of λ and of h such that $\gamma_0<\gamma(\lambda,h)<\gamma_1$ and in particular $\gamma(0,h)=1/3$,

(ii) if $|\lambda/h|\to\infty$,

$$\gamma(\lambda,h)=\begin{cases}O(e^{-2S_1(\lambda)/h}+e^{-2S_2(\lambda)/h}) & (\lambda>0),\\ 1-O(e^{(S_1(\lambda)+S_2(\lambda))/h}) & (\lambda<0).\end{cases}$$

Proof: With (2.7), one obtains

$$\gamma(\lambda,h)=\frac{\sqrt{1+e^{-2S_1(\lambda)/h}}\sqrt{1+e^{-2S_2(\lambda)/h}}-1}{\sqrt{1+e^{-2S_1(\lambda)/h}}\sqrt{1+e^{-2S_2(\lambda)/h}}+1},$$

and the lemma follows easily. In particular this function has singularities at the points satisfying $S_j(\lambda)=(n+1/2)\pi ih$ $(j=1,2)$. □

Proposition 2.8 *There exists a positive constant C such that $R_I(\lambda,h)$ and $R_{II}(\lambda,h)$ are holomorphic with respect to λ in $D(Ch)$ and in $D(Ch/\log(1/h))$ respectively for sufficiently small h.*

Proof: The functions $\theta(E,h)$ and $\sigma_{\rm e}(\lambda,h)$ are holomorphic in the required domain by Lemma 2.1 and Proposition 2.6. It remains toshow that the last term of R_{II} is also holomorphic in $D(Ch/\log(1/h))$. Let us calculate the derivative:

$$h\left\{\tan^{-1}\left(\gamma\tan\frac{\sigma_{\rm i}}{h}\right)\right\}' = \frac{\gamma\sigma_{\rm i}' + h\gamma'\cos(\sigma_{\rm i}/h)\sin(\sigma_{\rm i}/h)}{(1-\gamma^2)\cos^2(\sigma_{\rm i}/h)+\gamma^2}. \qquad (2.10)$$

Both γ and $\sigma_{\rm i}$ being holomorphic, it suffices to see that the denominator $D(\lambda,h) = (1-\gamma^2)\cos^2(\sigma_{\rm i}/h)+\gamma^2$ does not vanish in $D(Ch/\log(1/h))$. First we see that for real λ in this domain, $d(\lambda,h)$ is real and bounded from below by a positive constant independent of both λ and h. Next for complex λ, we see

$$\gamma(\lambda,h)\to\frac{1}{3},\quad |\operatorname{Im}\frac{\sigma_{\rm i}}{h}| \le C\frac{\rho_1+\rho_2}{2} + O\left(\frac{1}{\log(1/h)}\right),$$

as h tends to 0. Hence, by continuity, $d(\lambda,h)$ stays away from 0 for sufficiently small C and h. □

Proposition 2.8 enables us to estimate the derivatives of R_I and R_{II} in terms of themselves by Cauchy's integral formula; if a function $R(\lambda)$ is holomorphic in $\overline{D(r)}$, then its derivative is bounded from above in $D(r/2)$ by $2\sup_{D(r)}|R(\lambda)|/r$. Recalling that $R_I = O(h^2\log(1/h))$ and $R_{II} = O(h^2\log(1/h))$, we obtain

$$\frac{dR_I}{d\lambda} = O(h\log h^{-1}),\quad \frac{dR_{II}}{d\lambda} = O(h(\log h)^2).$$

On the other hand, we know from Proposition 2.6 that

$$\frac{d\sigma_{\rm e,i}}{dE} = \frac{\rho_1+\rho_2}{2}\log\frac{1}{h} + O(1)$$

and since $hd\gamma/dE = O(1)$

$$h\frac{d}{dE}\left\{\tan^{-1}\left(\gamma\tan\frac{\sigma_{\rm i}}{h}\right)\right\} = \frac{\rho_1+\rho_*}{2}\frac{\gamma}{(1-\gamma^2)\cos^2(\sigma_{\rm i}/h)+\gamma^2}\log\frac{1}{h} + O(1)$$

This completes the proof of Corollary 1.6.

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