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CONVERGENCE OF FORMAL SOLUTIONS OF SINGULAR FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF TOTALLY CHARACTERISTIC TYPE

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1. INTRODUCTION

Let \((t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n) \in \mathbb{C}^d \times \mathbb{C}^n\) be \((d+n)\)-dimensional complex variables \((d \geq 1, n \geq 1)\).

We consider the following first order nonlinear partial differential equation:

\[
\left\{ \begin{array}{l}
\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} u + \sum_{k=1}^{n} b_k(x) \partial_{x_k} u + c(x) u \\
\quad = \sum_{|l|=K} d_l(x) t^l + f_{K+1}(t, x, u, \{\partial_{t_j} u\}, \{\partial_{x_k} u\}), \\
u(t, x) = O(|t|^K), \end{array} \right.
\]

where \(|t| = t_1 + \cdots + t_d\), \(K\) is a fixed positive integer satisfying \(K \geq 2\) and \(a_{ij}(x), b_k(x), c(x)\) and \(d_l(x)\) are holomorphic in a neighbourhood of the origin, and \(f_{K+1}(t, x, u, \tau, \xi)\) \((\tau = (\tau_j) \in \mathbb{C}^d, \xi = (\xi_k) \in \mathbb{C}^n)\) is also holomorphic in a neighbourhood of the origin with the following Taylor expansion:

\[
f_{K+1}(t, x, u, \tau, \xi) = \sum_{|p|+K|q|+(K-1)|r|+K|s| \geq K+1} f_{pqrs}(x) t^p u^q \tau^r \xi^s,
\]

where \(q \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}, p = (p_1, \ldots, p_d) \in (\mathbb{Z}_{\geq 0})^d, r = (r_1, \ldots, r_d) \in (\mathbb{Z}_{\geq 0})^d,\)

\(s = (s_1, \ldots, s_n) \in (\mathbb{Z}_{\geq 0})^n,\)

\(|p| = p_1 + \cdots + p_d, \quad |r| = r_1 + \cdots + r_d, \quad |s| = s_1 + \cdots + s_n,\)

and

\[
t^p = \prod_{j=1}^{d} t_j^{p_j}, \quad \tau^r = \prod_{j=1}^{d} \tau_j^{r_j}, \quad \xi^s = \prod_{k=1}^{n} \xi_k^{s_k}.
\]

This equation seems to be a natural extension of totally characteristic type studied by Chen-Tahara ([CT]) to several time-space variables.
Here we remark that the assumption $K \geq 2$ implies $\partial_{t_j}u(0,0) = 0$ ($j = 1, 2, \ldots, d$) which assures that $(0,0,u(0,0),\{\partial_{t_j}u(0,0)\},\{\partial_{x_k}u(0,0)\})$ belongs to the domain of definition of $f_{K+1}(t,x,u,\tau,\xi)$.

Now our first theorem is stated as follows:

**Theorem 1.** Let $\{\lambda_j\}_{j=1}^{d}$ be the eigenvalues of the matrix $(a_{ij}(0))$. We assume that $b_k(x) \neq 0$ and $b_k(0) = 0$ for $k = 1, 2, \ldots, n$, and let $\{\mu_k\}_{k=1}^{n}$ be the eigenvalues of Jacobi matrix of $(b_1(x), \ldots, b_n(x))$ at $x = 0$. Then the formal power series solution of (1.1) exists uniquely and converges if the following conditions are satisfied:

There exists a positive constant $\sigma_0 > 0$, such that

$$\left| \sum_{j=1}^{d}\lambda_j l_j + \sum_{k=1}^{n}\mu_k m_k \right| \geq \sigma_0(|l| + |m|) \quad \text{(Poincaré condition)},$$

and

$$\sum_{j=1}^{d}\lambda_j l_j + \sum_{k=1}^{n}\mu_k m_k + c(0) \neq 0 \quad \text{(Non-resonance condition)}$$

hold for all $(l,m) \in (\mathbb{Z}_{\geq 0})^{d} \times (\mathbb{Z}_{\geq 0})^{n}$ with $|l| \geq K$ and $|m| \geq 0$.

**Remark 1.** It is easy to show the following proposition.

The conditions (1.2) and (1.3) imply that

$$\left| \sum_{j=1}^{d}\lambda_j l_j + \sum_{k=1}^{n}\mu_k m_k + c(0) \right| \geq \sigma(|l| + |m|)$$

holds by some positive constant $\sigma > 0$ for all $(l,m) \in (\mathbb{Z}_{\geq 0})^{d} \times (\mathbb{Z}_{\geq 0})^{n}$ with $|l| \geq K$ and $|m| \geq 0$. In the proof of Theorem 1, this condition will be used instead of (1.2) and (1.3).

