CONVERGENCE OF FORMAL SOLUTIONS OF SINGULAR FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF TOTALLY CHARACTERISTIC TYPE

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1. INTRODUCTION

Let $(t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n) \in \mathbb{C}^d \times \mathbb{C}^n$ be $(d+n)$-dimensional complex variables $(d \geq 1, n \geq 1)$. We consider the following first order nonlinear partial differential equation:

\[
\begin{aligned}
&\left\{ \sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} u + \sum_{k=1}^{n} b_k(x) \partial_{x_k} u + c(x) u \\
&= \sum_{|l|=K} d_l(x) t^l + f_{K+1}(t, x, u, \{\partial_{t_j} u\}, \{\partial_{x_k} u\}), \right.
\\
&u(t, x) = O(|t|^K),
\end{aligned}
\]
(1.1)

where $|t| = t_1 + \cdots + t_d$, $K$ is a fixed positive integer satisfying $K \geq 2$ and $a_{ij}(x)$, $b_k(x)$, $c(x)$ and $d_l(x)$ are holomorphic in a neighbourhood of the origin, and $f_{K+1}(t, x, u, \tau, \xi)$ $(\tau = (\tau_j) \in \mathbb{C}^d$, $\xi = (\xi_k) \in \mathbb{C}^n)$ is also holomorphic in a neighbourhood of the origin with the following Taylor expansion:

\[
f_{K+1}(t, x, u, \tau, \xi) = \sum_{|p|+|q|+|Kq+(K-1)r|+|s| \geq K+1} f_{pqrs}(x) t^p u^q \tau^r \xi^s,
\]

where $q \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, $p = (p_1, \ldots, p_d) \in (\mathbb{Z}_{\geq 0})^d$, $r = (r_1, \ldots, r_d) \in (\mathbb{Z}_{\geq 0})^d$, $s = (s_1, \ldots, s_n) \in (\mathbb{Z}_{\geq 0})^n$,

\[
|p| = p_1 + \cdots + p_d, \quad |r| = r_1 + \cdots + r_d, \quad |s| = s_1 + \cdots + s_n,
\]

and

\[
t^p = \prod_{j=1}^{d} t_j^{p_j}, \quad \tau^r = \prod_{j=1}^{d} \tau_j^{r_j}, \quad \xi^s = \prod_{k=1}^{n} \xi_k^{s_k}.
\]

This equation seems to be a natural extension of totally characteristic type studied by Chen-Tahara ([CT]) to several time-space variables.
Here we remark that the assumption $K \geq 2$ implies $\partial_{t_j}u(0,0) = 0$ ($j = 1, 2, \ldots, d$) which assures that $(0,0,u(0,0),\{\partial_{t_j}u(0,0)\},\{\partial_{x_k}u(0,0)\})$ belongs to the domain of definition of $f_{K+1}(t,x,u,\tau,\xi)$.

Now our first theorem is stated as follows:

**Theorem 1.** Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of the matrix $(a_{ij}(0))$. We assume that $b_k(x) \not\equiv 0$ and $b_k(0) = 0$ for $k = 1, 2, \ldots, n$, and let $\{\mu_k\}_{k=1}^n$ be the eigenvalues of Jacobi matrix of $(b_1(x),\ldots,b_n(x))$ at $x = 0$. Then the formal power series solution of (1.1) exists uniquely and converges if the following conditions are satisfied:

There exists a positive constant $\sigma_0 > 0$, such that

\begin{equation}
|\sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k| \geq \sigma_0 (|l| + |m|) \tag{Poincaré condition},
\end{equation}

and

\begin{equation}
\sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + c(0) \neq 0 \tag{Non-resonance condition}
\end{equation}

hold for all $(l,m) \in (\mathbb{Z}_{\geq 0})^d \times (\mathbb{Z}_{\geq 0})^n$ with $|l| \geq K$ and $|m| \geq 0$.

**Remark 1.** It is easy to show the following proposition.

The conditions (1.2) and (1.3) imply that

\begin{equation}
|\sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + c(0)| \geq \sigma (|l| + |m|)
\end{equation}

holds by some positive constant $\sigma > 0$ for all $(l,m) \in (\mathbb{Z}_{\geq 0})^d \times (\mathbb{Z}_{\geq 0})^n$ with $|l| \geq K$ and $|m| \geq 0$. In the proof of Theorem 1, this condition will be used instead of (1.2) and (1.3). $\square$

Next, we consider the following general equation:

\begin{equation}
\begin{cases}
f(t,x,u(t,x),\{\partial_{t_j}u(t,x)\},\{\partial_{x_k}u(t,x)\}) = 0, \\
u(0,x) \equiv 0.
\end{cases}
\end{equation}

**Assumption 1.** $f(t,x,u,\tau,\xi)$ ($\tau = (\tau_j) \in \mathbb{C}^d$, $\xi = (\xi_k) \in \mathbb{C}^n$) is holomorphic in a neighbourhood of the origin, and is an entire function in $\tau$ variables for any fixed $t, x, u$ and $\xi$. Moreover we assume that

\begin{equation}
f(0,x,0,\tau,0) \equiv 0
\end{equation}

for $x \in \mathbb{C}^n$ near the origin and $\tau \in \mathbb{C}^d$, which is a generalization of the definition of singular equations defined in [MS].
For the equation (1.5), we do not know whether or not the equation has a formal solution in general. Therefore, we assume the following:

