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ON EXPONENTIAL CALCULUS OF SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS OF MINIMUM TYPE

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1. Introduction

Let X and Y be n- and m- dimensional complex manifolds, respectively.

$$S^*X := (T^*X - X)/\mathbb{R}^+, S^*Y := (T^*Y - Y)/\mathbb{R}^+.$$

We define the mapping γ as

$$\gamma : \overset{\circ\circ}{T^*}(X \times Y) \ni (z, w; \xi, \eta) \longmapsto (z; \frac{\xi}{|\xi|}) \times (w; \frac{\eta}{|\eta|}) \in S^*X \times S^*Y,$$

where

$$\overset{\circ\circ}{T^*}(X \times Y) := T^*(X \times Y) \backslash \{(T^*X \times Y) \cup (X \times T^*Y)\}.$$

For $d > 0$ and an open subset U of $S^*X \times S^*Y$ we denote

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$$

by $\gamma^{-1}(U; d, d)$.
Hereafter we write (z, ξ, w, η) for coordinates $(z, w; \xi, \eta)$.

2. Symbols of product type

Let K be a compact subset of $S^*X \times S^*Y$.

Definition 2.1. $P(z, \xi, w, \eta)$ is said to be a symbol of product type on K if the following hold:

(1) There are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ such that $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$.
(2) For each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{\varepsilon(|\xi| + |\eta|)} \quad \text{on } \gamma^{-1}(U; d, d). \tag{2.1}$$

We denote by $S(K)$ the set of all such symbols on K.
$S(K)$ becomes a commutative ring with the usual sum and product.

Definition 2.2. We denote by $R(K)$ the set of all $P(z,\xi,w,\eta) \in S(K)$ satisfying the following;
there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0,\infty)$ such that

$$|P(z,\xi,w,\eta)| \leq C(|\xi|/|\eta|)e^{-\delta \min\{|\xi|,|\eta|\}}$$

on $\gamma^{-1}(U;d,d)$.

We call an element of $R(K)$ a symbol of 0-class.

Definition 2.3. A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta)$ is called a formal symbol of product type on K if the following hold:

(1) There are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U;(j+1)d,(k+1)d)$ for each $j,k \geq 0$.

(2) For each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|P_{j,k}(z,\xi,w,\eta)| \leq C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)} \quad \text{on } \gamma^{-1}(U;(j+1)d,(k+1)d) \tag{2.2}$$

for each $j,k \geq 0$.

We denote by $\widehat{S}(K)$ the set of such formal symbols on K.
We often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)$, in indeterminants t_1 and t_2 for $\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta)$.

We can easily obtain the following.

Proposition 2.4. *$\widehat{S}(K)$ becomes a commutative ring with the sum and the product as formal power series in t_1 and t_2.*

$S(K)$ is identified with a subring of $\widehat{S}(K)$ as follows:
$S(K) \simeq \widehat{S}(K)|_{t_1=0,\, t_2=0} = \{P = \sum t_1^j t_2^k P_{j,k}; P_{j,k} \equiv 0 \text{ for all } (j,k) \neq (0,0)\}$.

Definition 2.5. We denote by $\widehat{R}(K)$ the set of all $P(t_1,t_2;z,\xi,w,\eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)$ in $\widehat{S}(K)$ such that there are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

for each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left| \sum_{\substack{0 \le j \le s \\ 0 \le k \le t}} P_{j,k}(z,\xi,w,\eta) \right| \le C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t \ge 0$.

We call an element of $\widehat{R}(K)$ a formal symbol of zero class.

Proposition 2.6. *Under the previous identification, $S(K) \cap \widehat{R}(K) = R(K)$ holds.*