Next, we consider the following general equation:

$$\left\{ \begin{array}{l}
 f(t,x,u(t,x),\{\partial_{t_j}u(t,x)\},\{\partial_{x_k}u(t,x)\}) = 0, \\
 u(0,x) \equiv 0.
\end{array} \right.$$

**Assumption 1.** $f(t,x,u,\tau,\xi)$ ($\tau = (\tau_j) \in \mathbb{C}^d$, $\xi = (\xi_k) \in \mathbb{C}^n$) is holomorphic in a neighbourhood of the origin, and is an entire function in $\tau$ variables for any fixed $t$, $x$, $u$ and $\xi$. Moreover we assume that

$$f(0,x,0,\tau,0) \equiv 0$$

for $x \in \mathbb{C}^n$ near the origin and $\tau \in \mathbb{C}^d$, which is a generalization of the definition of singular equations defined in [MS].
For the equation (1.5), we do not know whether or not the equation has a formal solution in general. Therefore, we assume the following:

**Assumption 2.** The equation (1.5) has a formal solution of the form

\[ u(t, x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|l| \geq 2, |m| \geq 0} u_{lm} t^l x^m \in \mathbb{C}[t, x]. \]

By the existence of a formal solution, \{\varphi_j(x)\} satisfy the following system formally:

\[ f(0, x, 0, \{\varphi_j(x)\}, 0) \equiv 0 \quad (\text{trivial relation}), \]

and

\[ \left. \frac{\partial}{\partial t_i} f(t, x, u(t, x), \{\partial_{t_j} u(t, x)\}, \{\partial_{x_k} u(t, x)\}) \right|_{t=0} = \frac{\partial f}{\partial t_i}(0, x, 0, \{\varphi_j(x)\}, 0) \varphi_i(x) + \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0, x, 0, \{\varphi_j(x)\}, 0) \frac{\partial \varphi_i}{\partial x_k}(x) = 0, \text{ for } i = 1, 2, \ldots, d. \]

The formal solution of this system is not convergent in general. Therefore, we assume

**Assumption 3.** The coefficients \{\varphi_j(x)\} are all holomorphic in a neighbourhood of the origin of \(\mathbb{C}^n\).

**Remark 2.** In the case \(d = 1\) (\(d\) is the dimension of \(t\) variables), a sufficient condition for the formal solution of (1.9) to converge has been already obtained by Miyake-Shirai [MS]. In the case \(d \geq 2\), we give a sufficient condition for the formal solution of system (1.9) to be convergent, which will be given by Theorem 3 in Section 5, but for a while we consider the problem under Assumption 3 for simplicity of our arguments.

Now we put \(a(x) = (0, x, 0, \{\varphi_j(x)\}, 0)\) for simplicity, and define

\[ A_{ij}(x) := \frac{\partial^2 f}{\partial t_i \partial t_j}(a(x)) + \frac{\partial^2 f}{\partial u \partial t_j}(a(x)) \varphi_i(x) + \sum_{k=1}^{n} \frac{\partial^2 f}{\partial \tau_j \partial \xi_k}(a(x)) \frac{\partial \varphi_i}{\partial x_k}(x), \]

for \(i, j = 1, 2, \ldots, d\). Moreover we define

\[ B_k(x) := \frac{\partial f}{\partial \xi_k}(a(x)), \quad \text{for } k = 1, 2, \ldots, n. \]

**Remark 3.** The functions \(A_{ij}(x)\) and \(B_k(x)\) correspond to \(a_{ij}(x)\) and \(b_k(x)\) in Theorem 1, respectively (see (1.13) below).

Here we assume that the equation is of totally characteristic type, that is,
Assumption 4. $B_k(x) \not\equiv 0$ and $B_k(0) = 0$, for $k = 1, 2, \ldots, n$.

Now our second theorem which is our main result is stated as follows:

**Theorem 2.** Suppose Assumptions 1, 2, 3 and 4. Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of $(A_{ij}(0))$, and let $\{\mu_k\}_{k=1}^n$ be the eigenvalues of Jacobi matrix of the vector $(B_k(x))$ at $x = 0$. Then the formal solution (1.7) is convergent if the following condition is satisfied:

There exists a positive constant $\sigma > 0$, such that,

\[(1.12) \quad \left| \sum_{j=1}^d \lambda_j l_j + \sum_{k=1}^n \mu_k m_k + \frac{\partial f}{\partial u}(a(0)) \right| \geq \sigma (|l| + |m|),\]

holds for all $(l, m) \in (\mathbb{Z}_{\geq 0})^d \times (\mathbb{Z}_{\geq 0})^n$ with $|l| \geq 2$, $|m| \geq 0$.

**Remark 4.** We put $v(t, x) = u(t, x) - \sum_{j=1}^d \varphi_j(x) t_j$ as an new unknown function. By Assumptions 1, 2, 3 and 4, we can easily see that $v(t, x)$ satisfies the equation of the following form:

\[(1.13) \quad \begin{cases} \sum_{i,j=1}^d A_{ij}(x) t_i \partial_{t_j} v + \sum_{k=1}^n B_k(x) \partial_{x_k} v + \frac{\partial f}{\partial u}(a(x)) v \\ = \sum_{|l|=2} d_l(x) t^l + f_3(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}) \end{cases},
\]

$v(t, x) = O(|t|^2)$.

This is an equation considered in Theorem 1 in the case $K = 2$. Therefore, it is sufficient to prove Theorem 1 in order to prove Theorem 2.

\[\square\]

2. Reduction of the equation

As is mentioned in Remark 4, it is sufficient to study the equation (1.1).