**Assumption 2.** The equation (1.5) has a formal solution of the form

\[
(1.7) \quad u(t, x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|l| \geq 2, |m| \geq 0} u_{lm} t^l x^m \in \mathbb{C}[t, x].
\]

By the existence of a formal solution, \( \{\varphi_j(x)\} \) satisfy the following system formally:

\[
(1.8) \quad f(0, x, 0, \{\varphi_j(x)\}, 0) \equiv 0 \quad \text{(trivial relation)},
\]

and

\[
(1.9) \quad \frac{\partial}{\partial t_i} f(t, x, u(t, x), \{\partial_{t_j} u(t, x)\}, \{\partial_{x_k} u(t, x)\}) \bigg|_{t=0} = \frac{\partial f}{\partial t_i}(0, x, 0, \{\varphi_j(x)\}, 0) \varphi_i(x)
\]

\[
+ \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0, x, 0, \{\varphi_j(x)\}, 0) \frac{\partial \varphi_i}{\partial x_k}(x) = 0, \quad \text{for } i = 1, 2, \ldots, d.
\]

The formal solution of this system is not convergent in general. Therefore, we assume

**Assumption 3.** The coefficients \( \{\varphi_j(x)\} \) are all holomorphic in a neighbourhood of the origin of \( \mathbb{C}^n \).

**Remark 2.** In the case \( d = 1 \) (\( d \) is the dimension of \( t \) variables), a sufficient condition for the formal solution of (1.9) to converge has been already obtained by Miyake-Shirai [MS]. In the case \( d \geq 2 \), we give a sufficient condition for the formal solution of system (1.9) to be convergent, which will be given by Theorem 3 in Section 5, but for a while we consider the problem under Assumption 3 for simplicity of our arguments.

Now we put \( a(x) = (0, x, 0, \{\varphi_j(x)\}, 0) \) for simplicity, and define

\[
(1.10) \quad A_{ij}(x) := \frac{\partial^2 f}{\partial t_i \partial t_j}(a(x)) + \frac{\partial^2 f}{\partial u \partial t_j}(a(x)) \varphi_i(x) + \sum_{k=1}^{n} \frac{\partial^2 f}{\partial t_j \partial \xi_k}(a(x)) \frac{\partial \varphi_i}{\partial x_k}(x),
\]

for \( i, j = 1, 2, \ldots, d \). Moreover we define

\[
(1.11) \quad B_k(x) := \frac{\partial f}{\partial \xi_k}(a(x)), \quad \text{for } k = 1, 2, \ldots, n.
\]

**Remark 3.** The functions \( A_{ij}(x) \) and \( B_k(x) \) correspond to \( a_{ij}(x) \) and \( b_k(x) \) in Theorem 1, respectively (see (1.13) below).

Here we assume that the equation is of totally characteristic type, that is,
Assumption 4. \( B_k(x) \not\equiv 0 \) and \( B_k(0) = 0 \), for \( k = 1, 2, \ldots, n \).

Now our second theorem which is our main result is stated as follows:

**Theorem 2.** Suppose Assumptions 1, 2, 3 and 4. Let \( \{\lambda_j\}_{j=1}^d \) be the eigenvalues of \((A_{ij}(0))\), and let \( \{\mu_k\}_{k=1}^n \) be the eigenvalues of Jacobi matrix of the vector \((B_k(x))\) at \( x = 0 \). Then the formal solution (1.7) is convergent if the following condition is satisfied:

There exists a positive constant \( \sigma > 0 \), such that,

\[
\sum_{j=1}^d \lambda_j t_j + \sum_{k=1}^n \mu_k m_k + \frac{\partial f}{\partial u}(a(0)) \geq \sigma(|l| + |m|),
\]

holds for all \((l, m) \in (\mathbb{Z}_{\geq 0})^d \times (\mathbb{Z}_{\geq 0})^n\) with \(|l| \geq 2, |m| \geq 0\).

**Remark 4.** We put \( v(t, x) = u(t, x) - \sum_{j=1}^{d} \varphi_j(x) t_j \) as a new unknown function. By Assumptions 1, 2, 3 and 4, we can easily see that \( v(t, x) \) satisfies the equation of the following form:

\[
(1.13) \quad \sum_{i,j=1}^d A_{ij}(x) t_i \partial_{t_j} v + \sum_{k=1}^n B_k(x) \partial_{x_k} v + \frac{\partial f}{\partial u}(a(x)) v = \sum_{|l|=2} d_l(x) t^l + f_3(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}) \quad \text{and} \quad v(t, x) = O(|t|^2).
\]

This is an equation considered in Theorem 1 in the case \( K = 2 \). Therefore, it is sufficient to prove Theorem 1 in order to prove Theorem 2.

\[\square\]

2. **Reduction of the Equation**

As is mentioned in Remark 4, it is sufficient to study the equation (1.1).

By the assumption of Theorem 1,

\[
(a_{ij}(0)) \sim \begin{pmatrix} \lambda_1 & \delta_1 \\ \lambda_2 & \ldots & \delta_2 \\ \vdots & \ddots & \ddots \\ \lambda_{d-1} & \delta_{d-1} & \ldots & \lambda_d \end{pmatrix}, \quad \frac{\partial (b_1, \ldots, b_n)}{\partial (x_1, \ldots, x_n)}|_{x=0} \sim \begin{pmatrix} \mu_1 & \nu_1 \\ \mu_2 & \vdots & \ddots \\ \mu_{n-1} & \nu_{n-1} & \ldots & \mu_n \end{pmatrix},
\]

where \( \delta_j, \nu_k = 0 \) or 1 \((1 \leq j \leq d - 1, 1 \leq k \leq n - 1)\).

