ON EXPONENTIAL CALCULUS OF SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS OF MINIMUM TYPE

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1. INTRODUCTION

Let $X$ and $Y$ be $n$- and $m$- dimensional complex manifolds, respectively.

$S^*X := (T^*X - X)/\mathbb{R}^+, S^*Y := (T^*Y - Y)/\mathbb{R}^+$.

We define the mapping $\gamma$ as

$\gamma: \mathcal{T}^\infty(X \times Y) \ni (z, w; \xi, \eta) \mapsto (z; \frac{\xi}{|\xi|}) \times (w; \frac{\eta}{|\eta|}) \in S^*X \times S^*Y$,

where

$\mathcal{T}^\infty(X \times Y) := T^*(X \times Y) \setminus \{(T^*X \times Y) \cup (X \times T^*Y)\}$.

For $d > 0$ and an open subset $U$ of $S^*X \times S^*Y$ we denote

$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$

by $\gamma^{-1}(U; d, d)$.

Hereafter we write $(z, \xi, w, \eta)$ for coordinates $(z, w; \xi, \eta)$.

2. SYMBOLS OF PRODUCT TYPE

Let $K$ be a compact subset of $S^*X \times S^*Y$.

Definition 2.1. $P(z, \xi, w, \eta)$ is said to be a symbol of product type on $K$ if the following hold:

1. There are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ such that $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$.

2. For each $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$|P(z, \xi, w, \eta)| \leq C_\epsilon e^{\epsilon(|\xi| + |\eta|)}$ on $\gamma^{-1}(U; d, d)$.
We denote by $S(K)$ the set of all such symbols on $K$. $S(K)$ becomes a commutative ring with the usual sum and product.

**Definition 2.2.** We denote by $R(K)$ the set of all $P(z, \xi, w, \eta) \in S(K)$ satisfying the following:
there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(|\xi|/|\eta|)e^{-\delta \min\{|\xi|,|\eta|\}}$$
on $\gamma^{-1}(U;d,d)$.

We call an element of $R(K)$ a symbol of 0-class.

**Definition 2.3.** A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$ is called a formal symbol of product type on $K$ if the following hold:
(1) There are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.
(2) For each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$(2.2) \quad |P_{j,k}(z, \xi, w, \eta)| \leq C_\epsilon A^{j+k}e^{\epsilon(|\xi|+|\eta|)}$$
on $\gamma^{-1}(U; (j+1)d, (k+1)d)$
for each $j, k \geq 0$.

We denote by $\hat{S}(K)$ the set of such formal symbols on $K$.

We can easily obtain the following.

**Proposition 2.4.** $\hat{S}(K)$ becomes a commutative ring with the sum and the product as formal power series in $t_1$ and $t_2$.

$S(K)$ is identified with a subring of $\hat{S}(K)$ as follows:
$S(K) \simeq \hat{S}(K)|_{t_1=0t_2=0} = \{P = \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}; P_{j,k} \equiv 0 \text{ for all } (j,k) \neq (0,0)\}$.

**Definition 2.5.** We denote by $\hat{R}(K)$ the set of all $P(t_1, t_2; z, \xi, w, \eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$ in $\hat{S}(K)$ such that there are $d > 0, 0 < A < 1$,
and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;
for each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left| \sum_{0 \leq j \leq s \atop 0 \leq k \leq t} P_{j,k}(z, \xi, w, \eta) \right| \leq C_\varepsilon A^{\min\{s, t\}} e^{\varepsilon(|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 0$.

We call an element of $\widehat{R}(K)$ a formal symbol of zero class.

**Proposition 2.6.** Under the previous identification, $S(K) \cap \widehat{R}(K) = R(K)$ holds.

**Proof.** Let $P(z, \xi, w, \eta)$ be in $S(K)$. Then $P(z, \xi, w, \eta) \in \widehat{R}(K)$ is equivalent to the following; there exist $d > 0, \delta > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ such that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{|\xi|, |\eta|\} + \varepsilon(|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; d, d)$.

(\subset) Using the fact that $(0, \infty) = \{\varepsilon := |\xi|; (z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)\}$, by the hypothesis, we obtain the following;

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{1, \frac{1}{t}\}|\xi| + \varepsilon(1 + \frac{1}{t})|\xi|}$$

for all $t := |\xi| \in (0, \infty)$ and $(z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)$.