By the assumption of Theorem 1,

\[(a_{ij}(0)) \sim \begin{pmatrix} \lambda_1 & \delta_1 \\ \lambda_2 & \cdots \\ \cdots & \delta_{d-1} \\ \lambda_d \end{pmatrix}, \quad \frac{\partial (b_1, \ldots, b_n)}{\partial (x_1, \ldots, x_n)}|_{x=0} \sim \begin{pmatrix} \mu_1 & \nu_1 \\ \mu_2 & \cdots \\ \cdots & \nu_{n-1} \\ \mu_n \end{pmatrix},\]

where $\delta_j$, $\nu_k = 0$ or 1 $(1 \leq j \leq d - 1, 1 \leq k \leq n - 1)$.

Then by transforming the variables, (1.1) is reduced to the following form:

\[(2.1) \quad (\Lambda + \Delta)v(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^d \beta_{ij}(x) t_i \partial_{t_j} v + \gamma(x) v + \sum_{k=1}^n \varphi_k(x) \partial_{x_k} v + f_{K+1}(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}),\]
with \( v(t, x) = O(|t|^K) \), where
\[
\Lambda = \sum_{j=1}^{d} \lambda_{j} t_{j} \partial_{t_{j}} + \sum_{k=1}^{n} \mu_{k} x_{k} \partial_{x_{k}} + c(0),
\]
and
\[
\Delta = \sum_{j=1}^{d-1} \delta_{j} t_{j+1} \partial_{t_{j+1}} + \sum_{k=1}^{n-1} \nu_{k} x_{k+1} \partial_{x_{k+1}},
\]
and \( \alpha_{l}(x) \), \( \beta_{ij}(x) \), \( \gamma(x) \) and \( \varphi_{k}(x) \) are holomorphic in a neighbourhood of the origin, and satisfy \( \beta_{ij}(x) = O(|x|) \), \( \gamma(x) = O(|x|) \) and \( \varphi_{k}(x) = O(|x|^2) \), and \( \tilde{f}_{K+1}(t, x, u, \tau, \xi) \) is a holomorphic function which has a similar Taylor expansion with \( f_{K+1}(t, x, u, \tau, \xi) \).

In the following sections, we shall prove the existence and convergence of the unique formal solution of (2.1).

3. PREPARATION TO PROVE THEOREM 1

Let \( C[t, x]_{L,M} \) be the set of homogeneous polynomial of degree \( L \) in \( t \) variables and of degree \( M \) in \( x \) variables, that is,
\[
C[t, x]_{L,M} = \left\{ f_{LM}(t, x) = \sum_{|l|=L, |m|=M} f_{lm} t^{l} x^{m} \mid f_{lm} \in C \right\}.
\]
For the operator \( \Lambda + \Delta \), the following lemma holds:

**Lemma 1.** For all \( L \geq K \) and \( M \geq 0 \), the operator
\[
\Lambda + \Delta : C[t, x]_{L,M} \longrightarrow C[t, x]_{L,M}
\]
is invertible. Moreover, if the majorant relation \( f_{LM}(t, x) \ll F \times (t_{1} + \cdots + t_{d})^{L}(x_{1} + \cdots + x_{n})^{M} \) \((f_{LM}(x) \in C[t, x]_{L,M}, F > 0) \) holds, then we obtain the following majorant relation:
\[
(3.1) \quad (\Lambda + \Delta)^{-1} f_{LM}(t, x) \ll \frac{C}{L+M} F \times (t_{1} + \cdots + t_{d})^{L}(x_{1} + \cdots + x_{n})^{M},
\]
where \( C > 0 \) is a positive constant independent of \( L \) and \( M \).

**Proof.** We define a norm of \( u_{LM}(t, x) \in C[t, x]_{L,M} \) by
\[
||u_{LM}|| := \inf \left\{ C > 0 \mid u_{LM}(t, x) \ll C(t_{1} + \cdots + t_{d})^{L}(x_{1} + \cdots + x_{n})^{M} \right\}.
\]
We remark that \( C[t, x]_{L,M} \) becomes a Banach space by this norm.

First, by (1.4) it is easily checked that \( \Lambda \) is invertible on \( C[t, x]_{L,M} \) and
\[
(3.2) \quad ||\Lambda^{-1}|| \leq \frac{1}{\sigma(L+M)}
\]
holds for the operator norm of \( \Lambda^{-1} \) on \( C[t, x]_{L,M} \).
Next, since $u_{LM}(t, x) \ll ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$, we have

$$\Delta u_{LM}(t, x) \ll \sum_{j=1}^{d-1} L|\delta_j| \cdot ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$$

$$+ \sum_{k=1}^{n-1} M|\nu_k| \cdot ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$$

$$\ll \left\{ L(d-1) \max_{j=1, \ldots, d-1} |\delta_j| + M(n-1) \max_{k=1, \ldots, n-1} |\nu_k| \right\} \times$$

$$\times ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M.$$

Here we make a change of variables by $t_j = \epsilon^{j-1} \tau_j$, $x_k = \epsilon^{k-1} y_k$, then $\delta_j$ and $\nu_k$ (the components of nilpotent part of Jordan canonical form) turns to $\epsilon \delta_j$ and $\epsilon \nu_k$, respectively. Therefore, by choosing $\epsilon$ sufficiently small, we may assume that the components of nilpotent part are small enough. Hence we may assume that

$$(3.3) \quad \max_{j=1, \ldots, d-1} |\delta_j| < \frac{\sigma}{2(d-1)}, \quad \max_{k=1, \ldots, n-1} |\nu_k| < \frac{\sigma}{2(n-1)}.$$