Then by transforming the variables, (1.1) is reduced to the following form:

\[
(2.1) \quad (\Lambda + \Delta)v(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^d \beta_{ij}(x) t_i \partial_{t_j} v + \gamma(x) v + \sum_{k=1}^n \varphi_k(x) \partial_{x_k} v + \tilde{f}_{K+1}(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}),
\]
with \( v(t, x) = O(|t|^K) \), where
\[
\Lambda = \sum_{j=1}^{d} \lambda_{j} t_{j} \partial_{t_{j}} + \sum_{k=1}^{n} \mu_{k} x_{k} \partial_{x_{k}} + c(0),
\]
\[
\Delta = \sum_{j=1}^{d-1} \delta_{j} t_{j} \partial_{t_{j+1}} + \sum_{k=1}^{n-1} \nu_{k} x_{k} \partial_{x_{k+1}},
\]
and \( \alpha_{l}(x) \), \( \beta_{ij}(x) \), \( \gamma(x) \) and \( \varphi_{k}(x) \) are holomorphic in a neighbourhood of the origin, and satisfy \( \beta_{ij}(x) = O(|x|) \), \( \gamma(x) = O(|x|) \) and \( \varphi_{k}(x) = O(|x|^2) \), and \( \tilde{f}_{K+1}(t, x, u, \tau, \xi) \) is a holomorphic function which has a similar Taylor expansion with \( f_{K+1}(t, x, u, \tau, \xi) \).

In the following sections, we shall prove the existence and convergence of the unique formal solution of (2.1).

3. Preparation to prove Theorem 1

Let \( \mathbb{C}[t, x]_{L,M} \) be the set of homogeneous polynomial of degree \( L \) in \( t \) variables and of degree \( M \) in \( x \) variables, that is,
\[
\mathbb{C}[t, x]_{L,M} = \left\{ f_{LM}(t, x) = \sum_{|l|=L, |m|=M} f_{lm} t^l x^m | f_{lm} \in \mathbb{C} \right\}.
\]
For the operator \( \Lambda + \Delta \), the following lemma holds:

Lemma 1. For all \( L \geq K \) and \( M \geq 0 \), the operator \( \Lambda + \Delta : \mathbb{C}[t, x]_{L,M} \rightarrow \mathbb{C}[t, x]_{L,M} \) is invertible. Moreover, if the majorant relation \( f_{LM}(t, x) \ll F \times (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M \) \( (f_{LM}(x) \in \mathbb{C}[t, x]_{L,M}, \ F > 0) \) holds, then we obtain the following majorant relation:
\[
(3.1) \quad (\Lambda + \Delta)^{-1} f_{LM}(t, x) \ll \frac{C}{L+M} F \times (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M,
\]
where \( C > 0 \) is a positive constant independent of \( L \) and \( M \).

Proof. We define a norm of \( u_{LM}(t, x) \in \mathbb{C}[t, x]_{L,M} \) by
\[
||u_{LM}|| := \inf \left\{ C > 0 \mid u_{LM}(t, x) \ll C (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M \right\}.
\]
We remark that \( \mathbb{C}[t, x]_{L,M} \) becomes a Banach space by this norm.

First, by (1.4) it is easily checked that \( \Lambda \) is invertible on \( \mathbb{C}[t, x]_{L,M} \) and
\[
(3.2) \quad ||\Lambda^{-1}|| \leq \frac{1}{\sigma(L+M)}
\]
holds for the operator norm of \( \Lambda^{-1} \) on \( \mathbb{C}[t, x]_{L,M} \).
Next, since $u_{LM}(t, x) \ll ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$, we have

\[
\Delta u_{LM}(t, x) \ll \sum_{j=1}^{d-1} L|\delta_j| \cdot ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M \\
+ \sum_{k=1}^{n-1} M|\nu_k| \cdot ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M
\]

\[
\ll \left\{ L(d-1) \max_{j=1, \ldots, d-1} |\delta_j| + M(n-1) \max_{k=1, \ldots, n-1} |\nu_k| \right\} \times \\
||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M.
\]

Here we make a change of variables by $t_j = \epsilon^{j-1} \tau_j$, $x_k = \epsilon^{k-1} y_k$, then $\delta_j$ and $\nu_k$ (the components of nilpotent part of Jordan canonical form) turns to $\epsilon \delta_j$ and $\epsilon \nu_k$, respectively. Therefore, by choosing $\epsilon$ sufficiently small, we may assume that the components of nilpotent part are small enough. Hence we may assume that

\[
\max_{j=1, \ldots, d-1} |\delta_j| < \frac{\sigma}{2(d-1)}, \quad \max_{k=1, \ldots, n-1} |\nu_k| < \frac{\sigma}{2(n-1)}.
\]

Then

\[
\Delta u_{LM}(t, x) \ll \frac{\sigma(L + M)}{2} ||u_{LM}||(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M
\]

holds, and we obtain

\[
||\Delta|| \leq \frac{\sigma(L + M)}{2}.
\]

Therefore, the operator norm of $\Delta \Lambda^{-1}$ is estimated by

\[
||\Delta \Lambda^{-1}|| \leq \frac{1}{\sigma(L + M)} \cdot \frac{\sigma(L + M)}{2} = \frac{1}{2} < 1.
\]

By using the Neumann's series, we can see that $\Lambda + \Delta$ is invertible and the norm of the inverse operator is estimated by

\[
||\left( \Lambda + \Delta \right)^{-1}|| \leq \frac{2}{\sigma(L + M)},
\]

which we want to prove since $C = 2/\sigma$ is independent of $L$ and $M$.

