<table>
<thead>
<tr>
<th>Title</th>
<th>CONVERGENCE OF FORMAL SOLUTIONS OF SINGULAR FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF TOTALLY CHARACTERISTIC TYPE (Integral representations and twisted cohomology in the theory of differential equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Shirai, Akira</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2001, 1212: 116-132</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41152">http://hdl.handle.net/2433/41152</a></td>
</tr>
<tr>
<td>Right</td>
<td>Type</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
1. Introduction

Let \((t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n) \in \mathbb{C}^d \times \mathbb{C}^n\) be \((d + n)\)-dimensional complex variables \((d \geq 1, n \geq 1)\).

We consider the following first order nonlinear partial differential equation:

\[
\begin{aligned}
\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} u + \sum_{k=1}^{n} b_k(x) \partial_{x_k} u + c(x) u &= \sum_{|l|=K} d_l(x) t^l + f_{K+1}(t, x, u, \{\partial_{t_j} u\}, \{\partial_{x_k} u\}), \\
u(t, x) &= O(|t|^K),
\end{aligned}
\]

(1.1)

where \(|t| = t_1 + \cdots + t_d\), \(K\) is a fixed positive integer satisfying \(K \geq 2\) and \(a_{ij}(x), b_k(x), c(x)\) and \(d_l(x)\) are holomorphic in a neighbourhood of the origin, and \(f_{K+1}(t, x, u, \tau, \xi)\) \((\tau = (\tau_j) \in \mathbb{C}^d, \xi = (\xi_k) \in \mathbb{C}^n)\) is also holomorphic in a neighbourhood of the origin with the following Taylor expansion:

\[
f_{K+1}(t, x, u, \tau, \xi) = \sum_{|p|+|q|+(K-1)|r|+K|s| \geq K+1} f_{pqrs}(x) t^p u^q \tau^r \xi^s,
\]

where \(q \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}\), \(p = (p_1, \ldots, p_d) \in (\mathbb{Z}_{\geq 0})^d\), \(r = (r_1, \ldots, r_d) \in (\mathbb{Z}_{\geq 0})^d\), \(s = (s_1, \ldots, s_n) \in (\mathbb{Z}_{\geq 0})^n\),

\(|p| = p_1 + \cdots + p_d, \ |r| = r_1 + \cdots + r_d, \ |s| = s_1 + \cdots + s_n,\)

and

\[
t^p = \prod_{j=1}^{d} t_j^{p_j}, \quad \tau^r = \prod_{j=1}^{d} \tau_j^{r_j}, \quad \xi^s = \prod_{k=1}^{n} \xi_k^{s_k}.
\]

This equation seems to be a natural extension of totally characteristic type studied by Chen-Tahara ([CT]) to several time-space variables.
Here we remark that the assumption $K \geq 2$ implies $\partial_{t_j}u(0,0) = 0$ ($j = 1, 2, \ldots, d$) which assures that $(0, 0, u(0,0), \{\partial_{t_j}u(0,0)\}, \{\partial_{x_k}u(0,0)\})$ belongs to the domain of definition of $f_{K+1}(t, x, u, \tau, \xi)$.

Now our first theorem is stated as follows:

**Theorem 1.** Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of the matrix $(a_{ij}(0))$. We assume that $b_k(x) \neq 0$ and $b_k(0) = 0$ for $k = 1, 2, \ldots, n$, and let $\{\mu_k\}_{k=1}^n$ be the eigenvalues of Jacobi matrix of $(b_1(x), \ldots, b_n(x))$ at $x = 0$. Then the formal power series solution of (1.1) exists uniquely and converges if the following conditions are satisfied:

There exists a positive constant $\sigma_0 > 0$, such that

(1.2) \[ \left| \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k \right| \geq \sigma_0(|l| + |m|) \quad \text{(Poincaré condition)}, \]

and

(1.3) \[ \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + c(0) \neq 0 \quad \text{(Non-resonance condition)} \]

hold for all $(l, m) \in \mathbb{Z}_{\geq 0}^d \times \mathbb{Z}_{\geq 0}^n$ with $|l| \geq K$ and $|m| \geq 0$.

**Remark 1.** It is easy to show the following proposition.

The conditions (1.2) and (1.3) imply that

(1.4) \[ \left| \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + c(0) \right| \geq \sigma(|l| + |m|) \]

holds by some positive constant $\sigma > 0$ for all $(l, m) \in \mathbb{Z}_{\geq 0}^d \times \mathbb{Z}_{\geq 0}^n$ with $|l| \geq K$ and $|m| \geq 0$. In the proof of Theorem 1, this condition will be used instead of (1.2) and (1.3).

Next, we consider the following general equation:

(1.5) \[ \left\{ \begin{array}{l} f(t, x, u(t, x), \{\partial_{t_j}u(t, x)\}, \{\partial_{x_k}u(t, x)\}) = 0, \\
 u(0, x) \equiv 0. \end{array} \right. \]

**Assumption 1.** $f(t, x, u, \tau, \xi)$ ($\tau = (\tau_j) \in \mathbb{C}^d$, $\xi = (\xi_k) \in \mathbb{C}^n$) is holomorphic in a neighbourhood of the origin, and is an entire function in $\tau$ variables for any fixed $t, x, u$ and $\xi$. Moreover we assume that

(1.6) \[ f(0, x, 0, \tau, 0) \equiv 0 \]

for $x \in \mathbb{C}^n$ near the origin and $\tau \in \mathbb{C}^d$, which is a generalization of the definition of singular equations defined in [MS].
For the equation (1.5), we do not know whether or not the equation has a formal solution in general. Therefore, we assume the following:

**Assumption 2.** The equation (1.5) has a formal solution of the form

\[(1.7) \quad u(t,x) = \sum_{j=1}^{d} \varphi_{j}(x) t_{j} + \sum_{|l| \geq 2, |m| \geq 0} u_{lm} t^{l} x^{m} \in \mathbb{C}[t,x].\]

By the existence of a formal solution, \{\varphi_{j}(x)\} satisfy the following system formally:

\[(1.8) \quad f(0,x,0,\varphi_{j}(x),0) \equiv 0 \quad \text{(trivial relation)},\]

and

\[(1.9) \quad \frac{\partial}{\partial t_{i}} f(t,x,u(t,x),\partial_{t_{j}} u(t,x)) \bigg|_{t=0} = \frac{\partial f}{\partial t_{i}}(0,x,0,\varphi_{j}(x),0) + \frac{\partial f}{\partial u}(0,x,0,\varphi_{j}(x),0) \varphi_{i}(x) + \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_{k}}(0,x,0,\varphi_{j}(x),0) \frac{\partial \varphi_{i}}{\partial x_{k}}(x) = 0, \quad \text{for } i = 1, 2, \ldots, d.\]

The formal solution of this system is not convergent in general. Therefore, we assume

**Assumption 3.** The coefficients \{\varphi_{j}(x)\} are all holomorphic in a neighbourhood of the origin of \(\mathbb{C}^{n}\).

**Remark 2.** In the case \(d = 1\) (\(d\) is the dimension of \(t\) variables), a sufficient condition for the formal solution of (1.9) to converge has been already obtained by Miyake-Shirai [MS]. In the case \(d \geq 2\), we give a sufficient condition for the formal solution of system (1.9) to be convergent, which will be given by Theorem 3 in Section 5, but for a while we consider the problem under Assumption 3 for simplicity of our arguments.

Now we put \(a(x) = (0,x,0,\varphi_{j}(x),0)\) for simplicity, and define

\[(1.10) \quad A_{ij}(x) := \frac{\partial^{2} f}{\partial t_{i} \partial \tau_{j}}(a(x)) + \frac{\partial^{2} f}{\partial u \partial \tau_{j}}(a(x)) \varphi_{i}(x) + \sum_{k=1}^{n} \frac{\partial^{2} f}{\partial \tau_{j} \partial \xi_{k}}(a(x)) \frac{\partial \varphi_{i}}{\partial x_{k}}(x),\]

for \(i, j = 1, 2, \ldots, d\). Moreover we define

\[(1.11) \quad B_{k}(x) := \frac{\partial f}{\partial \xi_{k}}(a(x)), \quad \text{for } k = 1, 2, \ldots, n.\]

**Remark 3.** The functions \(A_{ij}(x)\) and \(B_{k}(x)\) correspond to \(a_{ij}(x)\) and \(b_{k}(x)\) in Theorem 1, respectively (see (1.13) below).

Here we assume that the equation is of totally characteristic type, that is,
Assumption 4. $B_k(x) \not\equiv 0$ and $B_k(0) = 0$, for $k = 1, 2, \ldots, n$.

Now our second theorem which is our main result is stated as follows:

**Theorem 2.** Suppose Assumptions 1, 2, 3 and 4. Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of $(A_{ij}(0))$, and let $\{\mu_k\}_{k=1}^n$ be the eigenvalues of Jacobi matrix of the vector $(B_k(x))$ at $x = 0$. Then the formal solution (1.7) is convergent if the following condition is satisfied:

There exists a positive constant $\sigma > 0$, such that,

$$(1.12) \quad \left| \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + \frac{\partial f}{\partial u}(a(0)) \right| \geq \sigma(|l| + |m|),$$

holds for all $(l, m) \in (\mathbb{Z}_{\geq 0})^d \times (\mathbb{Z}_{\geq 0})^n$ with $|l| \geq 2$, $|m| \geq 0$.

**Remark 4.** We put $v(t, x) = u(t, x) - \sum_{j=1}^{d} \varphi_j(x) t_j$ as a new unknown function. By Assumptions 1, 2, 3 and 4, we can easily see that $v(t, x)$ satisfies the equation of the following form:

$$(1.13) \quad \left\{ \begin{array}{l}
\sum_{i,j=1}^{d} A_{ij}(x) t_i \partial_{t_j} v + \sum_{k=1}^{n} B_k(x) \partial_{x_k} v + \frac{\partial f}{\partial u}(a(x)) v \\
= \sum_{|l|=2} d_l(x) t^l + f_3(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}) \\
v(t, x) = O(|t|^2).
\end{array} \right.$$

This is an equation considered in Theorem 1 in the case $K = 2$. Therefore, it is sufficient to prove Theorem 1 in order to prove Theorem 2.

\[ \square \]

2. REDUCTION OF THE EQUATION

As is mentioned in Remark 4, it is sufficient to study the equation (1.1).

By the assumption of Theorem 1,

$$(a_{ij}(0)) \sim \begin{pmatrix}
\lambda_1 & \delta_1 \\
\lambda_2 & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\lambda_d & & & & \lambda
\end{pmatrix}, \quad \begin{pmatrix}
\frac{\partial(b_1, \ldots, b_n)}{\partial(x_1, \ldots, x_n)}|_{x=0} \sim (\begin{array}{llll}
\mu_1 & \nu_1 & \mu_2 & \ddots \\
& \mu_3 & \ddots & \ddots \\
& & \ddots & \ddots & \mu_n
\end{array})
\end{pmatrix},$$

where $\delta_j, \nu_k = 0$ or 1 ($1 \leq j \leq d - 1, 1 \leq k \leq n - 1$).