Proof. Let $P(z,\xi,w,\eta)$ be in $S(K)$. Then $P(z,\xi,w,\eta) \in \widehat{R}(K)$ is equivalent to the following;
there exist $d > 0, \delta > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ such that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying

$$|P(z,\xi,w,\eta)| \le C_\varepsilon e^{-\delta \min\{|\xi|,|\eta|\}+\varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U;d,d)$.
($\subset$) Using the fact that $(0,\infty) = \{t := \frac{|\xi|}{|\eta|}; (z,\xi,w,\eta) \in \gamma^{-1}(U;d,d)\}$, by the hypothesis,we obtain the following;

$$|P(z,\xi,w,\eta)| \le C_\varepsilon e^{-\delta \min\{1,\frac{1}{t}\}|\xi|+\varepsilon(1+\frac{1}{t})|\xi|}$$

for all $t := \frac{|\xi|}{|\eta|} \in (0,\infty)$ and $(z,\xi,w,\eta) \in \gamma^{-1}(U;d,d)$.
We fix any $\varepsilon > 0$ such that $0 < \varepsilon < 1$ and $\varepsilon \le \frac{\delta}{3}$.
Then for every $t \in [\frac{3}{\delta}\varepsilon, 1]$

$$\begin{aligned} |P(z,\xi,w,\eta)| &\le C_\varepsilon e^{-\delta|\xi|+\varepsilon(1+\frac{1}{t})|\xi|} \le C_\varepsilon e^{-\delta|\xi|+\varepsilon(1+\frac{\delta}{3\varepsilon})|\xi|} \\ &= C_\varepsilon e^{(\varepsilon-\frac{2}{3}\delta)|\xi|} \le C_\varepsilon e^{-\frac{1}{3}\delta|\xi|}. \end{aligned}$$

On the other hand, for any sequence ε_n such that $\min\{1,\frac{\delta}{3}\} > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$, we define a function $C(\cdot)$ on $(0,1]$ as

$$C(t) := \begin{cases} C_{\frac{3}{\delta}\varepsilon_1}, & \varepsilon_1 < t \le 1, \\ C_{\frac{3}{\delta}\varepsilon_{n+1}}, & \varepsilon_{n+1} < t \le \varepsilon_n. \end{cases}$$

Then $C(\cdot)$ is locally bounded on $(0,1]$ and

$$|P(z,\xi,w,\eta)| \le C(\frac{|\xi|}{|\eta|}) e^{-\frac{1}{3}\delta|\xi|}$$

on $\gamma^{-1}(U;d,d)\cap\{|\xi|\le|\eta|\}$.
In like manners,

$$|P(z,\xi,w,\eta)|\le C(\frac{|\eta|}{|\xi|})e^{-\frac{1}{3}\delta|\eta|}$$

on $\gamma^{-1}(U;d,d)\cap\{|\xi|\ge|\eta|\}$.
Here,we define a function $C'(\cdot)$ on $(0,\infty)$ as $C'(t)=C(\min\{t,\frac{1}{t}\})$.
Then $C'(t)$ is locally bounded on $(0,\infty)$
and $|P(z,\xi,w,\eta)|\le C'(\frac{|\xi|}{|\eta|})e^{-\frac{1}{3}\delta\cdot\min\{|\xi|,|\eta|\}}$ on $\gamma^{-1}(U;d,d)$.
That is,$P(z,\xi,w,\eta)\in R(K)$.
($\supset$) Let $P(z,\xi,w,\eta)\in R(K)$.
Then there are $d>0,\delta>0,U\supset K$ open in $S^*X\times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0,\infty)$ such that

$$|P(z,\xi,w,\eta)|\le C(\frac{|\xi|}{|\eta|})e^{-\delta\min\{|\xi|,|\eta|\}}$$

on $\gamma^{-1}(U;d,d)$. We fix any ε such that $0<\varepsilon<1$. Then,

$$|P(z,\xi,w,\eta)|\le\max_{\varepsilon\le t\le 1}C(t)\cdot e^{-\delta\min\{|\xi|,|\eta|\}}$$

on $\gamma^{-1}(U;d,d)\cap\{\varepsilon\le\frac{|\xi|}{|\eta|}=:t\le 1\}$. We put $C'_\varepsilon:=\max_{\varepsilon\le t\le 1}C(t)$.
On the other hand, since $P(z,\xi,w,\eta)\in S(K)$, there exists $C''_\varepsilon>0$ such that

$$|P(z,\xi,w,\eta)|\le C''_\varepsilon e^{\varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U;d,d)$.