Now, we define some notations, which are used in the proof of Theorem 1.

**Definition** (1) Let $(t, x) \in \mathbb{C}^d \times \mathbb{C}^n$ ($d \geq 0$, $n \geq 0$) be complex variables. For formal power series $f(t, x) = \sum_{|l| \geq 0, |m| \geq 0} f_{l,m} t^l x^m$, we define

\[
|f|(t, x) = \sum_{|l| \geq 0, |m| \geq 0} |f_{l,m}| t^l x^m.
\]
(2) Let \((t, X) \in \mathbb{C}^d \times \mathbb{C} \ (d \geq 0)\) be complex variables. For formal power series 
\[ f(t, X) = \sum_{|l| \geq 0, \ M \geq 0} f_{l, M} t^l X^M, \]
we define the shift operator \(S\) by
\[
S(f)(t, X) = \sum_{|l| \geq 0, \ M \geq 0} f_{l, M+1} t^l X^M = \frac{f(t, X) - f(t, 0)}{X}.
\]

Remark 5. The following facts are easily shown:

- \(f(t, x) \ll |f|(t, x)\);
- If \(f(t, x)\) and \(g(t, X)\) are convergent power series, then \(|f|(t, x)\) and \(S(g)(t, X)\) are also convergent.

4. PROOF OF THEOREM 1

First, we prove a unique existence of formal power series solution.

Let
\[
u(t, x) = \sum_{|l| \geq K, |m| \geq 0} u_{lm} t^l x^m = \sum_{L \geq K} u_L(t, x) = \sum_{L \geq K, M \geq 0} u_{LM}(t, x)
\]
be a formal solution of (2.1), where
\[
u_{LM}(t, x) = \sum_{|l|=L, |m|=M} u_{lm} t^l x^m \in \mathbb{C}[t, x]_{L,M},
\]
\[
u_L(t, x) = \sum_{|l|=L} u_l(x) t^l = \sum_{M \geq 0} u_{LM}(t, x).
\]

We put \(P = \Lambda + \Delta\) for simplicity. We substitute \(u(t, x) = \sum_{L \geq K} u_L(t, x)\) into (2.1), then we have the following recursion formula:

\[
\begin{align*}
Pu_K(t, x) &= \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^d \beta_{ij}(x) t_i \partial_{t_j} u_K(t, x) \\
&\quad + \gamma(x) u_K(t, x) + \sum_{k=1}^n \varphi_k(x) \partial_{x_k} u_K(t, x), \\
Pu_L(t, x) &= \sum_{i,j=1}^d \beta_{ij}(x) t_i \partial_{t_j} u_L(t, x) + \gamma(x) u_L(t, x) + \sum_{k=1}^n \varphi_k(x) \partial_{x_k} u_L(t, x) \\
&\quad + G_L(t, x, \{u_p\}_{K \leq p < L}, \{\partial_{t_j} u_p\}_{K \leq p < L}, \{\partial_{x_k} u_p\}_{K \leq p < L}), \text{ for } L > K,
\end{align*}
\]

where \(G_L(t, x, \zeta, \tau, \xi)\) is a polynomial of \((t, \zeta, \tau, \xi)\).
First, we consider the case $L = K$. We substitute $u_K(t, x) = \sum_{M \geq 0} u_{KM}(t, x)$ into the above recursion formula, we have

$$
\begin{align*}
Pu_{K0}(t, x) &= \sum_{|l|=K} \alpha_l(0) t^l,
Pu_{KM}(t, x) &= \sum_{|l|=K} \alpha_l^M(x) t^l + \sum_{i,j=1}^{d} \sum_{p=1}^{M} \beta_{ij}^p(x) t_i \partial_{t_j} u_{K,M-p}(t, x)
&+ \sum_{p=1}^{M} \gamma^p(x) u_{K,M-p}(t, x) + \sum_{k=1}^{n} \sum_{p=2}^{M} \varphi_k^p(x) \partial_{x_k} u_{K,M-p+1}(t, x),
\end{align*}
$$

where we put

$$
\begin{align*}
\alpha_l(x) &= \sum_{M \geq 0} \alpha_l^M(x),
\alpha_l^M(x) &= \sum_{|m|=M} \alpha_{lm} x^m,
\beta_{ij}(x) &= \sum_{M \geq 1} \beta_{ij}^M(x),
\beta_{ij}^M(x) &= \sum_{|m|=M} \beta_{ijm} x^m,
\gamma(x) &= \sum_{M \geq 1} \gamma^M(x),
\gamma^M(x) &= \sum_{|m|=M} \gamma_{m} x^m,
\varphi_k(x) &= \sum_{M \geq 2} \varphi_k^M(x),
\varphi_k^M(x) &= \sum_{|m|=M} \varphi_{km} x^m.
\end{align*}
$$

By Lemma 1, we know that the solution sequence $\{u_{KM}(t, x)\}_{M \geq 0}$ exists uniquely. Moreover, by the same argument, we see that $\{u_{LM}(t, x)\}$ $(L > K)$ exist uniquely. These show that the formal solution exists uniquely.