We fix any $\varepsilon > 0$ such that $0 < \varepsilon < 1$ and $\varepsilon \leq \frac{\delta}{3}$.

Then for every $t \in [\frac{2}{3}\varepsilon, 1]$

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta|\xi| + \varepsilon(1 + \frac{1}{t})|\xi|} \leq C_\varepsilon e^{-\delta|\xi| + \varepsilon(1 + \frac{1}{\varepsilon})|\xi|}$$

$$= C_\varepsilon e^{(\varepsilon - \frac{2}{3}\delta)|\xi|} \leq C_\varepsilon e^{-\frac{1}{3}\delta|\xi|}.$$
on $\gamma^{-1}(U; d, d) \cap \{ |\xi| \leq |\eta| \}$.
In like manners,
\[
|P(z, \xi, w, \eta)| \leq C\left( \frac{|\eta|}{|\xi|} \right) e^{-\frac{1}{8} \delta |\eta|}
\]
on $\gamma^{-1}(U; d, d) \cap \{ |\xi| \geq |\eta| \}$.
Here, we define a function $C'(\cdot)$ on $(0, \infty)$ as $C'(t) = C\left( \min\{t, \frac{1}{t}\} \right)$.
Then $C'(t)$ is locally bounded on $(0, \infty)$ and $|P(z, \xi, w, \eta)| \leq C'(\frac{|\xi|}{|\eta|}) e^{-\frac{1}{3} \delta \min\{|\xi|,|\eta|\}}$ on $\gamma^{-1}(U; d, d)$.
That is, $P(z, \xi, w, \eta) \in R(K)$.

Proposition 2.7. $R(K)$ is an ideal in $S(K)$.

Proof. It is clear by the part $(\subset)$ of the proof of Proposition 2.6.

Proposition 2.8. $\hat{R}(K)$ is an ideal in $\hat{S}(K)$. 

Proof. Let $\sum P_{j,k}(z,\xi,w,\eta) \in \hat{R}(K)$ and $\sum Q_{j,k}(z,\xi,w,\eta) \in \hat{S}(K)$. Then there exist $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:

For each $\epsilon > 0$, we have some $C_\epsilon > 0$ such that

a) $|P_{s,t}(z,\xi,w,\eta)|, |Q_{s,t}(z,\xi,w,\eta)| \leq C_\epsilon A^{s+t}e^{\epsilon(|\xi|+|\eta|)}$

b) $\left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} P_{j,k}(z,\xi,w,\eta) \right| \leq C_\epsilon A^{\min\{s,t\}}e^{\epsilon(|\xi|+|\eta|)}$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s,t \geq 0$.

It suffices to show that $\sum R_{j,k} \in \hat{R}(K)$, where

$$ R_{j,k} := \sum_{j_1+j_2=j \atop k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2}.$$ 

Since we can easily estimate $\sum R_{j,k}$ for $st=0$,

we suppose $s \geq 1$ and $t \geq 1$.

Then we can obtain the following inequality:

$$ \left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} R_{j,k} \right| = \sum_{0 \leq j \leq s, 0 \leq k \leq t} \sum P_{j_1,k_1}Q_{j_2,k_2} $$

$$ \leq \left( \sum_{0 \leq j_1 \leq s \atop 0 \leq k_1 \leq t} P_{j_1,k_1} \right) \left( \sum_{0 \leq j_2 \leq s \atop 0 \leq k_2 \leq t} Q_{j_2,k_2} \right) + \sum_{s+1 \leq j \leq 2s \atop 0 \leq k \leq t} \sum_{k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2} $$

$$ + \sum_{0 \leq j \leq s \atop t+1 \leq k \leq 2t} \sum_{j_1+j_2=j \atop k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2} + \sum_{s+1 \leq j \leq 2s \atop 0 \leq k \leq t} \sum_{j_1+j_2=j \atop k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2}.$$ 

We shall estimate the four terms in the right side of the inequality, respectively.