Therefore, the following iequalities hold on $\gamma^{-1}(U;d,d)\cap\{\frac{|\xi|}{|\eta|}\le\varepsilon\}$.

$$\begin{aligned}|P(z,\xi,w,\eta)|&\le C''_\varepsilon e^{-\delta\min\{(|\xi|,|\eta|)\}+\delta\min\{(|\xi|,|\eta|)\}+\varepsilon(|\xi|+|\eta|)}\\&\le C''_\varepsilon e^{-\delta\min\{(|\xi|,|\eta|)\}+\varepsilon(1+\delta)(|\xi|+|\eta|)}.\end{aligned}$$

If we put $C_\varepsilon:=\max\{C'_\varepsilon,C''_\varepsilon\}$,

$$|P(z,\xi,w,\eta)|\le C_\varepsilon e^{-\delta\min\{(|\xi|,|\eta|)\}+\varepsilon(1+\delta)(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U;d,d)\cap\{|\xi|\le|\eta|\}$.
That is, $P(z,\xi,w,\eta)\in\widehat{R}(K)$.

Proposition 2.7. *$R(K)$ is an ideal in $S(K)$.*

Proof. It is clear by the part ($\subset$) of the proof of Proposition 2.6.

Proposition 2.8. *$\widehat{R}(K)$ is an ideal in $\widehat{S}(K)$.*

Proof. Let $\sum P_{j,k}(z,\xi,w,\eta) \in \widehat{R}(K)$ and $\sum Q_{j,k}(z,\xi,w,\eta) \in \widehat{S}(K)$. Then there exist $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:
For each $\varepsilon > 0$, we have some $C_\varepsilon > 0$ such that

$$\text{a)} \quad |P_{s,t}(z,\xi,w,\eta)|, |Q_{s,t}(z,\xi,w,\eta)| \le C_\varepsilon A^{s+t} e^{\varepsilon(|\xi|+|\eta|)}$$

$$\text{b)} \quad \left| \sum_{\substack{0\le j\le s \\ 0\le k\le t}} P_{j,k}(z,\xi,w,\eta) \right| \le C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \ge 0$.
It suffices to show that $\sum R_{j,k} \in \widehat{R}(K)$, where

$$R_{j,k} := \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2}.$$

Since we can easily estimate $\displaystyle\sum_{\substack{0\le j\le s \\ 0\le k\le t}} R_{j,k}$ for $st = 0$,
we suppose $s \ge 1$ and $t \ge 1$.
Then we can obtain the follwing inequality:

$$\left| \sum_{\substack{0\le j\le s \\ 0\le k\le t}} R_{j,k} \right| = \left| \sum_{\substack{0\le j\le s \\ 0\le k\le t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right|$$

$$\le \left| \Big(\sum_{\substack{0\le j_1\le s \\ 0\le k_1\le t}} P_{j_1,k_1} \Big) \Big(\sum_{\substack{0\le j_2\le s \\ 0\le k_2\le t}} Q_{j_2,k_2} \Big) \right| + \left| \sum_{\substack{s+1\le j\le 2s \\ t+1\le k\le 2t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right|$$

$$+ \left| \sum_{\substack{0\le j\le s \\ t+1\le k\le 2t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right| + \left| \sum_{\substack{s+1\le j\le 2s \\ 0\le k\le t}} \sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} P_{j_1,k_1} Q_{j_2,k_2} \right|.$$

We shall estimate the four terms in the right side of the inequality, respectively.

$$\text{the first term} \le C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)} \cdot \sum_{\substack{0\le j_2\le s \\ 0\le k_2\le t}} C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)}$$

$$\le C_\varepsilon \cdot C_\varepsilon \cdot A^{\min\{s,t\}} e^{2\varepsilon(|\xi|+|\eta|)} \cdot \frac{1}{1-A} \cdot \frac{1}{1-A}$$

on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t\geq 1$.

the 2nd term

$$\leq \sum_{\substack{s+1\leq j\leq 2s\\ t+1\leq k\leq 2t}} \sum_{\substack{j_1+j_2=j\\ k_1+k_2=k}} C_\varepsilon A^{j_1+k_1} e^{\varepsilon(|\xi|+|\eta|)} \cdot C_\varepsilon A^{j_2+k_2} e^{\varepsilon(|\xi|+|\eta|)}$$

$$= C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot \Big(\sum_{s+1\leq j\leq 2s} \sum_{j_1+j_2=j} A^j\Big)\Big(\sum_{t+1\leq k\leq 2t} \sum_{k_1+k_2=k} A^k\Big).$$