Next, we prove the convergence of the formal solution. We put $U(t, x) = Pu(t, x)$ as a new unknown function. By Lemma 1, the equation (2.1) is reduced to the following equation:

$$
(4.1) \quad U(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} P^{-1} U(t, x)
+ \gamma(x) P^{-1} U(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} P^{-1} U(t, x)
+ \tilde{f}_{K+1}(t, x, P^{-1} U(t, x), \{ \partial_{t_j} P^{-1} U(t, x) \}, \{ \partial_{x_k} P^{-1} U(t, x) \}).
$$

We know that (4.1) has a unique formal solution of the form

$$
U(t, x) = \sum_{|l| \geq K, |m| \geq 0} U_{lm} t^l x^m = \sum_{L \geq K} U_L(t, x) = \sum_{L \geq K, M \geq 0} U_{LM}(t, x).
$$
In order to get a majorant series of $U(t, x)$, we prepare some majorant series for the coefficients of (4.1). We put $T = t_1 + \cdots + t_d$, $X = x_1 + \cdots + x_n$, and choose
\[ \sum_{|l|=K} \alpha_l(x) t^l \ll A(X)T^K, \quad \beta_{ij}(x) \ll |\beta_{ij}|(X, \ldots, X) =: XB_{ij}(X), \]
\[ \gamma(x) \ll |\gamma|(X, \ldots, X) =: XG(X), \quad \varphi_k(x) \ll |\varphi_k|(X, \ldots, X) =: X^2\Phi_k(X), \]
\[ \overline{f}_{K+1}(t, x, u, \tau, \xi) \ll |\tilde{f}_{K+1}|(T, \ldots, T, X, \ldots, X, u, \tau, \xi) =: F_{K+1}(T, X, u, \tau, \xi). \]

where $A(X)$, $B_{ij}(X)$, $G(X)$ and $\Phi_k(X)$ are holomorphic in a neighbourhood of $X = 0$, and $F_{K+1}(T, X, u, \tau, \xi)$ is also holomorphic near $(T, X, u, \tau, \xi) = (0, 0, 0, 0, 0)$.

Now, we consider the following equation:

\begin{equation}
(4.2) \quad w(T, X) = A(X)T^K + C \sum_{i, j=1}^d XB_{ij}(X)w(T, X) \\
+ CXG(X)w(T, X) + C \sum_{k=1}^n X^2\Phi_k(x)(t, x)S(w)(T, X) \\
+ F_{K+1} \left(T, X, u, \frac{Cw}{T}, \{CS(w)\} \right),
\end{equation}

where $C$ is a positive constant appeared in Lemma 1.

Let $w(T, X) = \sum_{L \geq K, M \geq 0} w_{LM}(T, X)$ be the formal solution of (4.2). By the construction of (4.2), we can easily check that $U(t, x) \ll w(T, X)$ by the next lemma.

**Lemma 2.** For two formal power series $U(t, x)$ and $w(T, X)$ satisfying
\[ U(t, x) = \sum_{L \geq K, M \geq 0} U_{LM}(t, x) \ll w(T, X) = \sum_{L \geq K, M \geq 0} w_{LM}T^L X^M, \]
the following majorant relations hold:

1. $P^{-1}U(t, x) \ll Cw(T, X)$,
2. $t_i \partial t_j P^{-1}U(t, x) \ll Cw(T, X)$,
3. $\partial t_j P^{-1}U(t, x) \ll \frac{Cw(T, X)}{T}$,
4. $\partial x_k P^{-1}U(t, x) \ll CS(w)(T, X)$.

**Proof.** By using Lemma 1, we can prove this lemma easily. First, (1) is proved as follows:

\[ P^{-1}U(t, x) = \sum_{L \geq K, M \geq 0} P^{-1}U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{C}{L+M} w_{LM}T^L X^M \ll Cw(T, X). \]
Secondly, (2) and (3) is proved as follows:

\[ t_{i} \partial_{t_{j}} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 0} t_{i} \partial_{t_{j}} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{C L}{L + M} w_{LM} T^{L} X^{M} \ll C w(T, X); \]

\[ \partial_{t_{j}} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 0} \partial_{t_{j}} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{C L}{L + M} w_{LM} T^{L-1} X^{M} \ll \frac{C w(T, X)}{T}. \]

Finally, (4) is proved as follows:

\[ \partial_{x_{k}} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 1} \frac{C M}{L + M} w_{LM} T^{L} X^{M-1} \ll CS(w)(T, X). \]

This completes the proof. \( \square \)

Since \( w(T, X) \gg 0 \), we have

\[ (4.3) \quad XS(w)(T, X) = w(T, X) - w(T, 0) \ll w(T, X). \]

Let us consider the following equation:

\[ (4.4) \quad v(T, X) = A(X) T^{K} + CXh(X) v(T, X) + F_{K+1}(T, X, CTv, \{Cy\}, \{CTS(y)\}), \]

with \( v(T, X) = O(T^{K}) \), where \( h(X) = \sum_{i,j=1}^{d} B_{ij}(X) + G(X) + \sum_{k=1}^{n} \Phi_{k}(X) \). Then the following majorant relation is obvious:

\[ w(T, X) \ll v(T, X). \]

We put \( y(T, X) = v(T, X)/T \) as a new unknown function. By substituting this into (4.4), we see that \( y(T, X) \) satisfies

\[ (4.5) \quad y(T, X) = A(X) T^{K-1} + CXh(X) y(T, X) + \frac{1}{T} F_{K+1}(T, X, CTy, \{Cy\}, \{CTS(y)\}), \]

with \( y(T, X) = O(T^{K-1}) \).