Then by transforming the variables, (1.1) is reduced to the following form:

$$(2.1) \quad (\Lambda + \Delta)v(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_{t_j} v + \gamma(x) v + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} v + \tilde{f}_{K+1}(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}),$$
with \( v(t, x) = O(|t|^K) \), where
\[
\Lambda = \sum_{j=1}^{d} \lambda_j t_j \partial_{t_j} + \sum_{k=1}^{n} \mu_k x_k \partial_{x_k} + c(0),
\]
\[
\Delta = \sum_{j=1}^{d-1} \delta_j t_j \partial_{t_{j+1}} + \sum_{k=1}^{n-1} \nu_k x_k \partial_{x_{k+1}},
\]
and \( \alpha_l(x) \), \( \beta_{ij}(x) \), \( \gamma(x) \) and \( \varphi_k(x) \) are holomorphic in a neighbourhood of the origin, and satisfy \( \beta_{ij}(x) = O(|x|) \), \( \gamma(x) = O(|x|) \) and \( \varphi_k(x) = O(|x|^2) \), and \( \tilde{f}_{K+1}(t, x, u, \tau, \xi) \) is a holomorphic function which has a similar Taylor expansion with \( f_{K+1}(t, x, u, \tau, \xi) \).

In the following sections, we shall prove the existence and convergence of the unique formal solution of (2.1).

3. Preparation to prove Theorem 1

Let \( C[t, x]_{L,M} \) be the set of homogeneous polynomial of degree \( L \) in \( t \) variables and of degree \( M \) in \( x \) variables, that is,
\[
C[t, x]_{L,M} = \left\{ f_{LM}(t, x) = \sum_{|l|=L, |m|=M} f_{lm} t^l x^m \mid f_{lm} \in \mathbb{C} \right\}.
\]
For the operator \( \Lambda + \Delta \), the following lemma holds:

**Lemma 1.** For all \( L \geq K \) and \( M \geq 0 \), the operator
\[
\Lambda + \Delta : C[t, x]_{L,M} \longrightarrow C[t, x]_{L,M}
\]
is invertible. Moreover, if the majorant relation \( f_{LM}(t, x) \ll F \times (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M \) \( (f_{LM}(x) \in C[t, x]_{L,M}, F > 0) \) holds, then we obtain the following majorant relation:
\[
(3.1) \quad (\Lambda + \Delta)^{-1} f_{LM}(t, x) \ll \frac{C}{L+M} F \times (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M,
\]
where \( C > 0 \) is a positive constant independent of \( L \) and \( M \).

**Proof.** We define a norm of \( u_{LM}(t, x) \in C[t, x]_{L,M} \) by
\[
||u_{LM}|| := \inf \left\{ C > 0 \mid u_{LM}(t, x) \ll C (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M \right\}.
\]
We remark that \( C[t, x]_{L,M} \) becomes a Banach space by this norm.

First, by (1.4) it is easily checked that \( \Lambda \) is invertible on \( C[t, x]_{L,M} \) and
\[
(3.2) \quad ||\Lambda^{-1}|| \leq \frac{1}{\sigma(L+M)}
\]
holds for the operator norm of \( \Lambda^{-1} \) on \( C[t, x]_{L,M} \).
Next, since $u_{LM}(t, x) \ll \|u_{LM}\|(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M$, we have

$$
\Delta u_{LM}(t, x) \ll \sum_{j=1}^{d-1} L|\delta_j| \cdot \|u_{LM}\|(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M
+ \sum_{k=1}^{n-1} M|\nu_k| \cdot \|u_{LM}\|(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M
\ll \left\{ L(d-1) \max_{j=1,\ldots,d-1} |\delta_j| + M(n-1) \max_{k=1,\ldots,n-1} |\nu_k| \right\} \times \|u_{LM}\|(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M.
$$

Here we make a change of variables by $t_j = \epsilon^{j-1} \tau_j$, $x_k = \epsilon^{k-1} y_k$, then $\delta_j$ and $\nu_k$ (the components of nilpotent part of Jordan canonical form) turns to $\epsilon \delta_j$ and $\epsilon \nu_k$, respectively. Therefore, by choosing $\epsilon$ sufficiently small, we may assume that the components of nilpotent part are small enough. Hence we may assume that

$$
\max_{j=1,\ldots,d-1} |\delta_j| < \frac{\sigma}{2(d-1)}, \quad \max_{k=1,\ldots,n-1} |\nu_k| < \frac{\sigma}{2(n-1)}.
$$

Then

$$
\Delta u_{LM}(t, x) \ll \frac{\sigma(L + M)}{2} \|u_{LM}\|(t_1 + \cdots + t_d)^L(x_1 + \cdots + x_n)^M
$$
holds, and we obtain

$$
\|\Delta\| \leq \frac{\sigma(L + M)}{2}.
$$

Therefore, the operator norm of $\Delta \Lambda^{-1}$ is estimated by

$$
\|\Delta \Lambda^{-1}\| \leq \frac{1}{\sigma(L + M)} \cdot \frac{\sigma(L + M)}{2} = \frac{1}{2} < 1.
$$

By using the Neumann’s series, we can see that $\Lambda + \Delta$ is invertible and the norm of the inverse operator is estimated by

$$
\|(\Lambda + \Delta)^{-1}\| \leq \frac{2}{\sigma L + M},
$$
which we want to prove since $C = 2/\sigma$ is independent of $L$ and $M$.

