Title

CONVERGENCE OF FORMAL SOLUTIONS OF SINGULAR FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF TOTALLY CHARACTERISTIC TYPE (Integral representations and twisted cohomology in the theory of differential equations)

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1. INTRODUCTION

Let \((t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n) \in \mathbb{C}^d \times \mathbb{C}^n\) be \((d+n)\)-dimensional complex variables \((d \geq 1, n \geq 1)\).

We consider the following first order nonlinear partial differential equation:

\[
\begin{aligned}
\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} u + \sum_{k=1}^{n} b_k(x) \partial_{x_k} u + c(x) u &= \sum_{|l|=K} d_l(x) t^l + f_{K+1}(t, x, u, \{\partial_{t_j} u\}, \{\partial_{x_k} u\}), \\
u(t, x) &= O(|t|^K),
\end{aligned}
\]

where \(|t| = t_1 + \cdots + t_d\), \(K\) is a fixed positive integer satisfying \(K \geq 2\) and \(a_{ij}(x), b_k(x), c(x)\) and \(d_l(x)\) are holomorphic in a neighbourhood of the origin, and \(f_{K+1}(t, x, u, \tau, \xi)\) \((\tau = (\tau_j) \in \mathbb{C}^d, \xi = (\xi_k) \in \mathbb{C}^n)\) is also holomorphic in a neighbourhood of the origin with the following Taylor expansion:

\[
f_{K+1}(t, x, u, \tau, \xi) = \sum_{|p|+Kq+(K-1)|r|+K|s| \geq K+1} f_{pqrs}(x) t^p u^q \tau^r \xi^s,
\]

where \(q \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}, p = (p_1, \ldots, p_d) \in (\mathbb{Z}_{\geq 0})^d, r = (r_1, \ldots, r_d) \in (\mathbb{Z}_{\geq 0})^d, s = (s_1, \ldots, s_n) \in (\mathbb{Z}_{\geq 0})^n,\)

\(|p| = p_1 + \cdots + p_d, \ |r| = r_1 + \cdots + r_d, \ |s| = s_1 + \cdots + s_n,\)

and

\[
t^p = \prod_{j=1}^{d} t_j^{p_j}, \quad \tau^r = \prod_{j=1}^{d} \tau_j^{r_j}, \quad \xi^s = \prod_{k=1}^{n} \xi_k^{s_k}.
\]

This equation seems to be a natural extension of totally characteristic type studied by Chen-Tahara ([CT]) to several time-space variables.
Here we remark that the assumption $K \geq 2$ implies $\partial_{t_{j}}u(0,0) = 0$ ($j = 1, 2, \ldots, d$) which assures that $(0, 0, u(0,0), \{\partial_{t_{j}}u(0,0)\}, \{\partial_{x_{k}}u(0,0)\})$ belongs to the domain of definition of $f_{K+1}(t,x,u,\tau,\xi)$.

Now our first theorem is stated as follows:

**Theorem 1.** Let $\{\lambda_{j}\}_{j=1}^{d}$ be the eigenvalues of the matrix $(a_{ij}(0))$. We assume that $b_{k}(x) \not\equiv 0$ and $b_{k}(0) = 0$ for $k = 1, 2, \ldots, n$, and let $\{\mu_{k}\}_{k=1}^{n}$ be the eigenvalues of Jacobi matrix of $(b_{1}(x), \ldots, b_{n}(x))$ at $x = 0$. Then the formal power series solution of (1.1) exists uniquely and converges if the following conditions are satisfied:

There exists a positive constant $\sigma_{0} > 0$, such that

\[
(1.2) \quad \left| \sum_{j=1}^{d} \lambda_{j}l_{j} + \sum_{k=1}^{n} \mu_{k}m_{k} \right| \geq \sigma_{0}(|l| + |m|) \quad \text{(Poincaré condition)},
\]

and

\[
(1.3) \quad \sum_{j=1}^{d} \lambda_{j}l_{j} + \sum_{k=1}^{n} \mu_{k}m_{k} + c(0) \neq 0 \quad \text{(Non-resonance condition)}
\]

hold for all $(l,m) \in (\mathbb{Z}_{\geq 0})^{d} \times (\mathbb{Z}_{\geq 0})^{n}$ with $|l| \geq K$ and $|m| \geq 0$.

**Remark 1.** It is easy to show the following proposition.

The conditions (1.2) and (1.3) imply that

\[
(1.4) \quad \left| \sum_{j=1}^{d} \lambda_{j}l_{j} + \sum_{k=1}^{n} \mu_{k}m_{k} + c(0) \right| \geq \sigma(|l| + |m|)
\]

holds by some positive constant $\sigma > 0$ for all $(l,m) \in (\mathbb{Z}_{\geq 0})^{d} \times (\mathbb{Z}_{\geq 0})^{n}$ with $|l| \geq K$ and $|m| \geq 0$. In the proof of Theorem 1, this condition will be used instead of (1.2) and (1.3).