We decompose the formal solution \( y(T, X) \) as follows:

\[ y(T, X) = y_{1}(X) T^{K-1} + y_{2}(X) T^{K} + T^{K} z(T, X). \]
We remark that \( y_1(X) \) and \( y_2(X) \) are holomorphic functions in a neighbourhood of \( X = 0 \). Indeed, \( y_1(X) \) and \( y_2(X) \) are given by

\[
y_1(X) = \frac{A(X)}{1 - CXh(X)},
\]

\[
y_2(X) = \frac{1}{1 - CXh(X)} \sum_{|p| + Kq + (K-1)|r| + K|s| = K+1} F_{pqrs}(X) \{Cy_1(X)\}^{q + |r|} \{CS(y_1)(X)\}^{s}.
\]

These are holomorphic functions in a neighbourhood of \( X = 0 \).

In this case, \( z(T, X) \) satisfies the following equation:

\[
\begin{cases}
z(T, X) = CXh(X)z(T, X) + H(T, X, Tz(T, X), TS(z)(T, X)), \\
z(0, X) \equiv 0,
\end{cases}
\]

where

\[
H(T, X, \eta_1, \eta_2) = \frac{1}{T^{K+1}} \left[ F_{K+1}(T, X, Cy_1(X)T^K + Cy_2(X)T^{K+1} + CT^K \eta_1, \\
\{Cy_1(X)T^{K-1} + Cy_2(X)T^K + CT^{K-1}\eta_1\}, \\
\{CS(y_1)(X)T^K + CS(y_2)(X)T^{K+1} + CT^K \eta_2\}) \right] \\
- \sum_{|p| + Kq + (K-1)|r| + K|s| = K+1} F_{pqrs}(X) \{Cy_1(X)\}^{q + |r|} \{CS(y_1)(X)\}^{s}.
\]

**Remark 6.** The order of zeros in \( T \) variable of \( H(T, X, CTz(T, X), CTS(z)(T, X)) \) is greater than or equal to 1. \( \square \)

In order to prove the convergence of \( z(T, X) \), it is sufficient to show the following:

**Lemma 3.** There exists a small positive constant \( \epsilon > 0 \) such that \( z_\epsilon(\rho) = z(\epsilon \rho, \rho) \) is convergent in a neighbourhood of \( \rho = 0 \).

**Proof.** We substitute \( T = \epsilon \rho \) and \( X = \rho \) into (4.6) and by using the relation (4.3), we have

\[
\rho S(z)(\epsilon \rho, \rho) \ll z_\epsilon(\rho).
\]

By this relation, the following majorant relation also holds,

\[
TS(z)(T, X)|_{T=\epsilon \rho, X=\rho} = \epsilon \rho S(z)(\epsilon \rho, \rho) \ll \epsilon z_\epsilon(\rho).
\]

Here we consider

\[
(4.7) \quad \psi(\rho) = C\rho h(\rho)\psi(\rho) + H(\epsilon \rho, \rho, \epsilon \rho \psi(\rho), \epsilon \psi(\rho)).
\]
In the right hand side of (4.7), we decompose $H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho))$ into the term of $\psi(\rho)$ and otherwise as follows:

$$H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)) = \epsilon \frac{\partial H}{\partial \eta_2}(0, 0, 0, 0)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).$$

We remark that the following fact holds:

$$\frac{\partial \overline{H}}{\partial \psi}(\epsilon\rho, \rho, \epsilon\rho\psi, \epsilon\psi)|_{\rho=0, \psi=0} = 0.$$

We put $(\partial H/\partial \eta_2)(0, 0, 0, 0) = K_0 \geq 0$, then (4.7) is rewritten by

$$(1-\epsilon K_0)\psi(\rho) = C \rho h(\rho)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).$$

We choose $\epsilon > 0$ with $1-\epsilon K_0 > 0$. Then by using the implicit function theorem, we can see that (5.8) has a unique holomorphic solution $\psi(\rho)$ with $\psi(0) = 0$ in a neighbourhood of $\rho = 0$. Moreover we can see $z_\epsilon(\rho) \ll \psi(\rho)$.

Thus we complete the proof of Lemma 3. \(\square\)

5. SOLVABILITY OF THE SYSTEM (1.9)

In this section, we give a sufficient condition for the formal solution of the system (1.9) to be convergent. Recall that (1.9) is

$$\begin{align*}
\frac{\partial f}{\partial t_i}(0, x, 0, \{\varphi_j(x)\}, 0) + \frac{\partial f}{\partial u}(0, x, 0, \{\varphi_j(x)\}, 0)\varphi_i(x) \\
+ \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0, x, 0, \{\varphi_j(x)\}, 0)\frac{\partial \varphi_i(x)}{\partial x_k} = 0, \quad i = 1, 2, \ldots, d.
\end{align*}$$

By Assumption 4 of Theorem 2, the condition

$$\frac{\partial f}{\partial \xi_k}(0, 0, 0, \{\varphi_j(0)\}, 0) = 0, \quad k = 1, 2, \ldots, n$$

was assumed.