Now, we define some notations, which are used in the proof of Theorem 1.

**Definition** (1) Let $(t, x) \in \mathbb{C}^d \times \mathbb{C}^n$ $(d \geq 0, \ n \geq 0)$ be complex variables. For formal power series $f(t, x) = \sum_{|l|\geq 0, \ |m|\geq 0} f_{l,m} t^l x^m$, we define

$$
|f|(t, x) = \sum_{|l|\geq 0, \ |m|\geq 0} |f_{l,m}| t^l x^m.
$$
(2) Let \((t, X) \in \mathbb{C}^d \times \mathbb{C} (d \geq 0)\) be complex variables. For formal power series \(f(t, X) = \sum_{|l| \geq 0, M \geq 0} f_{l, M} t^l X^M\), we define the shift operator \(S\) by

\[
S(f)(t, X) = \sum_{|l| \geq 0, M \geq 0} f_{l, M+1} t^l X^M = \frac{f(t, X) - f(t, 0)}{X}.
\]

**Remark 5.** The following facts are easily shown:

- \(f(t, x) \ll |f|(t, x)\);
- If \(f(t, x)\) and \(g(t, X)\) are convergent power series, then \(|f|(t, x)\) and \(S(g)(t, X)\) are also convergent. \(\square\)

### 4. Proof of Theorem 1

First, we prove a unique existence of formal power series solution.

Let

\[
u(t, x) = \sum_{|l| \geq K, |m| \geq 0} u_{lm} t^l x^m = \sum_{L \geq K} u_L(t, x) = \sum_{L \geq K, M \geq 0} u_{LM}(t, x)\]

be a formal solution of (2.1), where

\[
u_{LM}(t, x) = \sum_{|l|=L, |m|=M} u_{lm} t^l x^m \in \mathbb{C}[t, x]_{L, M},
\]

\[
u_L(t, x) = \sum_{|l|=L} u_l(x) t^l = \sum_{M \geq 0} u_{LM}(t, x).
\]

We put \(P = \Lambda + \Delta\) for simplicity. We substitute \(u(t, x) = \sum_{L \geq K} u_L(t, x)\) into (2.1), then we have the following recursion formula:

\[
\begin{cases}
Pu_K(t, x) = \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^d \beta_{ij}(x) t_i \partial_{t_j} u_K(t, x) \\
\quad + \gamma(x) u_K(t, x) + \sum_{k=1}^n \varphi_k(x) \partial_{x_k} u_K(t, x),
\end{cases}
\]

\[
Pu_L(t, x) = \sum_{i,j=1}^d \beta_{ij}(x) t_i \partial_{t_j} u_L(t, x) + \gamma(x) u_L(t, x) + \sum_{k=1}^n \varphi_k(x) \partial_{x_k} u_L(t, x)
\]

\[
\quad + G_L(t, x, \{u_p\}_{K \leq p < L}, \{\partial_{t_j} u_p\}_{K \leq p < L}, \{\partial_{x_k} u_p\}_{K \leq p < L}), \text{ for } L > K,
\]

where \(G_L(t, x, \zeta, \tau, \xi)\) is a polynomial of \((t, \zeta, \tau, \xi)\).
First, we consider the case $L = K$. We substitute $u_K(t, x) = \sum_{M \geq 0} u_{KM}(t, x)$ into the above recursion formula, we have

\[
\begin{aligned}
Pu_{K0}(t, x) &= \sum_{|l|=K} \alpha_l(0) t^l, \\
Pu_{KM}(t, x) &= \sum_{|l|=K} \alpha_l^M(x) t^l + \sum_{i,j=1}^{d} \sum_{p=1}^{M} \beta_{ij}^p(x) t_i \partial_j u_{K,M-p}(t, x) \\
&\quad + \sum_{p=1}^{M} \gamma^p(x) u_{K,M-p}(t, x) + \sum_{k=1}^{n} \sum_{p=2}^{M} \varphi_k^p(x) \partial_{x_k} u_{K,M-p+1}(t, x),
\end{aligned}
\]

where we put

\[
\begin{aligned}
\alpha_l(x) &= \sum_{M \geq 0} \alpha_l^M(x), \quad \alpha_l^M(x) = \sum_{|m|=M} \alpha_{lm} x^m, \\
\beta_{ij}(x) &= \sum_{M \geq 1} \beta_{ij}^M(x), \quad \beta_{ij}^M(x) = \sum_{|m|=M} \beta_{ijm} x^m, \\
\gamma(x) &= \sum_{M \geq 1} \gamma^M(x), \quad \gamma^M(x) = \sum_{|m|=M} \gamma_{m} x^m, \\
\varphi_k(x) &= \sum_{M \geq 2} \varphi_k^M(x), \quad \varphi_k^M(x) = \sum_{|m|=M} \varphi_{km} x^m.
\end{aligned}
\]

By Lemma 1, we know that the solution sequence $\{u_{KM}(t, x)\}_{M \geq 0}$ exists uniquely. Moreover, by the same argument, we see that $\{u_{LM}(t, x)\} (L > K)$ exist uniquely. These show that the formal solution exists uniquely.