Next, we consider the following general equation:

\[
(1.5) \quad \begin{cases} 
  f(t,x,u(t,x),\{\partial_{t_{j}}u(t,x)\},\{\partial_{x_{k}}u(t,x)\}) = 0, \\
  u(0,x) \equiv 0.
\end{cases}
\]

**Assumption 1.** $f(t,x,u,\tau,\xi)$ ($\tau = (\tau_{j}) \in \mathbb{C}^{d}$, $\xi = (\xi_{k}) \in \mathbb{C}^{n}$) is holomorphic in a neighbourhood of the origin, and is an entire function in $\tau$ variables for any fixed $t$, $x$, $u$ and $\xi$. Moreover we assume that

\[
(1.6) \quad f(0,x,0,\tau,0) \equiv 0
\]

for $x \in \mathbb{C}^{n}$ near the origin and $\tau \in \mathbb{C}^{d}$, which is a generalization of the definition of singular equations defined in [MS].
For the equation (1.5), we do not know whether or not the equation has a formal solution in general. Therefore, we assume the following:

**Assumption 2.** The equation (1.5) has a formal solution of the form

\[
(1.7) \quad u(t, x) = \sum_{j=1}^{d} \varphi_j(x)t_j + \sum_{|l|\geq 2, |m|\geq 0} u_{lm}t^lx^m \in \mathbb{C}[t, x].
\]

By the existence of a formal solution, \( \{\varphi_j(x)\} \) satisfy the following system formally:

\[
(1.8) \quad f(0, x, 0, \{\varphi_j(x)\}, 0) \equiv 0 \quad \text{(trivial relation),}
\]

and

\[
(1.9) \quad \left. \frac{\partial}{\partial t_i} f(t, x, u(t, x), \{\partial_{t_j} u(t, x)\}, \{\partial_{x_k} u(t, x)\}) \right|_{t=0} = \left. \frac{\partial f}{\partial t_i}(0, x, 0, \{\varphi_j(x)\}, 0) \varphi_i(x) + \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0, x, 0, \{\varphi_j(x)\}, 0) \frac{\partial \varphi_i}{\partial x_k}(x) \right|_{t=0} = 0, \text{ for } i = 1, 2, \ldots, d.
\]

The formal solution of this system is not convergent in general. Therefore, we assume **Assumption 3.** The coefficients \( \{\varphi_j(x)\} \) are all holomorphic in a neighbourhood of the origin of \( \mathbb{C}^n \).

**Remark 2.** In the case \( d = 1 \) \((d \) is the dimension of \( t \) variables), a sufficient condition for the formal solution of (1.9) to converge has been already obtained by Miyake-Shirai [MS]. In the case \( d \geq 2 \), we give a sufficient condition for the formal solution of system (1.9) to be convergent, which will be given by Theorem 3 in Section 5, but for a while we consider the problem under Assumption 3 for simplicity of our arguments.

Now we put \( a(x) = (0, x, 0, \{\varphi_j(x)\}, 0) \) for simplicity, and define

\[
(1.10) \quad A_{ij}(x) := \frac{\partial^2 f}{\partial t_i \partial t_j}(a(x)) + \frac{\partial^2 f}{\partial u \partial t_j}(a(x)) \varphi_i(x) + \sum_{k=1}^{n} \frac{\partial^2 f}{\partial t_j \partial \xi_k}(a(x)) \frac{\partial \varphi_i}{\partial x_k}(x),
\]

for \( i, j = 1, 2, \ldots, d \). Moreover we define

\[
(1.11) \quad B_k(x) := \frac{\partial f}{\partial \xi_k}(a(x)), \quad \text{for } k = 1, 2, \ldots, n.
\]

**Remark 3.** The functions \( A_{ij}(x) \) and \( B_k(x) \) correspond to \( a_{ij}(x) \) and \( b_k(x) \) in Theorem 1, respectively (see (1.13) below).

Here we assume that the equation is of totally characteristic type, that is,
Assumption 4. $B_k(x) \not\equiv 0$ and $B_k(0) = 0$, for $k = 1, 2, \ldots, n$.

Now our second theorem which is our main result is stated as follows:

**Theorem 2.** Suppose Assumptions 1, 2, 3 and 4. Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of $(A_{ij}(0))$, and let $\{\mu_k\}_{k=1}^n$ be the eigenvalues of Jacobi matrix of the vector $(B_k(x))$ at $x = 0$. Then the formal solution (1.7) is convergent if the following condition is satisfied:

There exists a positive constant $\sigma > 0$, such that,

\[
\left| \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + \frac{\partial f}{\partial u}(a(0)) \right| \geq \sigma(|l| + |m|),
\]

holds for all $(l, m) \in (\mathbb{Z}_{\geq 0})^d \times (\mathbb{Z}_{\geq 0})^n$ with $|l| \geq 2$, $|m| \geq 0$.

**Remark 4.** We put $v(t, x) = u(t, x) - \sum_{j=1}^{d} \varphi_j(x) t_j$ as a new unknown function. By Assumptions 1, 2, 3 and 4, we can easily see that $v(t, x)$ satisfies the equation of the following form:

\[
\sum_{i,j=1}^{d} A_{ij}(x) t_i \partial_{t_j} v + \sum_{k=1}^{n} B_k(x) \partial_{x_k} v + \frac{\partial f}{\partial u}(a(x)) v = \sum_{|l|=2}^{d} d_l(x) t^l + f_3(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}),
\]

This is an equation considered in Theorem 1 in the case $K = 2$. Therefore, it is sufficient to prove Theorem 1 in order to prove Theorem 2.

\[\square\]

2. Reduction of the Equation

As is mentioned in Remark 4, it is sufficient to study the equation (1.1).