Let $\varphi(x) = \{\varphi_1(x), \ldots, \varphi_d(x)\}$ be the unknown functions. We put $\varphi(0) = \{\varphi_1^0, \ldots, \varphi_d^0\} \in \mathbb{C}^d$ as the constant term of $\varphi(x)$. We substitute $\varphi_j(x) = \varphi_j^0 + \psi_j(x)$ into the system (1.9), and by restricting at $x = 0$, we see that $\{\varphi_j^0\}$ satisfies the following system:

$$\begin{align*}
\frac{\partial f}{\partial t_i}(0, 0, 0, \{\varphi_j^0\}, 0) + \frac{\partial f}{\partial u}(0, 0, 0, \{\varphi_j^0\}, 0)\varphi_i^0 = 0, \quad i = 1, 2, \ldots, d.
\end{align*}$$

This system has some roots by the assumption of the existence of a formal solution, and we fix such $\{\varphi_j^0\}$. 
For such fixed \( \{\varphi_{j}^{0}\} \), we see that \( \{\psi_{j}(x)\} \) satisfies the system of the followir

\[
(5.2) \quad \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0,0,0,\{\varphi_{j}^{0}\},0)x_l \frac{\partial \psi_i}{\partial x_k}(x) \\
+ \sum_{k=1}^{n} \sum_{p=1}^{d} \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0,0,0,\{\varphi_{j}^{0}\},0) \psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x) \\
+ \frac{\partial f}{\partial u}(0,0,0,\{\varphi_{j}^{0}\},0) \psi_i(x) \\
+ \sum_{p=1}^{d} \left\{ \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0,0,0,\{\varphi_{j}^{0}\},0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0,0,0,\{\varphi_{j}^{0}\},0) \varphi_{i}^{0} \right\} \psi_p(x) \\
+ \sum_{l=1}^{n} \left\{ \frac{\partial^2 f}{\partial t_i \partial x_l}(0,0,0,\{\varphi_{j}^{0}\},0) + \frac{\partial^2 f}{\partial u \partial x_l}(0,0,0,\{\varphi_{j}^{0}\},0) \varphi_{i}^{0} \right\} x_l = 0 \\
\text{degree in } x \text{ is greater than or equal to } 2, \quad i = 1,2,\ldots,d.
\]

This system is written as follows for simplicity,

\[
(5.3) \quad \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \frac{\partial \psi_i}{\partial x_k}(x) + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x) \\
+ c \psi_i(x) + \sum_{p=1}^{d} d_{ip} \psi_p(x) + \sum_{l=1}^{n} e_{il} x_l = 0 \\
\text{degree in } x \text{ is greater than or equal to } 2, \quad i = 1,2,\ldots,d,
\]

where

\[ a_{kl} := \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0,0,0,\{\varphi_{j}^{0}\},0), \quad b_{kp} := \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0,0,0,\{\varphi_{j}^{0}\},0), \]

\[ c := \frac{\partial f}{\partial u}(0,0,0,\{\varphi_{j}^{0}\},0), \]

\[ d_{ip} := \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0,0,0,\{\varphi_{j}^{0}\},0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0,0,0,\{\varphi_{j}^{0}\},0) \varphi_{i}^{0}, \]

\[ e_{il} := \frac{\partial^2 f}{\partial t_i \partial x_l}(0,0,0,\{\varphi_{j}^{0}\},0) + \frac{\partial^2 f}{\partial u \partial x_l}(0,0,0,\{\varphi_{j}^{0}\},0) \varphi_{i}^{0}. \]
Here we decompose $\psi_i(x)$ into $\psi_i(x) = \overline{\psi}_i(x) + \eta_i(x)$ ($\overline{\psi}_i(x) = \sum_{k=1}^{n} \psi_{ik} x_k$, $\eta_i(x) = O(|x|^2)$). We substitute this into the system (5.3) and obtain

\begin{align*}
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \left( \frac{\partial \overline{\psi}_i(x)}{\partial x_k} + \frac{\partial \eta_i(x)}{\partial x_k} \right) \\
+ \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} (\overline{\psi}_p(x) + \eta_p(x)) \left( \frac{\partial \overline{\psi}_i(x)}{\partial x_k} + \frac{\partial \eta_i(x)}{\partial x_k} \right) \\
+ c(\overline{\psi}_i(x) + \eta_i(x)) + \sum_{p=1}^{d} d_{ip} (\overline{\psi}_p(x) + \eta_p(x)) + \sum_{l=1}^{n} e_{il} x_l \\
= (\text{degree in } x \text{ is greater than or equal to } 2), \quad i = 1, 2, \ldots, d.
\end{align*}

By picking up the degree 1 part on the both sides, we see that $\{\overline{\psi}_i(x)\}$ satisfy the following system:

\begin{align*}
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \frac{\partial \overline{\psi}_i(x)}{\partial x_k} + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \overline{\psi}_p(x) \frac{\partial \overline{\psi}_i(x)}{\partial x_k} \\
+ c(\overline{\psi}_i(x) + \eta_i(x)) + \sum_{p=1}^{d} d_{ip} \overline{\psi}_p(x) + \sum_{l=1}^{n} e_{il} x_l = 0,
\end{align*}

for $i = 1, 2, \ldots, d$.

By the existence of a formal solution, (5.5) has some solutions $\{\overline{\psi}_i(x)\}$ of linear functions, and we fix such $\{\overline{\psi}_i(x)\}$.