Next, we prove the convergence of the formal solution. We put $U(t, x) = Pu(t, x)$ as a new unknown function. By Lemma 1, the equation (2.1) is reduced to the following equation:

\[
\begin{aligned}
(4.1) \quad U(t, x) &= \sum_{|l|=K} \alpha_l(x) t^l + \sum_{i,j=1}^{d} \beta_{ij}(x) t_i \partial_j P^{-1}U(t, x) \\
&\quad + \gamma(x) P^{-1}U(t, x) + \sum_{k=1}^{n} \varphi_k(x) \partial_{x_k} P^{-1}U(t, x) \\
&\quad + \tilde{f}_{K+1}(t, x, P^{-1}U(t, x), \{\partial_{j} P^{-1}U(t, x)\}, \{\partial_{x_k} P^{-1}U(t, x)\}).
\end{aligned}
\]

We know that (4.1) has a unique formal solution of the form

\[
U(t, x) = \sum_{|l| \geq K, |m| \geq 0} U_{lm} t^l x^m = \sum_{L \geq K} U_L(t, x) = \sum_{L \geq K, M \geq 0} U_{LM}(t, x).
\]
In order to get a majorant series of \( U(t, x) \), we prepare some majorant series for the coefficients of (4.1). We put \( T = t_1 + \cdots + t_d \), \( X = x_1 + \cdots + x_n \), and choose
\[
\sum_{|l|=K} \alpha_l(x) t^l \ll A(X) T^K, \quad \beta_{ij}(x) \ll \beta_{ij}(X, \ldots, X) =: XB_{ij}(X),
\]
\[
\gamma(x) \ll \gamma(X, \ldots, X) =: XG(X), \quad \varphi_k(x) \ll \varphi_k(X, \ldots, X) =: X^2 \Phi_k(X),
\]
\[
\bar{f}_{K+1}(t, x, u, \tau, \xi) \ll |\bar{f}_{K+1}(T, \ldots, T, X, \ldots, X, u, \tau, \xi)| =: F_{K+1}(T, X, u, \tau, \xi).
\]
where \( A(X) \), \( B_{ij}(X) \), \( G(X) \) and \( \Phi_k(X) \) are holomorphic in a neighbourhood of \( X = 0 \), and \( F_{K+1}(T, X, u, \tau, \xi) \) is also holomorphic near \( (T, X, u, \tau, \xi) = (0, 0, 0, 0, 0) \).

Now, we consider the following equation:
\[
(4.2) \quad w(T, X) = A(X) T^K + C \sum_{i,j=1}^d X B_{ij}(X) w(T, X)
+ CXG(X) w(T, X) + C \sum_{k=1}^n X^2 \Phi_k(x) S(w)(T, X)
+ F_{K+1}
\]
where \( C \) is a positive constant appeared in Lemma 1.

Let \( w(T, X) = \sum_{L \geq K, M \geq 0} w_{LM}(T, X) \) be the formal solution of (4.2). By the construction of (4.2), we can easily check that \( U(t, x) \ll w(T, X) \) by the next lemma.

**Lemma 2.** For two formal power series \( U(t, x) \) and \( w(T, X) \) satisfying
\[
U(t, x) = \sum_{L \geq K, M \geq 0} U_{LM}(t, x) \ll w(T, X) = \sum_{L \geq K, M \geq 0} w_{LM} T^L X^M,
\]
the following majorant relations hold:
(1) \( P^{-1} U(t, x) \ll C w(T, X) \),
(2) \( t_i \partial t_i P^{-1} U(t, x) \ll C w(T, X) \),
(3) \( \partial t_i P^{-1} U(t, x) \ll \frac{C w(T, X)}{T} \),
(4) \( \partial x_k P^{-1} U(t, x) \ll C S(w)(T, X) \).

**Proof.** By using Lemma 1, we can prove this lemma easily. First, (1) is proved as follows:
\[
P^{-1} U(t, x) = \sum_{L \geq K, M \geq 0} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{C}{L + M} w_{LM} T^L X^M \ll C w(T, X).
\]
Secondly, (2) and (3) is proved as follows:

\[ t_i \partial_{t_j} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 0} t_i \partial_{t_j} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{CL}{L + M} w_{LM} T^L X^M \ll Cw(T, X); \]

\[ \partial_{t_j} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 0} \partial_{t_j} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 0} \frac{CL}{L + M} w_{LM} T^{L-1} X^M \ll \frac{Cw(T, X)}{T}. \]

Finally, (4) is proved as follows:

\[ \partial_{x_k} P^{-1} U(t, x) = \sum_{L \geq K, M \geq 1} \partial_{x_k} P^{-1} U_{LM}(t, x) \ll \sum_{L \geq K, M \geq 1} \frac{CM}{L + M} w_{LM} T^L X^{M-1} \ll CS(w)(T, X). \]

This completes the proof. \( \square \)

Since \( w(T, X) \gg 0 \), we have

(4.3) \( XS(w)(T, X) = w(T, X) - w(T, 0) \ll w(T, X). \)

Let us consider the following equation:

(4.4) \( v(T, X) = A(X) T^K + CX h(X) v(T, X) \)

\[ + F_{K+1} \left( T, X, Cv, \left\{ \frac{Cv}{T} \right\}, \{CS(v)\} \right), \]

with \( v(T, X) = O(T^K) \), where \( h(X) = \sum_{i,j=1}^d B_{ij}(X) + G(X) + \sum_{k=1}^n \Phi_k(X) \). Then the following majorant relation is obvious:

\( w(T, X) \ll v(T, X). \)

We put \( y(T, X) = v(T, X)/T \) as a new unknown function. By substituting this into (4.4), we see that \( y(T, X) \) satisfies

(4.5) \( y(T, X) = A(X) T^{K-1} + CX h(X) y(T, X) \)

\[ + \frac{1}{T} F_{K+1} (T, X, CTy, \{Cy\}, \{CTS(y)\}), \]

with \( y(T, X) = O(T^{K-1}) \).