By the assumption of Theorem 1,

\[
(a_{ij}(0)) \sim \begin{pmatrix}
\lambda_1 & \delta_1 \\
\vdots & \ddots \\
\lambda_d & \delta_{d-1}
\end{pmatrix}, \quad \frac{\partial(b_1, \ldots, b_n)}{\partial(x_1, \ldots, x_n)}|_{x=0} \sim \begin{pmatrix}
\mu_1 & \nu_1 \\
\vdots & \ddots \\
\mu_n & \nu_{n-1}
\end{pmatrix},
\]

where $\delta_j, \nu_k = 0$ or 1 ($1 \leq j \leq d - 1, 1 \leq k \leq n - 1$).

Then by transforming the variables, (1.1) is reduced to the following form:

\[
(\Lambda + \Delta) v(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} v + \gamma(x) v + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} v + \tilde{f}_{K+1}(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}),
\]
with $v(t, x) = O(|t|^K)$, where

$$
\Lambda = \sum_{j=1}^{d} \lambda_{j} t_{j} \partial_{t_{j}} + \sum_{k=1}^{n} \mu_{k} x_{k} \partial_{x_{k}} + c(0),
$$

$$
\Delta = \sum_{j=1}^{d-1} \delta_{j} t_{j} \partial_{t_{j+1}} + \sum_{k=1}^{n-1} \nu_{k} x_{k} \partial_{x_{k+1}},
$$

and $\alpha_{l}(x)$, $\beta_{ij}(x)$, $\gamma(x)$ and $\varphi_{k}(x)$ are holomorphic in a neighbourhood of the origin, and satisfy $\beta_{ij}(x) = O(|x|)$, $\gamma(x) = O(|x|)$ and $\varphi_{k}(x) = O(|x|^2)$, and $\hat{f}_{K+1}(t, x, u, \tau, \xi)$ is a holomorphic function which has a similar Taylor expansion with $f_{K+1}(t, x, u, \tau, \xi)$.

In the following sections, we shall prove the existence and convergence of the unique formal solution of (2.1).

3. Preparation to prove Theorem 1

Let $C[t, x]_{L,M}$ be the set of homogeneous polynomial of degree $L$ in $t$ variables and of degree $M$ in $x$ variables, that is,

$$
C[t, x]_{L,M} = \left\{ f_{LM}(t, x) = \sum_{|l|=L, |m|=M} f_{lm} t^{l} x^{m} \mid f_{lm} \in C \right\}.
$$

For the operator $\Lambda + \Delta$, the following lemma holds:

**Lemma 1.** For all $L \geq K$ and $M \geq 0$, the operator

$$
\Lambda + \Delta : C[t, x]_{L,M} \longrightarrow C[t, x]_{L,M}
$$

is invertible. Moreover, if the majorant relation $f_{LM}(t, x) \ll F \times (t_{1} + \cdots + t_{d})^{L}(x_{1} + \cdots + x_{n})^{M}$ ($f_{LM}(x) \in C[t, x]_{L,M}$, $F > 0$) holds, then we obtain the following majorant relation:

$$
(3.1) \quad (\Lambda + \Delta)^{-1} f_{LM}(t, x) \ll \frac{C}{L+M} F \times (t_{1} + \cdots + t_{d})^{L}(x_{1} + \cdots + x_{n})^{M},
$$

where $C > 0$ is a positive constant independent of $L$ and $M$.

**Proof.** We define a norm of $u_{LM}(t, x) \in C[t, x]_{L,M}$ by

$$
||u_{LM}|| := \inf \left\{ C > 0 \mid u_{LM}(t, x) \ll C (t_{1} + \cdots + t_{d})^{L}(x_{1} + \cdots + x_{n})^{M} \right\}.
$$

We remark that $C[t, x]_{L,M}$ becomes a Banach space by this norm.

First, by (1.4) it is easily checked that $\Lambda$ is invertible on $C[t, x]_{L,M}$ and

$$
(3.2) \quad ||\Lambda^{-1}|| \leq \frac{1}{\sigma(L+M)}
$$

holds for the operator norm of $\Lambda^{-1}$ on $C[t, x]_{L,M}$. 
Next, since $u_{LM}(t, x) \ll ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$, we have

$$\Delta u_{LM}(t, x) \ll \sum_{j=1}^{d-1} L|\delta_j| \cdot ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$$

$$+ \sum_{k=1}^{n-1} M|\nu_k| \cdot ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$$

$$\ll \left\{L(d-1)\max_{j=1,\ldots,d-1} |\delta_j| + M(n-1)\max_{k=1,\ldots,n-1} |\nu_k| \right\} \times$$

$$\times ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M.$$ 

Here we make a change of variables by $t_j = \varepsilon^{j-1}\tau_j$, $x_k = \varepsilon^{k-1}y_k$, then $\delta_j$ and $\nu_k$ (the components of nilpotent part of Jordan canonical form) turns to $\varepsilon\delta_j$ and $\varepsilon\nu_k$, respectively. Therefore, by choosing $\varepsilon$ sufficiently small, we may assume that the components of nilpotent part are small enough. Hence we may assume that

$$(3.3) \quad \max_{j=1,\ldots,d-1} |\delta_j| < \frac{\sigma}{2(d-1)}, \quad \max_{k=1,\ldots,n-1} |\nu_k| < \frac{\sigma}{2(n-1)}.$$ 