For fixed $\{\varphi_i^0\}$ and $\{\overline{\psi}_i(x)\}$, we see that $\{\eta_i(x)\}$ satisfy the following system:

\begin{align*}
\sum_{k=1}^{n} \sum_{l=1}^{n} \left( a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_{pl} \right) x_l \frac{\partial \eta_i(x)}{\partial x_k} \\
+ c(\overline{\psi}_i(x) + \eta_i(x)) + \sum_{p=1}^{d} \left( d_{ip} + \sum_{k=1}^{n} b_{kp} \psi_{ik} \right) \eta_p(x) \\
= (\text{degree in } x \text{ is greater than or equal to } 2.), \quad i = 1, 2, \ldots, d.
\end{align*}

We remark that the degree 2 part in the right hand side of this system does not include $\{\eta_i(x)\}$.

The following theorem holds:

**Theorem 3.** Let $(A_{kl})_{k,l=1,2,\ldots,n}$ be a matrix defined by

$$(A_{kl})_{k,l=1,2,\ldots,n} = \left( a_{kl} + \sum_{p=1}^{d} b_{kp} \psi_{pl} \right)_{k,l=1,2,\ldots,n}$$.
Let $\{\kappa_k\}_{k=1}^n$ be the eigenvalues of $(A_{kl})_{k,l=1,2,\ldots,n}$. If there exists a positive constant $\sigma_0$ such that the condition
\[
\left| \sum_{k=1}^n \kappa_k m_k \right| \geq \sigma_0|m|, \text{ (Poincaré condition)}
\]
holds for all $m = (m_1, \ldots, m_n) \in (\mathbb{Z}_{\geq 0})^n$ with $|m| \geq 2$, then the formal solution of the system (1.9) is convergent in a neighbourhood of the origin.

**Remark 7.** Let $(B_{ip})_{i,p=1,2,\ldots,d}$ be a matrix defined by
\[
(B_{ip})_{i,p=1,2,\ldots,d} = \left( d_{ip} + \sum_{k=1}^n b_{kp}\psi_{ik} \right),
\]
and let $\{\omega_j\}_{j=1}^d$ be the eigenvalues of $(B_{ip})_{i,p=1,2,\ldots,d}$.

By the same argument in Remark 1, we have
\[
(5.7) \quad \left| \sum_{k=1}^n \kappa_k m_k + c + \omega_j \right| \geq \sigma|m|, \text{ by some } \sigma > 0, \text{ and } j = 1, 2, \ldots, d,
\]
for large $m$, which will be used in the proof. $\blacksquare$

6. PROOF OF THEOREM 3

The proof of Theorem 3 is the same method of the proof of Theorem 1 in case that the unknown function is a vector values. However, there are some difference in the detail. Therefore, we introduce only the outline of the proof of Theorem 3 in this section.

Step 1. By taking a linear transformation of the independent variables and a linear transformation of the unknown functions, (5.6) is reduced to the following form:

\[
(\Lambda + \Delta + \mathbf{B}) \left( \begin{array}{c} w_1(x) \\ \vdots \\ w_d(x) \end{array} \right) = \left( \begin{array}{c} \Lambda_1 \\ \vdots \\ \Lambda_d \end{array} \right) + \left( \begin{array}{c} \Delta \\ \vdots \\ \Delta \end{array} \right) + \mathbf{B} \left( \begin{array}{c} w_1(x) \\ \vdots \\ w_d(x) \end{array} \right) = \left( \begin{array}{c} \sum_{|m|=2} a_{1,m}x^m + g_{3,1}(x, w(x), \partial_x w(x)) \\ \vdots \\ \sum_{|m|=2} a_{d,m}x^m + g_{3,d}(x, w(x), \partial_x w(x)) \end{array} \right),
\]
where \( w_j(x) \) \((j = 1, 2, \ldots, d)\) denote new unknown functions after linear transformations and

\[
\Lambda_j = \sum_{k=1}^{n} \kappa_k x_k \partial_{x_k} + c + \omega_j, \quad \Delta = \sum_{k=1}^{n-1} \varepsilon_k x_k \partial_{x_{k+1}}, \quad B = \begin{pmatrix}
0 & \varepsilon_1 \\
\vdots & \ddots \\
\varepsilon_{d-1} & 0
\end{pmatrix},
\]

where \( \varepsilon_j \) and \( e_j \) denote the nilpotent components of the Jordan canonical forms of the matrices \((A_{kl})\) and \((B_{ip})\), respectively, and

\[
g_{3,i}(x, \eta, \zeta) = \sum_{|\alpha|+2|\beta|+|\gamma| \geq 3} g_{\alpha\beta\gamma}^{(i)} x^\alpha \eta^\beta \zeta^\gamma.
\]

**Step 2.** We define \( C[x]_M \) by \( C[x]_M = \{ \sum_{|m|=M} u_n x^m ; u_n \in \mathbb{C} \} \), and define a norm of \( u(x) = (u_1(x), \ldots, u_d(x)) \in (C[x]_M)^d \) by

\[
||u|| := \inf\{ C > 0 ; u_i(x) \ll C(x_1 + \cdots + x_n)^M, \ i = 1, 2, \ldots, d \}.
\]

By the same argument in the proof of Lemma 1 and by Remark 7, we can prove the same results of Lemma 1 for the operator \( \Lambda + \Delta + B \).

**Step 3.** By the same method in the previous sections, we can construct a majorant equation whose formal solution is a majorant function of the all unknown functions of the system. Finally, by the implicit function theorem, we prove the convergence of the formal solution of the majorant equation.

**References**


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