We decompose the formal solution \( y(T, X) \) as follows:

\( y(T, X) = y_1(X) T^{K-1} + y_2(X) T^K + T^K z(T, X). \)
We remark that $y_1(X)$ and $y_2(X)$ are holomorphic functions in a neighbourhood of $X = 0$. Indeed, $y_1(X)$ and $y_2(X)$ are given by

\[ y_1(X) = \frac{A(X)}{1 - CXh(X)}, \]

\[ y_2(X) = \frac{1}{1 - CXh(X)} \sum_{|p|+Kq+(K-1)|r|+K|s|=K+1} F_{pqrs}(X) \{Cy_1(X)\}^{q+|r|}\{CS(y_1)(X)\}^{|s|}. \]

These are holomorphic functions in a neighbourhood of $X = 0$.

In this case, $z(T, X)$ satisfies the following equation:

\[ z(T, X) = CXh(X)z(T, X) + H(T, X, Tz(T, X), TS(z)(T, X)), \]

\[ z(0, X) \equiv 0, \]

where

\[ H(T, X, \eta_1, \eta_2) = \frac{1}{TK+1} \left[ F_{K+1}(T, X, Cy_1(X)T^K + Cy_2(X)T^{K+1} + CT^K \eta_1), \right. \]

\[ \left. \{Cy_1(X)T^{K-1} + Cy_2(X)T^K + CT^{K-1}\eta_1\}, \right. \]

\[ \left. \{CS(y_1)(X)T^K + CS(y_2)(X)T^{K+1} + CT^K \eta_2\} \right] - \sum_{|p|+Kq+(K-1)|r|+K|s|=K+1} F_{pqrs}(X) \{Cy_1(X)\}^{q+|r|}\{CS(y_1)(X)\}^{|s|}. \]

Remark 6. The order of zeros in $T$ variable of $H(T, X, CTz(T, X), CTS(z)(T, X))$ is greater than or equal to 1.

In order to prove the convergence of $z(T, X)$, it is sufficient to show the following:

Lemma 3. There exists a small positive constant $\epsilon > 0$ such that $z_\epsilon(\rho) = z(\epsilon \rho, \rho)$ is convergent in a neighbourhood of $\rho = 0$.

Proof. We substitute $T = \epsilon \rho$ and $X = \rho$ into (4.6) and by using the relation (4.3), we have

\[ \rho S(z)(\epsilon \rho, \rho) \ll z_\epsilon(\rho). \]

By this relation, the following majorant relation also holds,

\[ TS(z)(T, X)|_{T=\epsilon \rho, X=\rho} = \epsilon \rho S(z)(\epsilon \rho, \rho) \ll \epsilon z_\epsilon(\rho). \]

Here we consider

\[ (4.7) \quad \psi(\rho) = Cph(\rho)\psi(\rho) + H(\epsilon \rho, \rho, \epsilon \rho \psi(\rho), \epsilon \psi(\rho)). \]
In the right hand side of (4.7), we decompose $H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho))$ into the term of $\psi(\rho)$ and otherwise as follows:

$$H(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)) = \frac{\partial H}{\partial \eta_2}(0, 0, 0, 0)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).$$

We remark that the following fact holds:

$$\left. \frac{\partial \overline{H}}{\partial \psi}(\epsilon\rho, \rho, \epsilon\rho\psi, \epsilon\psi) \right|_{\rho=0, \psi=0} = 0.$$

We put $(\partial H/\partial \eta_2)(0, 0, 0, 0) = K_0 \geq 0$, then (4.7) is rewritten by

$$(4.8) \quad (1 - \epsilon K_0)\psi(\rho) = C\rho h(\rho)\psi(\rho) + \overline{H}(\epsilon\rho, \rho, \epsilon\rho\psi(\rho), \epsilon\psi(\rho)).$$

We choose $\epsilon > 0$ with $1 - \epsilon K_0 > 0$. Then by using the implicit function theorem, we can see that (5.8) has a unique holomorphic solution $\psi(\rho)$ with $\psi(0) = 0$ in a neighbourhood of $\rho = 0$. Moreover we can see $z_\epsilon(\rho) \ll \psi(\rho)$.

Thus we complete the proof of Lemma 3. \(\square\)

5. Solvability of the System (1.9)

In this section, we give a sufficient condition for the formal solution of the system (1.9) to be convergent. Recall that (1.9) is

$$(1.9) \quad \frac{\partial f}{\partial t_i}(0, x, 0, \{\varphi_j(x)\}, 0) + \frac{\partial f}{\partial u}(0, x, 0, \{\varphi_j(x)\}, 0)\varphi_i(x) + \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0, x, 0, \{\varphi_j(x)\}, 0)\frac{\partial \varphi_i(x)}{\partial x_k} = 0, \quad i = 1, 2, \ldots, d.$$ 

By Assumption 4 of Theorem 2, the condition

$$\frac{\partial f}{\partial \xi_k}(0, 0, 0, \{\varphi_j(0)\}, 0) = 0, \quad k = 1, 2, \ldots, n$$

was assumed.