Then

$$\Delta u_{LM}(t, x) \ll \frac{\sigma(L+M)}{2} ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$$

holds, and we obtain

$$||\Delta|| \leq \frac{\sigma(L+M)}{2}.$$ 

Therefore, the operator norm of $\Delta \Lambda^{-1}$ is estimated by

$$||\Delta \Lambda^{-1}|| \leq \frac{1}{\sigma(L+M)} \frac{\sigma(L+M)}{2} = \frac{1}{2} < 1.$$ 

By using the Neumann’s series, we can see that $\Lambda + \Delta$ is invertible and the norm of the inverse operator is estimated by

$$||(\Lambda + \Delta)^{-1}|| \leq \frac{2}{\sigma(L+M)},$$

which we want to prove since $C = 2/\sigma$ is independent of $L$ and $M$. \hfill \square

Now, we define some notations, which are used in the proof of Theorem 1.

**Definition** (1) Let $(t, x) \in \mathbb{C}^d \times \mathbb{C}^n (d \geq 0, \ n \geq 0)$ be complex variables. For formal power series $f(t, x) = \sum_{|l| \geq 0, \ |m| \geq 0} f_{l,m} t^l x^m$, we define

$$|f|(t, x) = \sum_{|l| \geq 0, \ |m| \geq 0} |f_{l,m}| t^l x^m.$$
(2) Let \((t, X) \in \mathbb{C}^d \times \mathbb{C}\) \((d \geq 0)\) be complex variables. For formal power series \(f(t, X) = \sum_{|l| \geq 0, M \geq 0} f_{l,M} t^l X^M\), we define the shift operator \(S\) by

\[
S(f)(t, X) = \sum_{|l| \geq 0, M \geq 0} f_{l,M+1} t^l X^M = \frac{f(t, X) - f(t, 0)}{X}.
\]

**Remark 5.** The following facts are easily shown:

- \(f(t, x) \ll |f|(t, x)\);
- If \(f(t, x)\) and \(g(t, X)\) are convergent power series, then \(|f|(t, x)\) and \(S(g)(t, X)\) are also convergent. \(\square\)

### 4. Proof of Theorem 1

First, we prove a unique existence of formal power series solution.

Let

\[
u(t, x) = \sum_{|l| \geq K, |m| \geq 0} u_{lm} t^l x^m = \sum_{L \geq K} u_L(t, x) = \sum_{L \geq K, M \geq 0} u_{LM}(t, x)\]

be a formal solution of (2.1), where

\[
u_{LM}(t, x) = \sum_{|l|=L, |m|=M} u_{lm} t^l x^m \in \mathbb{C}[t, x]_{L,M},
\]

\[
u_L(t, x) = \sum_{|l|=L} u_l(x) t^l = \sum_{M \geq 0} u_{LM}(t, x).
\]

We put \(P = \Lambda + \Delta\) for simplicity. We substitute \(u(t, x) = \sum_{L \geq K} u_L(t, x)\) into (2.1), then we have the following recursion formula:

\[
\begin{align*}
P_{K}(t, x) &= \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} u_{K}(t, x) \\
&\quad + \gamma(x) u_{K}(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} u_{K}(t, x), \\
\end{align*}
\]

\[
\begin{align*}
P_{L}(t, x) &= \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} u_{L}(t, x) + \gamma(x) u_{L}(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} u_{L}(t, x) \\
&\quad + G_{L}(t, x, \{u_p\}_{K \leq p < L}, \{\partial_{t_p} u_p\}_{K \leq p < L}, \{\partial_{x_k} u_p\}_{K \leq p < L}), \text{ for } L > K,
\end{align*}
\]

where \(G_L(t, x, \zeta, \tau, \xi)\) is a polynomial of \((t, \zeta, \tau, \xi)\).
First, we consider the case $L = K$. We substitute $u_K(t, x) = \sum_{M \geq 0} u_{KM}(t, x)$ into the above recursion formula, we have

$$
\begin{cases}
Pu_{K0}(t, x) = \sum_{|l|=K} \alpha_l(0)t^l, \\
Pu_{KM}(t, x) = \sum_{|l|=K} \alpha_l^M(x)t^l + \sum_M \sum_{i,j=1}^d \beta_{ij}^p(x) t_i \partial_{t_j} u_{K,M-p}(t, x) \\
\quad \quad \quad + \sum_{p=1}^M \gamma^p(x) u_{K,M-p}(t, x) + \sum_{k=1}^n \sum_{p=2}^M \varphi_k^p(x) \partial_{x_k} u_{K,M-p+1}(t, x),
\end{cases}
$$

where we put

$$
\alpha_l(x) = \sum_{M \geq 0} \alpha_l^M(x), \quad \alpha_l^M(x) = \sum_{m=M} \alpha_{lm} x^m,
$$

$$
\beta_{ij}(x) = \sum_{M \geq 1} \beta_{ij}^M(x), \quad \beta_{ij}^M(x) = \sum_{m=M} \beta_{ijm} x^m,
$$

$$
\gamma(x) = \sum_{M \geq 1} \gamma^M(x), \quad \gamma^M(x) = \sum_{m=M} \gamma_{m} x^m,
$$

$$
\varphi_k(x) = \sum_{M \geq 2} \varphi_k^M(x), \quad \varphi_k^M(x) = \sum_{m=M} \varphi_{km} x^m.
$$

By Lemma 1, we know that the solution sequence $\{u_{KM}(t, x)\}_{M \geq 0}$ exists uniquely. Moreover, by the same argument, we see that $\{u_{LM}(t, x)\}$ ($L > K$) exist uniquely. These show that the formal solution exists uniquely.