the first term $\leq C_\epsilon A^{\min\{s,t\}}e^{\epsilon(|\xi|+|\eta|)} \cdot \sum_{0 \leq j_2 \leq s \atop 0 \leq k_2 \leq t} C_\epsilon A^{j+k}e^{\epsilon(|\xi|+|\eta|)}$

$$ \leq C_\epsilon \cdot C_\epsilon \cdot A^{\min\{s,t\}}e^{2\epsilon(|\xi|+|\eta|)} \cdot \frac{1}{1-A} \cdot \frac{1}{1-A} $$
on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the 2nd term

$$
\leq \sum_{s+1 \leq j \leq 2s, j_1 + j_2 = j} \sum_{t+1 \leq k \leq 2t, k_1 + k_2 = k} C_{\epsilon} A^{j_1 + k_1} e^{\epsilon(\xi^1 + \eta)} \cdot C_{\epsilon} A^{j_2 + k_2} e^{\epsilon(\xi^2 + \eta)}
$$

$$
= C_{\epsilon} \cdot C_{\epsilon} \cdot e^{2\epsilon(|\xi^1 + \eta|)} \cdot (\sum_{s+1 \leq j \leq 2s, j_1 + j_2 = j} \sum_{t+1 \leq k \leq 2t, k_1 + k_2 = k} A^j) (\sum_{t+1 \leq k \leq 2t, k_1 + k_2 = k} A^k).
$$

If we choose any $B$ and $C$ such that $0 < B < 1$, $0 < C < 1$, and $BC \geq A$, we can get the following inequality:

$$
\sum_{s+1 \leq j \leq 2s, j_1 + j_2 = j} A^j \leq C_{s+1} (B^0 + B^1 + B^2 + \cdots)^2 = C_{s+1} \left( \frac{1}{1-B} \right)^2.
$$

Then,

the second term $\leq C_{\epsilon} \cdot C_{\epsilon} \cdot e^{2\epsilon(|\xi^1 + \eta|)} \cdot C_{s+1} \left( \frac{1}{1-A} \right)^2 \cdot C_{t+1} \left( \frac{1}{1-B} \right)^2$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the third term

$$
\leq \sum_{0 \leq j \leq s, j_1 + j_2 = j} \sum_{t+1 \leq k \leq 2t, k_1 + k_2 = k} C_{\epsilon} A^{j_1 + k_1} e^{\epsilon(\xi^1 + \eta)} \cdot C_{\epsilon} A^{j_2 + k_2} e^{\epsilon(\xi^2 + \eta)}
$$

$$
= C_{\epsilon} \cdot C_{\epsilon} \cdot e^{2\epsilon(|\xi^1 + \eta|)} \cdot (\sum_{0 \leq j \leq s, j_1 + j_2 = j} \sum_{t+1 \leq k \leq 2t, k_1 + k_2 = k} A^j) \cdot (\sum_{t+1 \leq k \leq 2t, k_1 + k_2 = k} A^k)
$$

$$
\leq C_{\epsilon} \cdot C_{\epsilon} \cdot e^{2\epsilon(|\xi^1 + \eta|)} \cdot \left( \frac{1}{1-A} \right)^2 \cdot C_{t+1} \left( \frac{1}{1-B} \right)^2
$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

In like manners,

the fourth term $\leq C_{\epsilon} \cdot C_{\epsilon} \cdot e^{2\epsilon(|\xi + \eta|)} \cdot C_{s+1} \left( \frac{1}{1-B} \right)^2 \cdot \left( \frac{1}{1-A} \right)^2$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

Hence, we conclude that $\sum R_{j,k} \in \hat{R}(k)$. 

$S(K)/R(K)$ becomes a commutative ring by Proposition 2.8. By Propositions 2.6 and 2.7, the inclusion $S(K) \hookrightarrow \hat{S}(K)$ induces the injective ring homomorphism
\[ \iota_K : S(K)/R(K) \hookrightarrow \hat{S}(K)/\hat{R}(K). \]
Conversely, we obtain the following.

**Theorem 2.9.** If \( \sum P_{j,k}(z, \xi, w, \eta) \in \hat{S}(K) \), there exists \( P(z, \xi, w, \eta) \in S(K) \) such that \( P - \sum P_{j,k} \in \hat{R}(K) \).

Thus, \( S(K)/R(K) \) is isomorphic to \( \hat{S}(K)/\hat{R}(K) \) in the sense of commutative rings.