Then

$$\Delta u_{LM}(t, x) \ll \frac{\sigma(L + M)}{2} ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$$

holds, and we obtain

$$||\Delta|| \leq \frac{\sigma(L + M)}{2}.$$

Therefore, the operator norm of $\Delta \Lambda^{-1}$ is estimated by

$$||\Delta \Lambda^{-1}|| \leq \frac{1}{\sigma(L + M)} \frac{\sigma(L + M)}{2} = \frac{1}{2} < 1.$$

By using the Neumann's series, we can see that $\Lambda + \Delta$ is invertible and the norm of the inverse operator is estimated by

$$||(\Lambda + \Delta)^{-1}|| \leq \frac{2}{\sigma \sigma M},$$

which we want to prove since $C = 2/\sigma$ is independent of $L$ and $M$.

Now, we define some notations, which are used in the proof of Theorem 1.

**Definition** (1) Let $(t, x) \in \mathbb{C}^d \times \mathbb{C}^n$ $(d \geq 0, \ n \geq 0)$ be complex variables. For formal power series $f(t, x) = \sum_{|l| \geq 0, \ |m| \geq 0} f_{l,m} t^l x^m$, we define

$$|f|(t, x) = \sum_{|l| \geq 0, \ |m| \geq 0} |f_{l,m}| t^l x^m.$$
(2) Let \((t, X) \in \mathbb{C}^d \times \mathbb{C} \ (d \geq 0)\) be complex variables. For formal power series 
\(f(t, X) = \sum_{|l|\geq 0, M \geq 0} f_{l,M} t^l X^M\), we define the shift operator \(S\) by
\[
S(f)(t, X) = \sum_{|l|\geq 0, M \geq 0} f_{l,M+1} t^l X^M = \frac{f(t, X) - f(t, 0)}{X}.
\]

**Remark 5.** The following facts are easily shown:
- \(f(t, x) \ll |f|(t, x)\);
- If \(f(t, x)\) and \(g(t, X)\) are convergent power series, then \(|f|(t, x)\) and \(S(g)(t, X)\) are also convergent. \(\Box\)

4. **Proof of Theorem 1**

First, we prove a unique existence of formal power series solution. Let
\[
u(t, x) = \sum_{|l|\geq K, |m| \geq 0} u_{lm} t^l x^m = \sum_{L \geq K} u_L(t, x) = \sum_{L \geq K, M \geq 0} u_{LM}(t, x)
\]
be a formal solution of (2.1), where
\[
u_{LM}(t, x) = \sum_{|l|=L, |m|=M} u_{lm} t^l x^m \in \mathbb{C}[t, x]_{L,M},
\]
\[
u_L(t, x) = \sum_{|l|=L} u_{l}(x) t^l = \sum_{M \geq 0} u_{LM}(t, x).
\]

We put \(P = \Lambda + \Delta\) for simplicity. We substitute \(u(t, x) = \sum_{L \geq K} u_L(t, x)\) into (2.1), then we have the following recursion formula:
\[
\begin{cases}
P u_K(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} u_K(t, x) \\
\quad + \gamma(x) u_K(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} u_K(t, x),
\end{cases}
\]
\[
P u_L(t, x) = \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} u_L(t, x) + \gamma(x) u_L(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} u_L(t, x)
\] 
\[+ G_L(t, x, \{u_p\}_{K \leq p < L}, \{\partial_{t_j} u_p\}_{K \leq p < L}, \{\partial_{x_k} u_p\}_{K \leq p < L}), \text{ for } L > K,
\]
where \(G_L(t, x, \zeta, \tau, \xi)\) is a polynomial of \((t, \zeta, \tau, \xi)\).
First, we consider the case $L = K$. We substitute $u_K(t, x) = \sum_{M \geq 0} u_{KM}(t, x)$ into the above recursion formula, we have

\[
\begin{cases}
Pu_{K0}(t, x) = \sum_{|l|=K} \alpha_l(0)t^l, \\
Pu_{KM}(t, x) = \sum_{|l|=K} \alpha^M_l(x)t^l + \sum_{i,j=1}^{d} \sum_{p=1}^{M} \beta^p_{ij}(x)t_i \partial_{t_j} u_{K,M-p}(t, x) \\
&+ \sum_{p=1}^{M} \gamma^p(x)u_{K,M-p}(t, x) + \sum_{k=1}^{n} \sum_{p=2}^{M} \varphi^p_k(x) \partial_{x_k} u_{K,M-p+1}(t, x),
\end{cases}
\]

where we put

\[
\alpha_l(x) = \sum_{M \geq 0} \alpha^M_l(x), \quad \alpha^M_l(x) = \sum_{|m|=M} \alpha_{lm} x^m,
\]

\[
\beta_{ij}(x) = \sum_{M \geq 1} \beta^M_{ij}(x), \quad \beta^M_{ij}(x) = \sum_{|m|=M} \beta_{ijm} x^m,
\]

\[
\gamma(x) = \sum_{M \geq 1} \gamma^M(x), \quad \gamma^M(x) = \sum_{|m|=M} \gamma_{m} x^m,
\]

\[
\varphi_k(x) = \sum_{M \geq 2} \varphi^M_k(x), \quad \varphi^M_k(x) = \sum_{|m|=M} \varphi_{km} x^m.
\]

By Lemma 1, we know that the solution sequence $\{u_{KM}(t, x)\}_{M \geq 0}$ exists uniquely. Moreover, by the same argument, we see that $\{u_{LM}(t, x)\}$ ($L > K$) exist uniquely. These show that the formal solution exists uniquely.