Next, we prove the convergence of the formal solution. We put $U(t, x) = Pu(t, x)$ as a new unknown function. By Lemma 1, the equation (2.1) is reduced to the following equation:

$$
(4.1) \quad U(t, x) = \sum_{|l|=K} \alpha_l(t) t^l + \sum_{i,j=1}^d \beta_{ij}(x) t_i \partial_{t_j} P^{-1} U(t, x) \\
\quad \quad \quad + \gamma(x) P^{-1} U(t, x) + \sum_{k=1}^n \varphi_k(x) \partial_{x_k} P^{-1} U(t, x) \\
\quad \quad \quad + f_{K+1}(t, x, P^{-1} U(t, x), \{\partial_{t_j} P^{-1} U(t, x)\}, \{\partial_{x_k} P^{-1} U(t, x)\}).
$$

We know that (4.1) has a unique formal solution of the form

$$
U(t, x) = \sum_{|l| \geq K, |m| \geq 0} U_{lm} t^l x^m = \sum_{L \geq K} U_L(t, x) = \sum_{L \geq K, M \geq 0} U_{LM}(t, x).
$$
In order to get a majorant series of $U(t, x)$, we prepare some majorant series for the coefficients of (4.1). We put $T = t_1 + \cdots + t_d$, $X = x_1 + \cdots + x_n$, and choose
\[
\sum_{|l|=K} \alpha_l(x)t^l \ll A(X)T^K, \quad \beta_{ij}(x) \ll |\beta_{ij}|(X, \ldots, X) =: XB_{ij}(X),
\]
\[
\gamma(x) \ll |\gamma|(X, \ldots, X) =: XG(X), \quad \varphi_k(x) \ll |\varphi_k|(X, \ldots, X) =: X^2\Phi_k(x),
\]
\[
\overline{f}_{K+1}(t, x, u, \tau, \xi) \ll \left| \tilde{f}_{K+1}(T, \ldots, T, X, \ldots, X, u, \tau, \xi) \right| =: F_{K+1}(T, X, u, \tau, \xi) = \sum_{|p|+Kq+(K-1)|r|+K|s|\geq K+1} F_{pqrs}(X)T^{|p|}u^q\tau^r\xi^s,
\]
where $A(X)$, $B_{ij}(X)$, $G(X)$ and $\Phi_k(x)$ are holomorphic in a neighbourhood of $X = 0$, and $F_{K+1}(T, X, u, \tau, \xi)$ is also holomorphic near $(T, X, u, \tau, \xi) = (0, 0, 0, 0, 0)$.

Now, we consider the following equation:
\[
(4.2) \quad w(T, X) = A(X)T^K + C \sum_{i,j=1}^d XB_{ij}(X)w(T, X) + C t_1 \sum_{i,j=1}^d \partial_{t_j}PB^{-1}U(t, x)w(T, X) + C X^2G(X)w(T, X) + C \sum_{k=1}^n X^2\Phi_k(x)(t, x)S(w)(T, X) + F_{K+1},
\]
where $C$ is a positive constant appeared in Lemma 1.

Let $w(T, X) = \sum_{L\geq K, M\geq 0} w_{LM}(T, X)$ be the formal solution of (4.2). By the construction of (4.2), we can easily check that $U(t, x) \ll w(T, X)$ by the next lemma.

**Lemma 2.** For two formal power series $U(t, x)$ and $w(T, X)$ satisfying
\[
U(t, x) = \sum_{L\geq K, M\geq 0} U_{LM}(t, x) \ll w(T, X) = \sum_{L\geq K, M\geq 0} w_{LM}T^LX^M,
\]
the following majorant relations hold:

1. $P^{-1}U(t, x) \ll Cw(T, X),$
2. $t_1 \partial_{t_j} P^{-1}U(t, x) \ll Cw(T, X),$
3. $\partial_{t_j} P^{-1}U(t, x) \ll \frac{Cw(T, X)}{T},$
4. $\partial_{x_k} P^{-1}U(t, x) \ll CS(w)(T, X).$

**Proof.** By using Lemma 1, we can prove this lemma easily. First, (1) is proved as follows:
\[
P^{-1}U(t, x) = \sum_{L\geq K, M\geq 0} P^{-1}U_{LM}(t, x) \ll \sum_{L\geq K, M\geq 0} \frac{C}{L+M} w_{LM} T^L X^M \ll Cw(T, X).
\]
Secondly, (2) and (3) is proved as follows:

\[ t_i \partial_{t_j} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 0} t_i \partial_{t_j} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{CL}{L + M} w_{LM} T^L X^M \ll C w(T, X); \]

\[ \partial_{t_j} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 0} \partial_{t_j} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{CL}{L + M} w_{LM} T^L X^M \ll \frac{C w(T, X)}{T}. \]