**Definition 2.10.** We call an element in the ring \( \hat{S}(K)/\hat{R}(K) \) a pseudo-differential operator of the product type on \( K \). We write \( \sum P_{j,k} \) : for the associated pseudo-differential operator of the product type on \( K \) using an element \( \sum P_{j,k} \) in \( \hat{S}(K) \).

The mapping \( \gamma \) is the composition of the following \( \gamma_1 \) and \( \gamma_2 \).

\[
\tilde{T}^\circ(X \times Y) \ni (z, w; \xi, \eta) \xrightarrow{\gamma_2} (z, w; \frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}) \in \tilde{S}^\circ(X \times Y),
\]
\[
\tilde{S}^\circ(X \times Y) \ni (z, w; \frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}) \xrightarrow{\gamma_1} (z, \frac{\xi}{|\xi|}) \times (w, \frac{\eta}{|\eta|}) \in S^*X \times S^*Y,
\]
where \( \tilde{S}^\circ(X \times Y) := S^*(X \times Y) \setminus \{(S^*X \times Y) \cup (X \times S^*Y)\} \).

**Proposition 2.11.** If \( P(z, \xi, w, \eta) \) is a symbol of product type on \( K \), \( P \) is a symbol on \( \gamma_1^{-1}(K) \) in the sense of Aoki's symbol.

**Proof.** By the hypothesis, there are \( d > 0 \) and \( U \supset K \) open in \( S^*X \times S^*Y \) satisfying the following:

a) \( P(z, \xi, w, \eta) \) is holomorphic in \( \gamma^{-1}(U; d, d) \), and

b) for each \( \varepsilon > 0 \) there is \( C_\varepsilon > 0 \) such that \( |P(z, \xi, w, \eta)| \leq C_\varepsilon e^{\varepsilon(|\xi| + |\eta|)} \) on \( \gamma^{-1}(U; d, d) \).

Let \( K' \) be compact in \( \tilde{S}^\circ(X \times Y) \) and \( \gamma_1^{-1}(K) \supset K' \).

Then there exist \( d' > 0 \) and \( U' \supset K' \) open in \( \tilde{S}^\circ(X \times Y) \) such that
\[
\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.
\]

In fact, for each \( (\tilde{z}, \tilde{w}; \tilde{\xi}, \tilde{\eta}) \in \gamma_1^{-1}(K) \) we can choose \( d' > 0 \) such that
\[
d' > \frac{d}{\min\{|\xi|, |\eta|\}}.
\]
Then there exists a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta}) \in \gamma^{-1}_1(K)$ in $\bar{S}(X \times Y)$ such that

$$
\gamma^{-1}(U) \cap \{ |\xi| > d, |\eta| > d \} \supset \gamma^{-1}_2(U') \cap \{ |\xi| + |\eta| > d' \}.
$$

By the compactness of $K'$, the proof is completed.

**Proposition 2.12.** If $P(z, \xi, w, \eta)$ is a symbol of product type of 0-class on $K$, that is, $P \in R(K)$, $P$ is a zero symbol on $\gamma^{-1}_1(K)$ in the sense of AOKI's symbol.

**Proof.** Let $K'$ be compact in $\bar{S}(X \times Y)$ and $\gamma^{-1}_1(K) \supset K'$. It suffices to show that $P$ is a zero symbol on $K'$ in the sense of AOKI's symbol. By the hypothesis, there exist $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$
|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right)e^{-\delta\min\{\xi, \eta\}}
$$
on $\gamma^{-1}(U) \cap \{ |\xi| > d, |\eta| > d \}$.

Let $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ be any point of $\gamma^{-1}_1(K)$. By Proposition 2.11, there exist $d' > 0$ and a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ in $\bar{S}(X \times Y)$ such that

$$
\gamma^{-1}(U) \cap \{ |\xi| > d, |\eta| > d \} \supset \gamma^{-1}_2(U') \cap \{ |\xi| + |\eta| > d' \},
$$
and that there exists $\delta' > 0$ satisfying

$$
\frac{|\xi|}{|\xi| + |\eta|}, \frac{|\eta|}{|\xi| + |\eta|} > \delta'
$$
on $\gamma^{-1}_2(U')$.