Next, we prove the convergence of the formal solution. We put $U(t, x) = Pu(t, x)$ as a new unknown function. By Lemma 1, the equation (2.1) is reduced to the following equation:

\[(4.1) \quad U(t, x) = \sum_{|l|=K} \alpha_l(x)t^l + \sum_{i,j=1}^{d} \beta^p_{ij}(x)t_i \partial_{t_j} P^{-1}U(t, x) \]
\[+ \gamma(x)P^{-1}U(t, x) + \sum_{k=1}^{n} \varphi^p_k(x) \partial_{x_k} P^{-1}U(t, x) \]
\[+ \tilde{f}_{K+1}(t, x, P^{-1}U(t, x), \{\partial_{t_j} P^{-1}U(t, x)\}, \{\partial_{x_k} P^{-1}U(t, x)\}).\]

We know that (4.1) has a unique formal solution of the form

\[U(t, x) = \sum_{|l| \geq K, |m| \geq 0} U_{lm} t^l x^m \quad \Rightarrow \quad U_L(t, x) = \sum_{L \geq K} U_{LM}(t, x).\]
In order to get a majorant series of $U(t, x)$, we prepare some majorant series for the coefficients of (4.1). We put $T = t_1 + \cdots + t_d$, $X = x_1 + \cdots + x_n$, and choose
\[
\sum_{|l|=K} \alpha_l(x)t^l \ll A(X)T^K, \quad \beta_{ij}(x) \ll |\beta_{ij}|(X, \ldots, X) =: XB_{ij}(X),
\]
\[
\gamma(x) \ll |\gamma|(X, \ldots, X) =: XG(X), \quad \varphi_k(x) \ll |\varphi_k|(X, \ldots, X) =: X^2\Phi_k(X),
\]
\[
\overline{f}_{K+1}(t, x, u, \tau, \xi) \ll |\tilde{f}_{K+1}|(T, \ldots, T, X, \ldots, X, u, \tau, \xi) =: F_{K+1}(T, X, u, \tau, \xi)
\]
where $A(X)$, $B_{ij}(X)$, $G(X)$ and $\Phi_k(X)$ are holomorphic in a neighbourhood of $X = 0$, and $F_{K+1}(T, X, u, \tau, \xi)$ is also holomorphic near $(T, X, u, \tau, \xi) = (0, 0, 0, 0, 0)$.

Now, we consider the following equation:
\[
(4.2) \quad w(T, X) = A(X)T^K + C \sum_{i,j=1}^{d} XB_{ij}(X)w(T, X)
\]
\[
+ CXG(X)w(T, X) + C \sum_{k=1}^{n} X^2\Phi_k(x)(t, x)S(w)(T, X)
\]
\[
+ F_{K+1}(T, X, u, \tau, \xi)
\]
where $C$ is a positive constant appeared in Lemma 1.

Let $w(T, X) = \sum_{L \geq K, M \geq 0} w_{LM}(T, X)$ be the formal solution of (4.2). By the construction of (4.2), we can easily check that $U(t, x) \ll w(T, X)$ by the next lemma.

**Lemma 2.** For two formal power series $U(t, x)$ and $w(T, X)$ satisfying
\[
U(t, x) = \sum_{L \geq K, M \geq 0} U_{LM}(t, x) \ll w(T, X) = \sum_{L \geq K, M \geq 0} w_{LM}T^L X^M,
\]
the following majorant relations hold:
(1) $P^{-1}U(t, x) \ll Cw(T, X)$,
(2) $t_i \partial_{t_i} P^{-1}U(t, x) \ll Cw(T, X)$,
(3) $\partial_{t_i} P^{-1}U(t, x) \ll \frac{Cw(T, X)}{T}$,
(4) $\partial_{x_k} P^{-1}U(t, x) \ll CS(w)(T, X)$.

**Proof.** By using Lemma 1, we can prove this lemma easily. First, (1) is proved as follows:
\[
P^{-1}U(t, x) = \sum_{L \geq K, M \geq 0} P^{-1}U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{C}{L + M} w_{LM}T^L X^M \ll Cw(T, X).
\]
Secondly, (2) and (3) is proved as follows:

\[
t_i \partial t_j P^{-1}U(t, x) = \sum_{L \geq K, M \geq 0} t_i \partial t_j P^{-1}U_{LM}(t, x) \\
\ll \sum_{L \geq K, M \geq 0} \frac{CL}{L+M} w_{LM} T^L X^M \ll Cw(T, X);
\]

\[
\partial t_j P^{-1}U(t, x) = \sum_{L \geq K, M \geq 0} \partial t_j P^{-1}U_{LM}(t, x) \\
\ll \sum_{L \geq K, M \geq 0} \frac{CL}{L+M} w_{LM} T^{L-1} X^M \ll \frac{Cw(T,X)}{T}.
\]