Finally, (4) is proved as follows:

\[ \partial_{x_k} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 1} \frac{CM}{L + M} w_{LM} T^L X^{M-1} \ll C S(w)(T, X). \]

This completes the proof. \( \square \)

Since \( w(T, X) \gg 0 \), we have

(4.3) \[ X S(w)(T, X) = w(T, X) - w(T, 0) \ll w(T, X). \]

Let us consider the following equation:

(4.4) \[ v(T, X) = A(X) T^K + C X h(X) v(T, X) \]
\[ + F_{K+1} \left( T, X, C v, \left\{ \frac{C v}{T} \right\}, \{ C S(v) \} \right), \]

with \( v(T, X) = O(T^K) \), where \( h(X) = \sum_{i,j=1}^d B_{ij}(X) + G(X) + \sum_{k=1}^n \Phi_k(X) \). Then the following majorant relation is obvious:

\[ w(T, X) \ll v(T, X). \]

We put \( y(T, X) = v(T, X)/T \) as a new unknown function. By substituting this into (4.4), we see that \( y(T, X) \) satisfies

(4.5) \[ y(T, X) = A(X) T^{K-1} + C X h(X) y(T, X) \]
\[ + \frac{1}{T} F_{K+1} (T, X, C T y, \{ C y \}, \{ C T S(y) \}), \]

with \( y(T, X) = O(T^{K-1}) \).

We decompose the formal solution \( y(T, X) \) as follows:

\[ y(T, X) = y_1(X) T^{K-1} + y_2(X) T^K + T^K z(T, X). \]
We remark that $y_1(X)$ and $y_2(X)$ are holomorphic functions in a neighbourhood of $X=0$. Indeed, $y_1(X)$ and $y_2(X)$ are given by

\[ y_1(X) = \frac{A(X)}{1 - CXh(X)}, \]

\[ y_2(X) = \frac{1}{1 - CXh(X)} \sum_{|p|+Kq+(K-1)|r|+K|s|=K+1} F_{pqrs}(X) \{ Cy_1(X) \}^{q+|r|} \{ CS(y_1)(X) \}^{|s|}. \]

These are holomorphic functions in a neighbourhood of $X=0$.

In this case, $z(T, X)$ satisfies the following equation:

\[
\begin{align*}
\{ z(T, X) &= CXh(X)z(T, X) + H(T, X, Tz(T, X), TS(z)(T, X)), \\
z(0, X) &\equiv 0,
\end{align*}
\]

where

\[
H(T, X, \eta_1, \eta_2) = \frac{1}{T^{K+1}} \left[ F_{K+1}(T, X, Cy_1(X)T^K + Cy_2(X)T^{K+1} + CT^K\eta_1, \right. \\
&\left. \{ Cy_1(X)T^{K-1} + Cy_2(X)T^K + CT^{K-1}\eta_1 \}, \right.
\]

\[
\left. \{ CS(y_1)(X)T^K + CS(y_2)(X)T^{K+1} + CT^K\eta_2 \} \right] \\
- \sum_{|p|+Kq+(K-1)|r|+K|s|=K+1} F_{pqrs}(X)(Cy_{1}(X))^{q+|t|}(CS(y_{1})(X))^{s}.
\]

**Remark 6.** The order of zeros in $T$ variable of $H(T, X, CTz(T, X), CTS(z)(T, X))$ is greater than or equal to 1. \( \square \)

In order to prove the convergence of $z(T, X)$, it is sufficient to show the following:

**Lemma 3.** There exists a small positive constant $\epsilon > 0$ such that $z_\epsilon(\rho) = z(\epsilon\rho, \rho)$ is convergent in a neighbourhood of $\rho = 0$.

**Proof.** We substitute $T = \epsilon\rho$ and $X = \rho$ into (4.6) and by using the relation (4.3), we have

\[ \rho S(z)(\epsilon\rho, \rho) \ll z_\epsilon(\rho). \]

By this relation, the following majorant relation also holds,

\[ TS(z)(T, X)|_{T=\epsilon\rho, X=\rho} = \epsilon\rho S(z)(\epsilon\rho, \rho) \ll \epsilon z_\epsilon(\rho). \]

Here we consider

\[
\begin{align*}
\psi(\rho) &= Cph(\rho)\psi(\rho) + H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).
\end{align*}
\]
In the right hand side of (4.7), we decompose $H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho))$ into the term of $\psi(\rho)$ and otherwise as follows:

$$H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)) = \epsilon \frac{\partial H}{\partial \eta_2}(0,0,0,0)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)),$$

We remark that the following fact holds:

$$\frac{\partial \overline{H}}{\partial \psi}(\epsilon\rho, \rho, \epsilon\rho\psi, \epsilon\psi)|_{\rho=0,\psi=0} = 0.$$

We put $(\partial H/\partial \eta_2)(0,0,0,0)=K_0 \geq 0$, then (4.7) is rewritten by

$$(1-\epsilon K_0)\psi(\rho) = C\rho h(\rho)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).$$

We choose $\epsilon > 0$ with $1-\epsilon K_0 > 0$. Then by using the implicit function theorem, we can see that (5.8) has a unique holomorphic solution $\psi(\rho)$ with $\psi(0)=0$ in a neighbourhood of $\rho=0$. Moreover we can see $z_\epsilon(\rho) \ll \psi(\rho)$.