Hence,

$$
|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right)e^{-\delta'(|\xi| + |\eta|)}
$$
on $\gamma^{-1}_2(U') \cap \{ |\xi| + |\eta| > d' \}$.

Since $K$ is compact, $P$ is a zero symbol on $\gamma^{-1}_1(K)$ in the sense of AOKI's symbol.

**Definition 2.13.** The canonical mapping $H_K$ is defined as follows;

$$
S(K)/R(K) \ni P : H_K [P] \in \lim_{U \supset \gamma^{-1}_1(K)} \mathcal{E}^R(U).
$$

**Proposition 2.14.** Suppose $K_1$ and $K_2$ are compact in $S^*X \times S^*Y$, respectively, and $K_1 \supset K_2$. Then, $H_{K_1}(\colon P \colon)|_{\gamma^{-1}_1(K_2)} = H_{K_2}(\colon P_{|K_2} \colon)$ for all $P \in S(K)/R(K)$.
Definition 2.15. We define the product $*$ of two elements of $\hat{S}(K)$ as follows:

$$ (\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)) * (\sum_{j,k=0}^{\infty} Q_{j,k}(z, \xi, w, \eta)) = \sum_{j,k=0}^{\infty} R_{j,k}(z, \xi, w, \eta), $$

where

$$ \sum_{j,k=0}^{\infty} t_1^j t_2^k R_{j,k}(z, \xi, w, \eta) := e^{t_1(\partial_\xi, \partial_z) + t_2(\partial_\eta, \partial_w)} (\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta) \times (\sum_{j,k=0}^{\infty} Q_{j,k}(z^*, \xi^*, w^*, \eta^*)) |_{w^*=w, \eta^*=\eta} z^*=z, \xi^*=\xi). $$

That is,

$$ R_{j,k}(z, \xi, w, \eta) := \sum_{j_1+j_2+|\alpha|=j, k_1+k_2+|\beta|=k} \frac{1}{\alpha!\beta!} \partial_{\xi}^\alpha \partial_{\eta}^\beta P_{j_1,k_1}(z, \xi, w, \eta) \times \partial_{z}^{j_1} \partial_{w}^{j_2} Q_{j_2,k_2}(z, \xi, w, \eta). $$

Then we obtain the following.

Lemma 2.16. If $\sum P_{j,k}$ and $\sum Q_{j,k}$ are formal symbols of product type on $K$, then $\sum R_{j,k}$ is also a formal symbol of product type on $K$.

Proposition 2.17. If $\sum P_{j,k} \in \hat{S}(K)$ and $\sum Q_{j,k} \in \hat{R}(K)$, otherwise $\sum P_{j,k} \in \hat{R}(K)$ and $\sum Q_{j,k} \in \hat{S}(K)$, $\sum R_{j,k}$ is also in $\hat{R}(K)$.

By Lemma 2.16 and Proposition 2.17, the following composition of two elements in $\hat{S}(K)/\hat{R}(K)$ is well-defined;

$$ : \sum P_{j,k} : \circ : \sum Q_{j,k} := (\sum P_{j,k}) * (\sum Q_{j,k}) :. $$

We can easily verify the associativity about the operation $\circ$. That is, $\hat{S}(K)/\hat{R}(K)$ becomes an associative algebra. Hence the mapping $H_K$ is a homomorphism about the operation $\circ$, $+$, and $\cdot$, where

$$ \mathcal{E}_{X \times Y}^{\mathbb{C}}(\gamma^{-1}(K)) \xrightarrow{H_K} \mathcal{E}_{X \times Y}^{\mathbb{C}}(\gamma^{-1}(K)). $$

Definition 2.18. The reverse of $\sum P_{j,k}$ in $\hat{S}(K)$ is defined as

$$ (\sum t_1^j t_2^k P_{j,k})^R := e^{t_1(\partial_\xi, \partial_z) + t_2(\partial_\eta, \partial_w)} (\sum t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)). $$

We can verify that if $\sum P_{j,k}$ is in $\hat{S}(K)$ ($\hat{R}(K)$) then $(\sum P_{j,k})^R$ is in $\hat{S}(K)$ ($\hat{R}(K)$), respectively.
3. Exponential Calculus of Symbols of Minimum Type

**Definition 3.1.** A function $\Lambda : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following hold:

1. $\Lambda$ is continuous,
2. for each $\alpha > 1$, $\Lambda(\alpha t) \leq \alpha \Lambda(t)$ on $(0, \infty)$,
3. $\Lambda$ is increasing,
4. $\lim_{t \to \infty} \frac{\Lambda(t)}{t} = 0$.