If we choose any B and C such that $0 < B < 1$, $0 < C < 1$, and $BC \geq A$, we can get the following inequality:

$$\sum_{s+1\leq j\leq 2s} \sum_{j_1+j_2=j} A^j \leq C^{s+1}(B^0 + B^1 + B^2 + \cdots)^2 = C^{s+1}\Big(\frac{1}{1-B}\Big)^2.$$

Then,

$$\text{the second term} \leq C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot C^{s+1}\Big(\frac{1}{1-B}\Big)^2 \cdot C^{t+1}\Big(\frac{1}{1-B}\Big)^2$$

on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t\geq 1$.

the third term

$$\leq \sum_{\substack{0\leq j\leq s\\ t+1\leq k\leq 2t}} \sum_{\substack{j_1+j_2=j\\ k_1+k_2=k}} C_\varepsilon A^{j_1+k_1} e^{\varepsilon(|\xi|+|\eta|)} \cdot C_\varepsilon A^{j_2+k_2} e^{\varepsilon(|\xi|+|\eta|)}$$

$$= C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \Big(\sum_{0\leq j\leq s} \sum_{j_1+j_2=j} A^j\Big)\Big(\sum_{t+1\leq k\leq 2t} \sum_{k_1+k_2=k} A^k\Big)$$

$$\leq C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot \Big(\frac{1}{1-A}\Big)^2 \cdot C^{t+1}\Big(\frac{1}{1-B}\Big)^2$$

on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t\geq 1$.
In like manners,

$$\text{the fourth term} \leq C_\varepsilon \cdot C_\varepsilon \cdot e^{2\varepsilon(|\xi|+|\eta|)} \cdot C^{s+1} \cdot \Big(\frac{1}{1-B}\Big)^2 \cdot \Big(\frac{1}{1-A}\Big)^2$$

on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t\geq 1$.
Hence, we conclude that $\sum R_{j,k} \in \widehat{R}(k)$.

$\widehat{S}(K)/\widehat{R}(K)$ becomes a commutative ring by Proposition 2.8. By Propsitions 2.6 and 2.7, the inclusion $S(K) \hookrightarrow \widehat{S}(K)$ induces the injective ring homomorphism

$$\iota_K : S(K)/R(K) \longrightarrow \widehat{S}(K)/\widehat{R}(K).$$

Conversely, we obtain the following.

Theorem 2.9. *If $\sum P_{j,k}(z,\xi,w,\eta) \in \widehat{S}(K)$, there exists $P(z,\xi,w,\eta) \in S(K)$ such that $P - \sum P_{j,k} \in \widehat{R}(K)$.*

Thus, $S(K)/R(K)$ is isomorphic to $\widehat{S}(K)/\widehat{R}(K)$ in the sense of commutative rings.

Definition 2.10. We call an element in the ring $\widehat{S}(K)/\widehat{R}(K)$ a pseudo-differential operator of the product type on K. We write $: \sum P_{j,k} :$ for the associated pseudo-differential operator of the product type on K using an element $\sum P_{j,k}$ in $\widehat{S}(K)$.

The mapping γ is the composition of the following γ_1 and γ_2.

$$\overset{\circ\circ}{T^*}(X \times Y) \ni (z,w;\xi,\eta) \overset{\gamma_2}{\longmapsto} (z,w;\frac{\xi}{|(\xi,\eta)|},\frac{\eta}{|(\xi,\eta)|}) \in \overset{\circ\circ}{S^*}(X \times Y),$$

$$\overset{\circ\circ}{S^*}(X\times Y) \ni (z,w;\frac{\xi}{|(\xi,\eta)|},\frac{\eta}{|(\xi,\eta)|}) \overset{\gamma_1}{\longmapsto} (z,\frac{\xi}{|\xi|})\times(w,\frac{\eta}{|\eta|}) \in S^*X\times S^*Y,$$

$$\text{where} \quad \overset{\circ\circ}{S^*}(X \times Y) := S^*(X \times Y)\backslash\{(S^*X \times Y) \cup (X \times S^*Y)\}.$$