Thus we complete the proof of Lemma 3. \hfill \square

5. SOLVABILITY OF THE SYSTEM (1.9)

In this section, we give a sufficient condition for the formal solution of the system (1.9) to be convergent. Recall that (1.9) is

$$(1.9) \quad \frac{\partial f}{\partial t_i}(0,x,0,\{\varphi_j(x)\},0) + \frac{\partial f}{\partial u}(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x) + \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0,x,0,\{\varphi_j(x)\},0)\frac{\partial \varphi_i(x)}{\partial x_k} = 0, \quad i = 1,2,\ldots,d.$$

By Assumption 4 of Theorem 2, the condition

$$\frac{\partial f}{\partial \xi_k}(0,0,0,\{\varphi_j(0)\},0) = 0, \quad k = 1,2,\ldots,n$$

was assumed.

Let $\varphi(x) = ^t(\varphi_1(x),\ldots,\varphi_d(x))$ be the unknown functions. We put $\varphi(0) = ^t(\varphi_1^0,\ldots,\varphi_d^0) \in \mathbb{C}^d$ as the constant term of $\varphi(x)$. We substitute $\varphi_j(x) = \varphi_j^0 + \psi_j(x)$ into the system (1.9), and by restricting at $x=0$, we see that $\{\varphi_j^0\}$ satisfies the following system:

$$(5.1) \quad \frac{\partial f}{\partial t_i}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 = 0, \quad i = 1,2,\ldots,d.$$

This system has some roots by the assumption of the existence of a formal solution, and we fix such $\{\varphi_j^0\}$. 
For such fixed \{\varphi_j^0\}, we see that \{\psi_j(x)\} satisfies the system of the followir

\begin{equation}
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0,0,0,\{\varphi_j^0\},0)x_l \frac{\partial \psi_i}{\partial x_k}(x)
+ \sum_{k=1}^{n} \sum_{p=1}^{d} \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0,0,0,\{\varphi_j^0\},0)\psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x)
+ \frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\psi_i(x)
+ \sum_{p=1}^{d} \left\{ \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 \right\} \psi_p(x)
+ \sum_{l=1}^{n} \left\{ \frac{\partial^2 f}{\partial t_i \partial x_l}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 \right\} x_l
= \text{(degree in } x \text{ is greater than or equal to 2), } i = 1, 2, \ldots, d.
\end{equation}

This system is written as follows for simplicity,

\begin{equation}
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \frac{\partial \psi_i}{\partial x_k}(x) + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x)
+ c \psi_i(x) + \sum_{p=1}^{d} d_{ip} \psi_p(x) + \sum_{l=1}^{n} e_l x_l
= \text{(degree in } x \text{ is greater than or equal to 2), } i = 1, 2, \ldots, d,
\end{equation}

where

\begin{align*}
a_{kl} &:= \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0,0,0,\{\varphi_j^0\},0),
b_{kp} &:= \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0,0,0,\{\varphi_j^0\},0),
c &:= \frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0),
d_{ip} &:= \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0,
e_l &:= \frac{\partial^2 f}{\partial t_i \partial x_l}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial^2 f}{\partial u \partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0.
\end{align*}
Here we decompose \( \psi_i(x) \) into \( \psi_i(x) = \psi_t(x) + \eta_i(x) \) \( (\psi_t(x) = \sum_{k=1}^{n} \psi_k x_k, \eta_i(x) = O(|x|^2)) \). We substitute this into the system (5.3) and obtain

\[
(5.4) \quad \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \left( \frac{\partial \psi_t}{\partial x_k} + \frac{\partial \eta_i}{\partial x_k} \right) + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} (\psi_p(x) + \eta_p(x)) \left( \frac{\partial \psi_t}{\partial x_k} + \frac{\partial \eta_i}{\partial x_k} \right) \\
+c(\psi_t(x) + \eta_i(x)) + \sum_{p=1}^{d} d_{ip} (\psi_p(x) + \eta_p(x)) + \sum_{l=1}^{n} e_{il} x_l = 0
\]

for \( i = 1, 2, \ldots, d \).

By picking up the degree 1 part on the both sides, we see that \( \{\psi_t(x)\} \) satisfy the following system:

\[
(5.5) \quad \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \frac{\partial \psi_t}{\partial x_k} + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \psi_p(x) \frac{\partial \psi_t}{\partial x_k} + c \psi_t(x) + \sum_{p=1}^{d} d_{ip} \psi_p(x) + \sum_{l=1}^{n} e_{il} x_l = 0
\]

for \( i = 1, 2, \ldots, d \).

By the existence of a formal solution, (5.5) has some solutions \( \{\psi_t(x)\} \) of linear functions, and we fix such \( \{\psi_t(x)\} \).