**Definition 3.2.** $P(z, \xi, w, \eta) \in S(K)$ is called a symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:

1. $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$,
2. $|P(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$ on $\gamma^{-1}(U; d, d)$.

**Example 3.3.** (by K. Kataoka)

$\Omega = \Omega' := \mathbb{C} \times \{\xi \in \mathbb{C}; |\arg \xi| < \delta, \xi \neq 0\}(0 < \delta < \frac{\pi}{2})$.

Let $K$ be any compact subset of $S^*\mathbb{C}_x \times S^*\mathbb{C}_w$ such that $\gamma^{-1}(K) \subset \Omega \times \Omega'$.

$P(z, \xi, w, \eta) := (\xi \eta)^{(1+\sigma)/2}/(\xi + \eta)$,

$\Lambda_1(t) = \Lambda_2(t) := t^\sigma$ with $0 < \sigma < 1$.

**Remark 3.4.** If $P$ is a symbol of minimum type on $K$, $e^P$ is a symbol of product type on $K$.

**Definition 3.5.** $\sum P_{j,k}$ in $\hat{S}(K)$ is called a formal symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, $0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:

1. $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$,
2. $|P_{j,k}(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\} \cdot A^{j+k}$ on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

**Remark 3.6.** If $\sum P_{j,k}$ is a formal symbol of minimum type on $K$, $e^{\sum P_{j,k}}$ is a formal symbol of product type on $K$. 

Proposition 3.7. If $P$ and $Q$ are in $S(K)$, then

\[
P(z, \xi, w, \eta) \ast (Q(z, \xi, w, \eta))^R = e^{t_1(\partial_\xi, \partial_z) + t_2(\partial_\eta, \partial_w) P(z, \xi, w, \eta) Q(z^*, \xi, w^*, \eta)}_{z^* = z, w^* = w}.
\]

Theorem 3.8. If $P$ and $Q$ are symbols of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$, there exists a formal symbol, $\sum R_{j,k}$, of minimum type on $K$ satisfying $e^P \ast e^Q = e^{\sum t_1^j t_2^k R_{j,k}}$.

Proof. $W(s, t; z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) := e^{s(\partial_\xi, \partial_z) + t(\partial_\eta, \partial_w)} \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*))$ is the unique formal series solution to the following system of partial differential equations:

\[
\begin{cases}
\partial_s W = (\partial_\xi, \partial_z) W, \\
\partial_t W = (\partial_\eta, \partial_w) W, \\
W_{s=t=0} = \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*)). 
\end{cases}
\]

If we put $W = \exp(\sum_{j,k}^\infty s^j t^k W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*))$, by the above system, we obtain the following recursive formulas about $\{W_{j,k}\}_{j,k \geq 0}$:

\[
\begin{cases}
W_{0,0} = P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*), \\
W_{j+1,k} = \frac{1}{j+1} \{(\partial_\xi, \partial_z) W_{j,k} + \sum_{j_1+j_2=j, k_1+k_2=k} (\partial_\xi W_{j_1,k_1}, \partial_z W_{j_2,k_2})\}, \\
W_{j,k+1} = \frac{1}{k+1} \{(\partial_\eta, \partial_w) W_{j,k} + \sum_{j_1+j_2=j, k_1+k_2=k} (\partial_\eta W_{j_1,k_1}, \partial_w W_{j_2,k_2})\}.
\end{cases}
\]

Then $R_{j,k} = W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*)_{z^* = z, \xi^* = \xi, w^* = w, \eta^* = \eta}$.

Suppose there exist $C_P = C_Q > 0$, $d > 0$, and an open subset $U(\supset K)$ of $S^*X \times S^*Y$ satisfying the following:

1. $P$ and $Q$ are holomorphic in $\gamma^{-1}(U; d, d)$,
2. $|P(z, \xi, w, \eta)|, |Q(z, \xi, w, \eta)| \leq C_P \Lambda(|\xi|, |\eta|)$ on $\gamma^{-1}(U; d, d)$

where $\Lambda(|\xi|, |\eta|) := \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$.