Proposition 2.11. *If $P(z,\xi,w,\eta)$ is a symbol of product type on K, P is a symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.*

Proof. By the hypothesis, there are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:
a) $P(z,\xi,w,\eta)$ is holomorphic in $\gamma^{-1}(U;d,d)$, and
b)for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $|P(z,\xi,w,\eta)| \le C_\varepsilon e^{\varepsilon(|\xi|+|\eta|)}$ on $\gamma^{-1}(U;d,d)$.
Let K' be compact in $\overset{\circ\circ}{S^*}(X \times Y)$ and $\gamma_1^{-1}(K) \supset K'$.
Then there exist $d' > 0$ and $U' \supset K'$ open in $\overset{\circ\circ}{S^*}(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.$$

In fact, for each $(\mathring{z},\mathring{w};\mathring{\xi},\mathring{\eta}) \in \gamma_1^{-1}(K)$ we can choose $d' > 0$ such that

$$d' > \frac{d}{\min\{|\mathring{\xi}|,|\mathring{\eta}|\}}.$$

Then there exists a neighborhood U' of $(\mathring{z}, \mathring{w}; \mathring{\xi}, \mathring{\eta}) \in \gamma_1^{-1}(K)$ in $\overset{\circ\circ}{S}{}^*(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.$$

By the compactness of K', the proof is completed.

Proposition 2.12. *If $P(z, \xi, w, \eta)$ is a symbol of product type of 0-class on K, that is, $P \in R(K)$, P is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.*

Proof. Let K' be compact in $\overset{\circ\circ}{S}{}^*(X \times Y)$ and $\gamma_1^{-1}(K) \supset K'$.
It suffices to show that P is a zero symbol on K' in the sense of AOKI's symbol. By the hypothesis, there exist $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|})e^{-\delta \min\{|\xi|,|\eta|\}}$$

on $\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$.
Let $(\mathring{z}, \mathring{w}; \mathring{\xi}, \mathring{\eta})$ be any point of $\gamma_1^{-1}(K)$. By Proposition 2.11, there exist $d' > 0$ and a neighborhood U' of $(\mathring{z}, \mathring{w}; \mathring{\xi}, \mathring{\eta})$ in $\overset{\circ\circ}{S}{}^*(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\},$$

and that there exists $\delta' > 0$ satisfying

$$\min\{\frac{|\xi|}{|\xi| + |\eta|}, \frac{|\eta|}{|\xi| + |\eta|}\} > \delta' \quad \text{on} \quad \gamma_2^{-1}(U').$$

Hence,

$$|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|})e^{-\delta\delta'(|\xi|+|\eta|)}$$

on $\gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}$.
Since K is compact, P is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.

Definition 2.13. Tne canonical mapping H_K is defined as follows;

$$S(K)/R(K) \ni: P :\overset{H_K}{\longmapsto} [P] \in \varinjlim_{U \supset \gamma_1^{-1}(K)} \mathcal{E}^{\mathbb{R}}(U).$$

Proposition 2.14. *Suppose K_1 and K_2 are compact in $S^*X \times S^*Y$, respectively, and $K_1 \supset K_2$. Then, $H_{K_1}(: P :)\Big|_{\gamma_1^{-1}(K_2)} = H_{K_2}(: P|_{K_2} :)$ for all $P \in S(K)/R(K)$.*

Definition 2.15. We define the product $*$ of two elements of $\widehat{S}(K)$. as follows:

$$(\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta)) * (\sum_{j,k=0}^{\infty} Q_{j,k}(z,\xi,w,\eta)) = \sum_{j,k=0}^{\infty} R_{j,k}(z,\xi,w,\eta),$$

where

$$\sum_{j,k=0}^{\infty} t_1^j t_2^k R_{j,k}(z,\xi,w,\eta) := e^{t_1\langle\partial_\xi,\partial_{z^*}\rangle + t_2\langle\partial_\eta,\partial_{w^*}\rangle}((\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta))$$

$$\times (\sum_{j,k=0}^{\infty} Q_{j,k}(z^*,\xi^*,w^*,\eta^*)))\Big|_{\substack{z^*=z,\xi^*=\xi \\ w^*=w,\eta^*=\eta}}.$$

That is,

$$R_{j,k}(z,\xi,w,\eta) := \sum_{\substack{j_1+j_2+|\alpha|=j \\ k_1+k_2+|\beta|=k}} \frac{1}{\alpha!\beta!}\partial_\xi^\alpha \partial_\eta^\beta P_{j_1k_1}(z,\xi,w,\eta)$$

$$\times \partial_z^\alpha \partial_w^\beta Q_{j_2k_2}(z,\xi,w,\eta).$$

Then we obtain the following.