Finally, (4) is proved as follows:

\[
\partial_{x_k} P^{-1}U(t, x) = \sum_{L \geq K, M \geq 1} \partial_{x_k} P^{-1}U_{LM}(t, x) \\
\ll \sum_{L \geq K, M \geq 1} \frac{CM}{L+M} w_{LM} T^L X^{M-1} \ll CS(w)(T, X).
\]

This completes the proof. \(\square\)

Since \(w(T, X) \gg 0\), we have

\[XS(w)(T, X) = w(T, X) - w(T, 0) \ll w(T, X).
\]

Let us consider the following equation:

\[
v(T, X) = A(X)T^K + CXh(X)v(T, X) \\
+ F_{K+1} \left(T, X, Cv, \left\{ \frac{Cv}{T} \right\}, \{CS(v)\}\right),
\]

with \(v(T, X) = O(T^K)\), where \(h(X) = \sum_{i,j=1}^d B_{ij}(X) + G(X) + \sum_{k=1}^n \Phi_k(X)\). Then the following majorant relation is obvious:

\[w(T, X) \ll v(T, X).
\]

We put \(y(T, X) = v(T, X)/T\) as a new unknown function. By substituting this into (4.4), we see that \(y(T, X)\) satisfies

\[
y(T, X) = A(X)T^{K-1} + CXh(X)y(T, X) \\
+ \frac{1}{T} F_{K+1} \left(T, X, CTy, \{Cy\}, \{CTS(y)\}\right),
\]

with \(y(T, X) = O(T^{K-1})\).

We decompose the formal solution \(y(T, X)\) as follows:

\[y(T, X) = y_1(X)T^{K-1} + y_2(X)T^K + T^K z(T, X).
\]
We remark that $y_1(X)$ and $y_2(X)$ are holomorphic functions in a neighbourhood of $X = 0$. Indeed, $y_1(X)$ and $y_2(X)$ are given by

$$y_1(X) = \frac{A(X)}{1 - CXh(X)},$$

$$y_2(X) = \frac{1}{1 - CXh(X)} \sum_{|p|+Kq+(K-1)r+Ks=K+1} F_{pqrs}(X) \{Cy_1(X)\}^{q+|r|} \{CS(y_1)(X)\}^{s}.$$  

These are holomorphic functions in a neighbourhood of $X = 0$.

In this case, $z(T, X)$ satisfies the following equation:

$$z(T, X) = CXh(X)z(T, X) + H(T, X, Tz(T, X), TS(z)(T, X)),
\quad z(0, X) \equiv 0,$$

where

$$H(T, X, \eta_1, \eta_2) = \frac{1}{T^{K+1}} \left[ F_{K+1}(T, X, Cy_1(X)T^K + Cy_2(X)T^{K+1} + CT^{K-1}\eta_1,
\{Cy_1(X)T^{K-1} + Cy_2(X)T^K + CT^{K-1}\eta_1\},
\{CS(y_1)(X)T^K + CS(y_2)(X)T^{K+1} + CT^K\eta_2\}\right] - \sum_{|p|+Kq+(K-1)r+Ks=K+1} F_{pqrs}(X) (Cy_1(X))^{q+|r|} (CS(y_1)(X))^{s}.$$  

**Remark 6.** The order of zeros in $T$ variable of $H(T, X, CTz(T, X), CTS(z)(T, X))$ is greater than or equal to 1. \qed

In order to prove the convergence of $z(T, X)$, it is sufficient to show the following:

**Lemma 3.** There exists a small positive constant $\varepsilon > 0$ such that $z_\varepsilon(\rho) = z(\varepsilon\rho, \rho)$ is convergent in a neighbourhood of $\rho = 0$.

*Proof.* We substitute $T = \varepsilon\rho$ and $X = \rho$ into (4.6) and by using the relation (4.3), we have

$$\rho S(z)(\varepsilon\rho, \rho) \ll z_\varepsilon(\rho).$$

By this relation, the following majorant relation also holds,

$$TS(z)(T, X)|_{T=\varepsilon\rho, X=\rho} = \varepsilon \rho S(z)(\varepsilon\rho, \rho) \ll \varepsilon z_\varepsilon(\rho).$$

Here we consider

$$\psi(\rho) = Cph(\rho)\psi(\rho) + H(\varepsilon\rho, \varepsilon\rho\psi(\rho), \varepsilon\psi(\rho)).$$  

(4.7)
In the right hand side of (4.7), we decompose $H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho))$ into the term of $\psi(\rho)$ and otherwise as follows:

$$H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)) = \frac{\partial H}{\partial \eta_2}(0, 0, 0, 0)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).$$

We remark that the following fact holds:

$$\frac{\partial \overline{H}}{\partial \psi}(\epsilon\rho, \rho, \epsilon\rho\psi, \epsilon\psi)|_{\rho=0,\psi=0}=0.$$

We put $(\partial H/\partial \eta_2)(0,0,0,0)=K_0 \geq 0$, then (4.7) is rewritten by

$$(1-\epsilon K_0)\psi(\rho) = C\rho h(\rho)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).$$

We choose $\epsilon > 0$ with $1-\epsilon K_0 > 0$. Then by using the implicit function theorem, we can see that (5.8) has a unique holomorphic solution $\psi(\rho)$ with $\psi(0) = 0$ in a neighbourhood of $\rho = 0$. Moreover we can see $z_\epsilon(\rho) \ll \psi(\rho)$.