$V := \gamma^{-1}(U; d, d) \times \gamma^{-1}(U; d, d)$,

$V^{\varepsilon_1, \varepsilon_2} := \{(z, w, \xi, \eta, z^*, w^*, \xi^*, \eta^*) \in V; |\xi' - \xi| \leq \varepsilon_1|\xi|, |z^* - z^*| \leq \varepsilon_1, |\eta' - \eta| \leq \varepsilon_2|\eta|, |w^* - w^*| \leq \varepsilon_2 \implies (z, w, \xi', \eta', z'^*, w'^*, \xi^*, \eta^*) \in V\}$.

Then we obtain the following lemma.
Lemma 3.9. Suppose \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfy the following conditions:

(1) 
\[
C_{j+1,k}^{(\mu,\nu)} \geq \frac{9ne^{10}}{j+1} \{C_{j,k}^{(\mu,\nu)}(j+1)^2 + \sum^*(j_1+1)(j_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)}\},
\]

(2) 
\[
C_{j,k+1}^{(\mu,\nu)} \geq \frac{9me^{10}}{k+1} \{C_{j,k}^{(\mu,\nu)}(k+1)^2 + \sum^{**}(k_1+1)(k_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\eta)}\},
\]

(3) 
\[
C_{0,0}^{0,0} \leq C_P + C_Q,
\]

(4) 
\[
C_{j,k}^{(\mu,\nu)} \geq 0 \ (j,k \geq 0, \ 0 \leq \mu \leq j, \ 0 \leq \nu \leq k),
\]

(5) 
\[
C_{j,k}^{(\mu,\nu)} = 0 \ (\text{otherwise}).
\]

Here, the sum \( \sum^* \), \( \sum^{**} \) mean \( \sum j_1+j_2=j, \sum j_1+j_2=j \), respectively.

Then for each \( \epsilon_1 \) and \( \epsilon_2 \) such that \( 0 < \epsilon_1 \ll 1 \) and \( 0 < \epsilon_2 \ll 1 \), the following hold:

(3.4) 
\[
|W_{j,k}| \leq \sum_{0 \leq \mu \leq j, \ 0 \leq \nu \leq k} \frac{C_{j,k}^{(\mu,\nu)}}{\epsilon_1^{2j} \epsilon_2^{2k} |\xi|^j |\eta|^k} (\tilde{\Lambda}(|\xi|, |\eta|) + \tilde{\Lambda}(|\xi^*|, |\eta^*|))
\]
\[
\times (\Lambda_1(|\xi|) + \Lambda_1(|\xi^*|))^{\mu} (\Lambda_2(|\eta|) + \Lambda_2(|\eta^*|))^{\nu}
\]
on \( V^{\epsilon_1,\epsilon_2} \) for all \( j,k \geq 0 \).

The following lemma guarantees the existence of \( C_{j,k}^{(\mu,\nu)} \) satisfying the conditions from (1) through (5) of the previous lemma.

Lemma 3.10. The following sequence \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfies the conditions from (1) through (5) of the previous lemma.

\[
C_{j,k}^{(\mu,\nu)} := \begin{cases} 
1B^{j+k}(j+1)^{-\mu-3}(k+1)^{-\nu-3}, & (0 \leq \mu \leq j, \ 0 \leq \nu \leq k) \\
0, & (\text{otherwise})
\end{cases}
\]

where \( l \) and \( B \) are constants satisfying \( l \geq \max\{C_P + C_Q, 1\} \) and \( B \geq 72l \cdot \max\{m,n\} \cdot e^{10} \cdot (c^2 + 1) \) and \( c \) is a constant satisfying the following T. Aoki's inequality:

(3.5) 
\[
\frac{1}{j+1} \sum_{\mu=0}^{\nu-1} \sum_{k=\mu}^{j-\nu+\mu+1} (k+1)^{j-\nu-2}(j-k+1)^{j-k-\nu+\mu-1} \leq c(j+1)^{j-\nu-2}
\]
for all $j, \nu$ such that $0 \leq \nu - 1 \leq j$.

(continued) We can prove the theorem using the above two lemmas.

**REFERENCES**