Lemma 2.16. *If $\sum P_{j,k}$ and $\sum Q_{j,k}$ are formal symbols of product type on K, then $\sum R_{j,k}$ is also a formal symbol of product type on K.*

Proposition 2.17. *If $\sum P_{j,k} \in \widehat{S}(K)$ and $\sum Q_{j,k} \in \widehat{R}(K)$, otherwise $\sum P_{j,k} \in \widehat{R}(K)$ and $\sum Q_{j,k} \in \widehat{S}(K)$, $\sum R_{j,k}$ is also in $\widehat{R}(K)$.*

By Lemma 2.16 and Proposition 2.17, the following composition of two elements in$\widehat{S}(K)/\widehat{R}(K)$ is well-defined;

$$:\sum P_{j,k} : \circ : \sum Q_{j,k} ::=: (\sum P_{j,k}) * (\sum Q_{j,k}) : .$$

We can easily verify the associativity about the operation $\circ$. That is, $\widehat{S}(K)/\widehat{R}(K)$ becomes an associative $\mathbb{C}$ algebra. Hence the mapping H_K is a homomorphism about the operation $\circ$, $+$, and $\cdot$, where

$$\mathcal{E}_{X\times Y}^{\mathbb{R},prod}(K) \equiv \widehat{S}(K)/\widehat{R}(K) \xrightarrow{H_K} \mathcal{E}_{X\times Y}^{\mathbb{R}}(\gamma^{-1}(K)).$$

Definition 2.18. The reverse of $\sum P_{j,k}$ in $\widehat{S}(K)$ is defined as

$$(\sum t_1^j t_2^k P_{j,k})^R := e^{t_1\langle\partial_\xi,\partial_z\rangle + t_2\langle\partial_\eta,\partial_w\rangle}(\sum t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)).$$

We can verify that if $\sum P_{j,k}$ is in $\widehat{S}(K)$ ($\widehat{R}(K)$) then $(\sum P_{j,k})^R$ is in $\widehat{S}(K)$ ($\widehat{R}(K)$) ,respectively.

3. Exponential calculus of symbols of minimum type

Definition 3.1. A function $\Lambda : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following hold;

(1) Λ is continuous,

(2) for each $\alpha > 1$, $\Lambda(\alpha t) \leq \alpha \Lambda(t)$ on $(0, \infty)$,

(3) Λ is increasing,

(4) $\lim_{t \to \infty} \dfrac{\Lambda(t)}{t} = 0$.

Definition 3.2. $P(z, \xi, w, \eta) \in S(K)$ is called a symbol of minimum type of growth order (Λ_1, Λ_2) on K if there exist constants $C > 0$, $d > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

(1) $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$, and

(2) $|P(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$ on $\gamma^{-1}(U; d, d)$.

Example 3.3. (by K. Kataoka)
$\Omega = \Omega' := \mathbb{C} \times \{\xi \in \mathbb{C}; |\arg \xi| < \delta,\ \ \xi \neq 0\}(0 < \delta < \frac{\pi}{2})$.
Let K be any compact subset of $S^*\mathbb{C}_z \times S^*\mathbb{C}_w$ such that $\gamma^{-1}(K) \subset \Omega \times \Omega'$.
$P(z, \xi, w, \eta) := (\xi\eta)^{(1+\sigma)/2}/(\xi + \eta)$,
$\Lambda_1(t) = \Lambda_2(t) := t^\sigma$ with $0 < \sigma < 1$.

Remark 3.4. If P is a symbol of minimum type on K, e^P is a symbol of product type on K.