Let $\varphi(x) = \{\varphi_1(x), \ldots, \varphi_d(x)\}$ be the unknown functions. We put $\varphi(0) = \{\varphi_1^0, \ldots, \varphi_d^0\} \in \mathbb{C}^d$ as the constant term of $\varphi(x)$. We substitute $\varphi_j(x) = \varphi_j^0 + \psi_j(x)$ into the system (1.9), and by restricting at $x = 0$, we see that $\{\varphi_j^0\}$ satisfies the following system:

$$(5.1) \quad \frac{\partial f}{\partial t_i}(0, 0, 0, \{\varphi_j^0\}, 0) + \frac{\partial f}{\partial u}(0, 0, 0, \{\varphi_j^0\}, 0)\varphi_i^0 = 0, \quad i = 1, 2, \ldots, d.$$ 

This system has some roots by the assumption of the existence of a formal solution, and we fix such $\{\varphi_j^0\}$. 
For such fixed $\{\varphi_j^0\}$, we see that $\{\psi_j(x)\}$ satisfies the system of the followir

\[
\begin{align*}
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0, 0, 0, \{\varphi_j^0\}, 0) & x_l \frac{\partial \psi_i}{\partial x_k}(x) \\
+ \sum_{k=1}^{n} \sum_{p=1}^{d} \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0, 0, 0, \{\varphi_j^0\}, 0) & \psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x) \\
+ \frac{\partial f}{\partial u}(0, 0, 0, \{\varphi_j^0\}, 0) & \psi_i(x) \\
+ \sum_{p=1}^{d} \left\{ \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0, 0, 0, \{\varphi_j^0\}, 0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0, 0, 0, \{\varphi_j^0\}, 0) \varphi_i^0 \right\} \psi_p(x) \\
+ \sum_{l=1}^{n} \left\{ \frac{\partial^2 f}{\partial t_i \partial x_l}(0, 0, 0, \{\varphi_j^0\}, 0) + \frac{\partial^2 f}{\partial u \partial x_l}(0, 0, 0, \{\varphi_j^0\}, 0) \varphi_i^0 \right\} x_l \\
= (\text{degree in } x \text{ is greater than or equal to 2}), \quad i = 1, 2, \ldots, d.
\end{align*}
\]

This system is written as follows for simplicity,

\[
\begin{align*}
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l & \frac{\partial \psi_i}{\partial x_k}(x) + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \psi_p(x) \frac{\partial \psi_i}{\partial x_k}(x) \\
+c & \psi_i(x) + \sum_{p=1}^{d} d_{ip} \psi_p(x) + \sum_{l=1}^{n} e_{il} x_l \\
= (\text{degree in } x \text{ is greater than or equal to 2}), \quad i = 1, 2, \ldots, d,
\end{align*}
\]

where

\[
\begin{align*}
a_{kl} & := \frac{\partial^2 f}{\partial \xi_k \partial x_l}(0, 0, 0, \{\varphi_j^0\}, 0), \quad b_{kp} := \frac{\partial^2 f}{\partial \xi_k \partial \tau_p}(0, 0, 0, \{\varphi_j^0\}, 0), \\
c & := \frac{\partial f}{\partial u}(0, 0, 0, \{\varphi_j^0\}, 0), \\
d_{ip} & := \frac{\partial^2 f}{\partial t_i \partial \tau_p}(0, 0, 0, \{\varphi_j^0\}, 0) + \frac{\partial^2 f}{\partial u \partial \tau_p}(0, 0, 0, \{\varphi_j^0\}, 0) \varphi_i^0, \\
e_{il} & := \frac{\partial^2 f}{\partial t_i \partial x_l}(0, 0, 0, \{\varphi_j^0\}, 0) + \frac{\partial^2 f}{\partial u \partial x_l}(0, 0, 0, \{\varphi_j^0\}, 0) \varphi_i^0.
\end{align*}
\]
Here we decompose $\psi_i(x)$ into $\bar{\psi}_i(x) + \eta_i(x)$ ($\bar{\psi}_i(x) = \sum_{k=1}^{n} \psi_{ik} x_k$, $\eta_i(x) = O(|x|^2)$). We substitute this into the system (5.3) and obtain

\begin{equation}
(5.4)
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \left( \frac{\partial \bar{\psi}_i}{\partial x_k}(x) + \frac{\partial \eta_i}{\partial x_k}(x) \right) \\
+ \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} (\bar{\psi}_p(x) + \eta_p(x)) \left( \frac{\partial \bar{\psi}_i}{\partial x_k}(x) + \frac{\partial \eta_i}{\partial x_k}(x) \right) \\
+ c(\bar{\psi}_i(x) + \eta_i(x)) + \sum_{p=1}^{d} d_{ip}(\bar{\psi}_p(x) + \eta_p(x)) + \sum_{l=1}^{n} e_{il} x_l \\
= (\text{degree in } x \text{ is greater than or equal to } 2), \ i = 1, 2, \ldots, d.
\end{equation}

By picking up the degree 1 part on the both sides, we see that \{\bar{\psi}_i(x)\} satisfy the following system:

\begin{equation}
(5.5)
\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} x_l \frac{\partial \bar{\psi}_i}{\partial x_k}(x) + \sum_{k=1}^{n} \sum_{p=1}^{d} b_{kp} \bar{\psi}_p(x) \frac{\partial \bar{\psi}_i}{\partial x_k}(x) \\
+ c\bar{\psi}_i(x) + \sum_{p=1}^{d} d_{ip}\bar{\psi}_p(x) + \sum_{l=1}^{n} e_{il} x_l = 0,
\end{equation}

for $i = 1, 2, \ldots, d$.

By the existence of a formal solution, (5.5) has some solutions \{\bar{\psi}_i(x)\} of linear functions, and we fix such \{\bar{\psi}_i(x)\}.