Thus we complete the proof of Lemma 3.

\[\square\]

5. SOLVABILITY OF THE SYSTEM (1.9)

In this section, we give a sufficient condition for the formal solution of the system (1.9) to be convergent. Recall that (1.9) is

$$\frac{\partial f}{\partial t_i}(0, x, 0, \{\varphi_j(x)\}, 0) + \frac{\partial f}{\partial u}(0, x, 0, \{\varphi_j(x)\}, 0)\varphi_i(x)$$

$$+ \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0, x, 0, \{\varphi_j(x)\}, 0)\frac{\partial \varphi_i(x)}{\partial x_k} = 0, \quad i=1,2,\ldots,d.$$  

By Assumption 4 of Theorem 2, the condition

$$\frac{\partial f}{\partial \xi_k}(0,0,0,\{\varphi_j(0)\},0)=0, \quad k=1,2,\ldots,n$$

was assumed.

Let $\varphi(x)={}^t(\varphi_1(x), \ldots, \varphi_d(x))$ be the unknown functions. We put $\varphi(0)={}^t(\varphi_1^0, \ldots, \varphi_d^0) \in C^d$ as the constant term of $\varphi(x)$. We substitute $\varphi_j(x) = \varphi_j^0 + \psi_j(x)$ into the system (1.9), and by restricting at $x=0$, we see that $\{\varphi_j^0\}$ satisfies the following system:

$$(5.1) \quad \frac{\partial f}{\partial t_i}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 = 0, \quad i=1,2,\ldots,d.$$ 

This system has some roots by the assumption of the existence of a formal solution, and we fix such $\{\varphi_j^0\}$. 
For such fixed \( \{\varphi_j^0\} \), we see that \( \{\psi_j(x)\} \) satisfies the system of the followin

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0,0,0,\{\varphi_j^0\},0)x_l \frac{\partial \psi_i}{\partial x_k}(x) + 
\sum_{k=1}^{n} \sum_{p=1}^{d} \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0,0,0,\{\varphi_j^0\},0)\psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x) + 
\frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\psi_i(x) + 
\sum_{p=1}^{d} \left\{ \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 \right\} \psi_p(x) + 
\sum_{l=1}^{n} \left\{ \frac{\partial^2 f}{\partial t_i \partial x_l}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 \right\} x_l = (\text{degree in } x \text{ is greater than or equal to } 2), \quad i = 1,2,\ldots,d.
\]

This system is written as follows for simplicity,

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \frac{\partial \psi_i}{\partial x_k}(x) + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x) + c \psi_i(x) + \sum_{p=1}^{d} d_{ip} \psi_p(x) + \sum_{l=1}^{n} e_l x_l = (\text{degree in } x \text{ is greater than or equal to } 2), \quad i = 1,2,\ldots,d,
\]

where

\[
a_{kl} := \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0,0,0,\{\varphi_j^0\},0), \quad b_{kp} := \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0,0,0,\{\varphi_j^0\},0),
\]

\[
c := \frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0),
\]

\[
d_{ip} := \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0,
\]

\[
e_l := \frac{\partial^2 f}{\partial t_i \partial x_l}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0.
\]
Here we decompose $\psi_i(x)$ into $\psi_i(x) = \tilde{\psi}_i(x) + \eta_i(x)$ ($\tilde{\psi}_i(x) = \sum_{k=1}^{n} \psi_{ik} x_k$, $\eta_i(x) = O(|x|^2)$). We substitute this into the system (5.3) and obtain

\[ (5.4) \]
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \left( \frac{\partial \tilde{\psi}_i}{\partial x_k} + \frac{\partial \eta_i}{\partial x_k} \right) + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} (\tilde{\psi}_p(x) + \eta_p(x)) \left( \frac{\partial \tilde{\psi}_i}{\partial x_k} + \frac{\partial \eta_i}{\partial x_k} \right) \\
+ c(\tilde{\psi}_i(x) + \eta_i(x)) + \sum_{p=1}^{d} d_{ip} (\tilde{\psi}_p(x) + \eta_p(x)) + \sum_{l=1}^{n} e_{il} x_l \\
= \text{(degree in } x \text{ is greater than or equal to } 2), \quad i = 1, 2, \ldots, d.
\]

By picking up the degree 1 part on the both sides, we see that $\{\tilde{\psi}_i(x)\}$ satisfy the following system:

\[ (5.5) \]
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \frac{\partial \tilde{\psi}_i}{\partial x_k} + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \tilde{\psi}_p(x) \frac{\partial \tilde{\psi}_i}{\partial x_k} + c\tilde{\psi}_i(x) + \sum_{p=1}^{d} d_{ip} \tilde{\psi}_p(x) + \sum_{l=1}^{n} e_{il} x_l = 0,
\]

for $i = 1, 2, \ldots, d$.

By the existence of a formal solution, (5.5) has some solutions $\{\tilde{\psi}_i(x)\}$ of linear functions, and we fix such $\{\tilde{\psi}_i(x)\}$.