Definition 3.5. $\sum P_{j,k}$ in $\widehat{S}(K)$ is called a formal symbol of minimum type of growth order (Λ_1, Λ_2) on K if there exist constants $C > 0$, $d > 0$, $0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

(1) $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$,
(2)

$$|P_{j,k}(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\} \cdot A^{j+k}$$

on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

Remark 3.6. If $\sum P_{j,k}$ is a formal symbol of minimum type on K, $e^{\sum P_{j,k}}$ is a formal symbol of product type on K.

Proposition 3.7. *If P and Q are in $S(K)$, then*

$$P(z,\xi,w,\eta) * (Q(z,\xi,w,\eta))^R = e^{t_1\langle\partial_\xi,\partial_{z^*}\rangle + t_2\langle\partial_\eta,\partial_{w^*}\rangle} P(,\xi,w,\eta)Q(z^*,\xi,w^*,\eta)\Big|_{\substack{z^*=z\\ w^*=w}} . \tag{3.1}$$

Theorem 3.8. *If P and Q are symbols of minimum type of growth order (Λ_1,Λ_2) on K, there exists a formal symbol, $\sum R_{j,k}$, of minimum type on K satisfying $e^P * e^Q = e^{\sum t_1^j t_2^k R_{j,k}}$.*

Proof. $W(s,t;z,\xi,w,\eta,z^*,\xi^*,w^*,\eta^*)$ $:= e^{s\langle\partial_\xi,\partial_{z^*}\rangle + t\langle\partial_\eta,\partial_{w^*}\rangle}\exp(P(z,\xi,w,\eta) + Q(z^*,\xi^*,w^*,\eta^*))$ is the unique formal series solution to the following system of partial differential equations:

$$\begin{cases} \partial_s W = \langle\partial_\xi,\partial_{z^*}\rangle W, \quad \partial_t W = \langle\partial_\eta,\partial_{w^*}\rangle W, \\ W_{s=t=0} = \exp(P(z,\xi,w,\eta) + Q(z^*,\xi^*,w^*,\eta^*)). \end{cases} \tag{3.2}$$

If we put $W = \exp(\sum_{j,k}^{\infty} s^j t^k W_{j,k}(z,\xi,w,\eta,z^*,\xi^*,w^*,\eta^*))$, by the above system, we obtain the following recursive formulas about $\{W_{j,k}\}_{j,k\geq 0}$:

$$\begin{cases} W_{0,0} = P(z,\xi,w,\eta) + Q(z^*,\xi^*,w^*,\eta^*), \\ W_{j+1,k} = \frac{1}{j+1}\{\langle\partial_\xi,\partial_{z^*}\rangle W_{j,k} + \sum\limits_{\substack{j_1+j_2=j\\ k_1+k_2=k}} \langle\partial_\xi W_{j_1,k_1}, \partial_{z^*} W_{j_2,k_2}\rangle\}, \\ W_{j,k+1} = \frac{1}{k+1}\{\langle\partial_\eta,\partial_{w^*}\rangle W_{j,k} + \sum\limits_{\substack{j_1+j_2=j\\ k_1+k_2=k}} \langle\partial_\eta W_{j_1,k_1}, \partial_{w^*} W_{j_2,k_2}\rangle\}. \end{cases} \tag{3.3}$$

Then $R_{j,k} = W_{j,k}(z,\xi,w,\eta,z^*,\xi^*,w^*,\eta^*)\Big|_{\substack{z^*=z,\xi^*=\xi\\ w^*=w,\eta^*=\eta}}$.

Suppose there exist $C_P(= C_Q) > 0$, $d > 0$, and an open subset $U(\supset K)$ of $S^*X \times S^*Y$ satisfying the following:

1. (1) P and Q are holomorphic in $\gamma^{-1}(U;d,d)$,
2. (2) $|P(z,\xi,w,\eta)|$, and $|Q(z,\xi,w,\eta)| \leq C_P\cdot\widetilde{\Lambda}(|\xi|,|\eta|)$ on $\gamma^{-1}(U;d,d)$, where $\widetilde{\Lambda}(|\xi|,|\eta|) := \min\{\Lambda_1(|\xi|),\Lambda_2(|\eta|)\}$.