For fixed \{\varphi_i^0\} and \{\bar{\psi}_i(x)\}, we see that \{\eta_i(x)\} satisfy the following system:

\begin{equation}
(5.6)
\sum_{k=1}^{n} \sum_{l=1}^{n} \left( a_{kl} + \sum_{p=1}^{d} b_{kp}\psi_{pl} \right) x_l \frac{\partial \eta_i}{\partial x_k}(x) + c\eta_i(x) + \sum_{p=1}^{d} \left( d_{ip} + \sum_{k=1}^{n} b_{kp}\psi_{ik} \right) \eta_p(x) \\
= (\text{degree in } x \text{ is greater than or equal to } 2.), \ i = 1, 2, \ldots, d.
\end{equation}

We remark that the degree 2 part in the right hand side of this system does not include \{\eta_i(x)\}.

The following theorem holds:

**Theorem 3.** Let $(A_{kl})_{k,l=1,2,\ldots,n}$ be a matrix defined by

$$(A_{kl})_{k,l=1,2,\ldots,n} = \left( a_{kl} + \sum_{p=1}^{d} b_{kp}\psi_{pl} \right)_{k,l=1,2,\ldots,n}. $$
Let \( \{\kappa_k\}_{k=1}^{n} \) be the eigenvalues of \((A_{kl})_{k,l=1,2,\ldots,n}\). If there exists a positive constant \( \sigma_0 \) such that the condition
\[
\left| \sum_{k=1}^{n} \kappa_k m_k \right| \geq \sigma_0 |m|, \quad \text{(Poincaré condition)}
\]
holds for all \( m = (m_1, \ldots, m_n) \in (\mathbb{Z}_{\geq 0})^n \) with \( |m| \geq 2 \), then the formal solution of the system (1.9) is convergent in a neighbourhood of the origin.

**Remark 7.** Let \((B_{ip})_{i,p=1,2,\ldots,d}\) be a matrix defined by
\[
(B_{ip})_{i,p=1,2,\ldots,d} = \left( d_{ip} + \sum_{k=1}^{n} b_{kp} \psi_{ik} \right)_{i,p=1,2,\ldots,d},
\]
and let \( \{\omega_j\}_{j=1}^{d} \) be the eigenvalues of \((B_{ip})_{i,p=1,2,\ldots,d}\).

By the same argument in Remark 1, we have
\[
\left| \sum_{k=1}^{n} \kappa_k m_k + c + \omega_j \right| \geq \sigma |m|, \quad \text{by some } \sigma > 0, \text{ and } j = 1, 2, \ldots, d,
\]
for large \( m \), which will be used in the proof. \( \square \)

### 6. PROOF OF THEOREM 3

The proof of Theorem 3 is the same method of the proof of Theorem 1 in case that the unknown function is a vector values. However, there are some difference in the detail. Therefore, we introduce only the outline of the proof of Theorem 3 in this section.

**Step 1.** By taking a linear transformation of the independent variables and a linear transformation of the unknown functions, (5.6) is reduced to the following form:

\[
(\Lambda + \Delta + \mathbf{B}) \begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x) \end{pmatrix} := \left\{ \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_d \end{pmatrix} + \begin{pmatrix} \Delta \\ \vdots \\ \Delta \end{pmatrix} + \mathbf{B} \right\} \begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x) \end{pmatrix}
= \begin{pmatrix} \sum_{|m|=2} a_{1,m} x^{m} + g_{3,1}(x, w(x), \partial_x w(x)) \\ \vdots \\ \sum_{|m|=2} a_{d,m} x^{m} + g_{3,d}(x, w(x), \partial_x w(x)) \end{pmatrix},
\]
where \( w_j(x) \) \((j = 1, 2, \ldots, d)\) denote new unknown functions after linear transformations and

\[
\Lambda_j = \sum_{k=1}^{n} \kappa_k x_k \partial_{x_k} + c + \omega_j, \quad \Delta = \sum_{k=1}^{n-1} \varepsilon_k x_k \partial_{x_{k+1}}, \quad B = \begin{pmatrix} 0 & e_1 \\ & \ddots & \ddots \\ & & 0 & e_{d-1} \end{pmatrix},
\]

where \( \varepsilon_j \) and \( e_j \) denote the nilpotent components of the Jordan canonical forms of the matrices \( (A_{kl}) \) and \( (B_{lp}) \), respectively, and

\[
g_{3,i}(x, \eta, \zeta) = \sum_{|\alpha|+2|\beta|+|\gamma|\geq 3} g_{\alpha\beta\gamma}^{(i)} x^\alpha \eta^\beta \zeta^\gamma.
\]

**Step 2.** We define \( C[x]_M \) by \( C[x]_M = \{ \sum_{|m|=M} u_m x^m ; u_m \in \mathbb{C} \} \), and define a norm of \( u(x) = (u_1(x), \ldots, u_d(x)) \in (C[x]_M)^d \) by

\[
||u|| := \inf \{ C > 0 ; u_i(x) \ll C(x_1 + \cdots + x_n)^M, \ i = 1, 2, \ldots, d \}.
\]

By the same argument in the proof of Lemma 1 and by Remark 7, we can prove the same results of Lemma 1 for the operator \( \Lambda + \Delta + B \).

**Step 3.** By the same method in the previous sections, we can construct a majorant equation whose formal solution is a majorant function of all unknown functions of the system. Finally, by the implicit function theorem, we prove the convergence of the formal solution of the majorant equation.

**References**


GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, HURO-CHO, CHIKUSA-KU, NAGOYA 464-8602, JAPAN

*E-mail address*: m96034q@math.nagoya-u.ac.jp