For fixed $\{\varphi_i^0\}$ and $\{\tilde{\psi}_i(x)\}$, we see that $\{\eta_i(x)\}$ satisfy the following system:

\[ (5.6) \]
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \left( a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_{pl} \right) x_l \frac{\partial \eta_i}{\partial x_k} + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \eta_i(x) + \left( d_{ip} + \sum_{k=1}^{n} b_{kp} \psi_{ik} \right) \eta_p(x) \\
= \text{(degree in } x \text{ is greater than or equal to } 2.), \quad i = 1, 2, \ldots, d.
\]

We remark that the degree 2 part in the right hand side of this system does not include $\{\eta_i(x)\}$.

The following theorem holds:

**Theorem 3.** Let $(A_{kl})_{k,l=1,2,\ldots,n}$ be a matrix defined by

\[
(A_{kl})_{k,l=1,2,\ldots,n} = \left( a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_{pl} \right)_{k,l=1,2,\ldots,n}.
\]
Let \( \{\kappa_k\}_{k=1}^{n} \) be the eigenvalues of \((A_{kl})_{k,l=1,2,...,n}\). If there exists a positive constant \( \sigma_0 \) such that the condition
\[
\left| \sum_{k=1}^{n} \kappa_k m_k \right| \geq \sigma_0 |m|, \quad \text{(Poincaré condition)}
\]
holds for all \( m = (m_1, \ldots, m_n) \in (\mathbb{Z}_{\geq 0})^n \) with \( |m| \geq 2 \), then the formal solution of the system (1.9) is convergent in a neighbourhood of the origin.

**Remark 7.** Let \((B_{ip})_{i,p=1,2,...,d}\) be a matrix defined by
\[
(B_{ip})_{i,p=1,2,...,d} = \left( d_{ip} + \sum_{k=1}^{n} b_{kp}\psi_{ik} \right),
\]
and let \( \{\omega_j\}_{j=1}^{d} \) be the eigenvalues of \((B_{ip})_{i,p=1,2,...,d}\).

By the same argument in Remark 1, we have
\[
(5.7) \quad \left| \sum_{k=1}^{n} \kappa_k m_k + c + \omega_j \right| \geq \sigma|m|, \quad \text{by some } \sigma > 0, \text{ and } j = 1, 2, \ldots, d,
\]
for large \( m \), which will be used in the proof. \( \square \)

6. **Proof of Theorem 3**

The proof of Theorem 3 is the same method of the proof of Theorem 1 in case that the unknown function is a vector values. However, there are some difference in the detail. Therefore, we introduce only the outline of the proof of Theorem 3 in this section.

**Step 1.** By taking a linear transformation of the independent variables and a linear transformation of the unknown functions, (5.6) is reduced to the following form:

\[
(\Lambda + \Delta + B) \begin{pmatrix} w_1(x) \\
\vdots \\
w_d(x) \end{pmatrix} := \begin{pmatrix} \Lambda_1 \\
\vdots \\
\Lambda_d \end{pmatrix} + \begin{pmatrix} \Delta \\
\vdots \\
\Delta \end{pmatrix} + B \begin{pmatrix} w_1(x) \\
\vdots \\
w_d(x) \end{pmatrix} = \begin{pmatrix} \sum_{|m|=2} a_{1,m} x^m + g_{3,1}(x, w(x), \partial_x w(x)) \\
\vdots \\
\sum_{|m|=2} a_{d,m} x^m + g_{3,d}(x, w(x), \partial_x w(x)) \end{pmatrix},
\]
where \( w_j(x) \) \((j = 1, 2, \ldots, d)\) denote new unknown functions after linear transformations and

\[
\Lambda_j = \sum_{k=1}^{n} \kappa_k x_k \partial_{x_k} + c + \omega_j, \quad \Delta = \sum_{k=1}^{n-1} \varepsilon_k x_k \partial_{x_{k+1}}, \quad B = \begin{pmatrix} 0 & e_1 \\ & \ddots \\ & & \ddots \\ & & & e_{d-1} \\ & & & & 0 \end{pmatrix},
\]

where \( \varepsilon_j \) and \( e_j \) denote the nilpotent components of the Jordan canonical forms of the matrices \((A_{kl})\) and \((B_{ip})\), respectively, and

\[
g_{3,i}(x, \eta, \zeta) = \sum_{|\alpha|+2|\beta|+|\gamma| \geq 3} g_{\alpha\beta\gamma}^{(i)} x^\alpha \eta^\beta \zeta^\gamma.
\]

**Step 2.** We define \( C[x]_M \) by \( C[x]_M = \{ \sum_{|m|=M} u_m x^m ; u_m \in C \} \), and define a norm of \( u(x) = ^t(u_1(x), \ldots, u_d(x)) \in (C[x]_M)^d \) by

\[
||u|| := \inf \{ C > 0 ; u_i(x) \ll C(x_1 + \cdots + x_n)^M, \ i = 1, 2, \ldots, d \}.
\]

By the same argument in the proof of Lemma 1 and by Remark 7, we can prove the same results of Lemma 1 for the operator \( \Lambda + \Delta + B \).

**Step 3.** By the same method in the previous sections, we can construct a majorant equation whose formal solution is a majorant function of the all unknown functions of the system. Finally, by the implicit function theorem, we prove the convergence of the formal solution of the majorant equation.

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