$V := \gamma^{-1}(U;d,d) \times \gamma^{-1}(U;d,d)$,
$V^{\varepsilon_1,\varepsilon_2} := \{(z,w,\xi,\eta,z^*,w^*,\xi^*,\eta^*) \in V; |\xi' - \xi| \leq \varepsilon_1|\xi|, |z^{*\prime} - z^*| \leq \varepsilon_1,$
$|\eta' - \eta| \leq \varepsilon_2|\eta|, |w^{*\prime} - w^*| \leq \varepsilon_2 \Longrightarrow (z,w,\xi',\eta',z^{*\prime},w^{*\prime},\xi^*,\eta^*) \in V\}$.
Then we obtain the following lemma.

Lemma 3.9. *Suppose* $\{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu\geq 0}$ *satisfy the following conditions:*
(1)

$$C_{j+1,k}^{(\mu,\nu)} \geq \frac{9ne^{10}}{j+1}\{C_{j,k}^{(\mu,\nu)}(j+1)^2 + \sum{}^{*}(j_1+1)(j_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)}\},$$

(2)

$$C_{j,k+1}^{(\mu,\nu)} \geq \frac{9me^{10}}{k+1}\{C_{j,k}^{(\mu,\nu)}(k+1)^2 + \sum{}^{**}(k_1+1)(k_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)}\},$$

(3) $C_{0,0}^{0,0} \leq C_P + C_Q$,
(4) $C_{j,k}^{(\mu,\nu)} \geq 0 \ \ (j,k\geq 0, \ \ 0\leq \mu \leq j, \ \ 0 \leq \nu \leq k)$,
(5) $C_{j,k}^{(\mu,\nu)} = 0$ *(otherwise).*
Here, the sum $\sum^{*}$, $\sum^{**}$ *mean* $\sum_{\substack{j_1+j_2=j,\\ k_1+k_2=k,\\ \mu_1+\mu_2=\mu-1,\\ \nu_1+\nu_2=\nu}}$, $\sum_{\substack{j_1+j_2=j,\\ k_1+k_2=k,\\ \mu_1+\mu_2=\mu,\\ \nu_1+\nu_2=\nu-1}}$, *respectively.*
Then for each ε_1 *and* ε_2 *such that* $0<\varepsilon_1\ll 1$ *and* $0<\varepsilon_2\ll 1$, *the following hold:*

$$(3.4)\quad |W_{j,k}| \leq \sum_{\substack{0\leq\mu\leq j,\\ 0\leq\nu\leq k}} \frac{C_{j,k}^{(\mu,\nu)}}{\varepsilon_1^{2j}\varepsilon_2^{2k}|\xi|^j|\eta|^k}(\widetilde{\Lambda}(|\xi|,|\eta|)+\widetilde{\Lambda}(|\xi^*|,|\eta^*|)) \times (\Lambda_1(|\xi|)+\Lambda_1(|\xi^*|))^{\mu}(\Lambda_2(|\eta|)+\Lambda_2(|\eta^*|))^{\nu}$$

on $V^{\varepsilon_1,\varepsilon_2}$ *for all* $j,k\geq 0$.

The following lemma guarantees the existence of $C_{j,k}^{(\mu,\nu)}$ satisfying the conditions from (1) through (5) of the previous lemma.

Lemma 3.10. *The following sequence* $\{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu\geq 0}$ *satisfies the conditions from (1) through (5) of the previous lemma.*

$$C_{j,k}^{(\mu,\nu)} := \begin{cases} lB^{j+k}(j+1)^{j-\mu-3}(k+1)^{k-\nu-3}, & (0\leq\mu\leq j, \ \ 0\leq\nu\leq k) \\ 0, & (otherwise) \end{cases},$$

where l *and* B *are constants satisfying* $l \geq \max\{C_P+C_Q, 1\}$ *and* $B \geq 72l\cdot\max\{m,n\}\cdot e^{10}\cdot(c^2+1)$ *and* c *is a constant satisfying the following T.Aoki's iequality;*

$$(3.5)\quad \frac{1}{j+1}\sum_{\mu=0}^{\nu-1}\sum_{k=\mu}^{j-\nu+\mu+1}(k+1)^{k-\mu-2}(j-k+1)^{j-k-\nu+\mu-1} \leq c(j+1)^{j-\nu-2}$$

for all j, ν *such that* $0 \leq \nu - 1 \leq j$.

(continued) We can prove the theorem using the above two lemmas.

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