For fixed \( \{\varphi_i^0\} \) and \( \{\psi_t(x)\} \), we see that \( \{\eta_i(x)\} \) satisfy the following system:

\[
(5.6) \quad \sum_{k=1}^{n} \sum_{l=1}^{n} \left( a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_p \right) x_l \frac{\partial \eta_i}{\partial x_k} + c \eta_i(x) + \sum_{p=1}^{d} \left( d_{ip} + \sum_{k=1}^{n} b_{kp} \psi_{ik} \right) \eta_p(x) = 0
\]

for \( i = 1, 2, \ldots, d \).

We remark that the degree 2 part in the right hand side of this system does not include \( \{\eta_i(x)\} \).

The following theorem holds:

**Theorem 3.** Let \( (A_{kl})_{k,l=1,2,\ldots,n} \) be a matrix defined by

\[
(A_{kl})_{k,l=1,2,\ldots,n} = \left( a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_p \right)
\]

for \( k, l = 1, 2, \ldots, n \).
Let \( \{\kappa_k\}_{k=1}^{n} \) be the eigenvalues of \((A_{kl})_{k,l=1,2,\ldots,n}\). If there exists a positive constant \( \sigma_0 \) such that the condition
\[
\left| \sum_{k=1}^{n} \kappa_k m_k \right| \geq \sigma_0 |m|,
\]
(Poincaré condition)
holds for all \( m = (m_1, \ldots, m_n) \in (\mathbb{Z}_{\geq 0})^n \) with \( |m| \geq 2 \), then the formal solution of the system (1.9) is convergent in a neighbourhood of the origin.

**Remark 7.** Let \((B_{ip})_{i,p=1,2,\ldots,d}\) be a matrix defined by
\[
(B_{ip})_{i,p=1,2,\ldots,d} = \left( d_{ip} + \sum_{k=1}^{n} b_{kp} \psi_{ik} \right)_{i,p=1,2,\ldots,d},
\]
and let \( \{\omega_j\}_{j=1}^{d} \) be the eigenvalues of \((B_{ip})_{i,p=1,2,\ldots,d}\).

By the same argument in Remark 1, we have
\[
\left( \sum_{k=1}^{n} \kappa_k m_k + c + \omega_j \right) \geq \sigma |m|, \quad \text{by some} \quad \sigma > 0, \quad \text{and} \quad j = 1, 2, \ldots, d,
\]
for large \( m \), which will be used in the proof. \( \square \)

### 6. Proof of Theorem 3

The proof of Theorem 3 is the same method of the proof of Theorem 1 in case that the unknown function is a vector values. However, there are some difference in the detail. Therefore, we introduce only the outline of the proof of Theorem 3 in this section.

**Step 1.** By taking a linear transformation of the independent variables and a linear transformation of the unknown functions, (5.6) is reduced to the following form:

\[
(\Lambda + \Delta + \mathbf{B}) \begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x) \end{pmatrix} := \left\{ \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_d \end{pmatrix} + \begin{pmatrix} \Delta \\ \vdots \\ \Delta \end{pmatrix} + \mathbf{B} \right\} \begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x) \end{pmatrix} = \left( \begin{array}{c} \sum_{|m|=2} a_{1,m} x^m + g_{3,1}(x, w(x), \partial_x w(x)) \\ \vdots \\ \sum_{|m|=2} a_{d,m} x^m + g_{3,d}(x, w(x), \partial_x w(x)) \end{array} \right),
\]
where $w_j(x) (j = 1, 2, \ldots, d)$ denote new unknown functions after linear transformations and

$$
\Lambda_j = \sum_{k=1}^{n} \kappa_k x_k \partial_{x_k} + c + \omega_j, \quad \Delta = \sum_{k=1}^{n-1} \varepsilon_k x_k \partial_{x_{k+1}}, \quad \mathbf{B} = \begin{pmatrix} 0 & e_1 \\ \vdots & \ddots & \ddots \\ & & 0 & e_{d-1} \end{pmatrix},
$$

where $\varepsilon_j$ and $e_j$ denote the nilpotent components of the Jordan canonical forms of the matrices $(A_{kl})$ and $(B_{ip})$, respectively, and

$$
g_{3,i}(x, \eta, \zeta) = \sum_{|\alpha|+2|\beta|+|\gamma| \geq 3} g_{\alpha\beta\gamma}^{(i)} x^\alpha \eta^\beta \zeta^\gamma.
$$

**Step 2.** We define $\mathbf{C}[x]_M$ by $\mathbf{C}[x]_M = \{\sum_{|m|=M} u_m x^m ; u_m \in \mathbf{C}\}$, and define a norm of $u(x) = \{u_1(x), \ldots, u_d(x)\} \in (\mathbf{C}[x]_M)^d$ by

$$
||u|| := \inf\{C > 0 ; u_i(x) \ll C(x_1 + \cdots + x_n)^M, \ i = 1, 2, \ldots, d\}.
$$

By the same argument in the proof of Lemma 1 and by Remark 7, we can prove the same results of Lemma 1 for the operator $\Lambda + \Delta + \mathbf{B}$.

**Step 3.** By the same method in the previous sections, we can construct a majorant equation whose formal solution is a majorant function of the all unknown functions of the system. Finally, by the implicit function theorem, we prove the convergence of the formal solution of the majorant equation.

**References**


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