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Kyoto University
ON EXPONENTIAL CALCULUS OF SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS OF MINIMUM TYPE

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1. INTRODUCTION

Let $X$ and $Y$ be $n$- and $m$-dimensional complex manifolds, respectively.

$$S^*X := (T^*X - X)/\mathbb{R}^+, \quad S^*Y := (T^*Y - Y)/\mathbb{R}^+.$$ 

We define the mapping $\gamma$ as

$$\gamma : \mathcal{T}^*(X \times Y) \ni (z, w; \xi, \eta) \mapsto (z; \frac{\xi}{|\xi|}) \times (w; \frac{\eta}{|\eta|}) \in S^*X \times S^*Y,$$

where

$$\mathcal{T}^*(X \times Y) := T^*(X \times Y) \backslash \{(T^*X \times Y) \cup (X \times T^*Y)\}.$$ 

For $d > 0$ and an open subset $U$ of $S^*X \times S^*Y$ we denote

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$$

by $\gamma^{-1}(U; d, d)$.

Hereafter we write $(z, \xi, w, \eta)$ for coordinates $(z, w; \xi, \eta)$.

2. SYMBOLS OF PRODUCT TYPE

Let $K$ be a compact subset of $S^*X \times S^*Y$.

**Definition 2.1.** $P(z, \xi, w, \eta)$ is said to be a symbol of product type on $K$ if the following hold:

1. There are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ such that $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$.
2. For each $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$|P(z, \xi, w, \eta)| \leq C_\epsilon e^{\epsilon(|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; d, d)$. 

\[2.1\]
We denote by $S(K)$ the set of all such symbols on $K$. $S(K)$ becomes a commutative ring with the usual sum and product.

**Definition 2.2.** We denote by $R(K)$ the set of all $P(z, \xi, w, \eta) \in S(K)$ satisfying the following:

there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(|\xi|/|\eta|)e^{-\delta \min\{|\xi|,|\eta|\}}$$

on $\gamma^{-1}(U; d, d)$.

We call an element of $R(K)$ a symbol of $0$-class.

**Definition 2.3.** A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$ is called a formal symbol of product type on $K$ if the following hold:

1. There are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.
2. For each $\epsilon > 0$, there is $C_{\epsilon} > 0$ such that

$$(2.2) \quad |P_{j,k}(z, \xi, w, \eta)| \leq C_{\epsilon} A^{j+k}e^{\epsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

We denote by $\hat{S}(K)$ the set of such formal symbols on $K$.

We often write a formal power series $\sum_{j,k=0}^{\infty} t_{1}^{j}t_{2}^{k}P_{j,k}(z, \xi, w, \eta)$, in indeterminants $t_{1}$ and $t_{2}$ for $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$.

We can easily obtain the following.

**Proposition 2.4.** $\hat{S}(K)$ becomes a commutative ring with the sum and the product as formal power series in $t_{1}$ and $t_{2}$.

$S(K)$ is identified with a subring of $\hat{S}(K)$ as follows:

$S(K) \simeq \hat{S}(K)|_{t_{1}=0, t_{2}=0} = \{P = \sum_{j,k=0}^{\infty} t_{1}^{j}t_{2}^{k}P_{j,k}; P_{j,k} \equiv 0 \text{ for all } (j, k) \neq (0,0)\}$.

**Definition 2.5.** We denote by $\hat{R}(K)$ the set of all $P(t_{1}, t_{2}; z, \xi, w, \eta) := \sum_{j,k=0}^{\infty} t_{1}^{j}t_{2}^{k}P_{j,k}(z, \xi, w, \eta)$ in $\hat{S}(K)$ such that there are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;
for each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} P_{j,k}(z, \xi, w, \eta) \right| \leq C_\varepsilon A^{\min\{s, t\}} e^{\varepsilon(|\xi| + |\eta|)}$$
on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 0$.

We call an element of $\hat{R}(K)$ a formal symbol of zero class.

**Proposition 2.6.** Under the previous identification, $S(K) \cap \hat{R}(K) = R(K)$ holds.

**Proof.** Let $P(z, \xi, w, \eta)$ be in $S(K)$. Then $P(z, \xi, w, \eta) \in \hat{R}(K)$ is equivalent to the following: there exist $d > 0, \delta > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ such that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{|\xi|, |\eta|\} + \varepsilon(|\xi| + |\eta|)}$$
on $\gamma^{-1}(U; d, d)$.

$(\subset)$ Using the fact that $(0, \infty) = \{t := \frac{|\xi|}{|\eta|}; (z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)\}$, by the hypothesis, we obtain the following;

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{1, \frac{1}{t}\} |\xi| + \varepsilon(1 + \frac{1}{t}) |\xi|}$$

for all $t := \frac{|\xi|}{|\eta|} \in (0, \infty)$ and $(z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)$.

We fix any $\varepsilon > 0$ such that $0 < \varepsilon < 1$ and $\varepsilon \leq \frac{\delta}{3}$.

Then for every $t \in \left[\frac{3}{\varepsilon}, 1\right]$,

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta |\xi| + \varepsilon(1 + \frac{1}{t}) |\xi|} \leq C_\varepsilon e^{-\delta |\xi| + \varepsilon(1 + \frac{\delta}{3}) |\xi|}$$

$$= C_\varepsilon e^{(\varepsilon - \frac{\delta}{3}) |\xi|} \leq C_\varepsilon e^{-\frac{\delta}{3} |\xi|}.$$

On the other hand, for any sequence $\varepsilon_n$ such that $\min\{1, \frac{\delta}{3}\} > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$, we define a function $C(\cdot)$ on $(0, 1]$ as

$$C(t) := \begin{cases} C_\varepsilon, & \varepsilon_1 < t \leq 1, \\ C_\varepsilon \varepsilon_n^{\varepsilon_{n+1}}, & \varepsilon_{n+1} < t \leq \varepsilon_n. \end{cases}$$

Then $C(\cdot)$ is locally bounded on $(0, 1]$ and

$$|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|}) e^{-\frac{\delta}{3} |\xi|}$$
on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$.

In like manners,

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\eta|}{|\xi|}\right)e^{-\frac{1}{3}\delta|\eta|}$$
on $\gamma^{-1}(U; d, d) \cap \{|\xi| \geq |\eta|\}$.

Here, we define a function $C'(\cdot)$ on $(0, \infty)$ as $C'(t) = C(\min\{t, \frac{1}{t}\})$.

Then $C'(t)$ is locally bounded on $(0, \infty)$ and $|P(z, \xi, w, \eta)| \leq C'(\frac{|\xi|}{|\eta|})e^{-\frac{1}{3}\delta\min\{|\xi|,|\eta|\}}$ on $\gamma^{-1}(U; d, d)$.

That is, $P(z, \xi, w, \eta) \in R(K)$.

(>) Let $P(z, \xi, w, \eta) \in R(K)$.

Then there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C'(t)\cdot e^{-\frac{1}{3}\delta\min\{|\xi|,|\eta|\}}$$
on $\gamma^{-1}(U; d, d)$.

We fix any $\epsilon$ such that $0 < \epsilon < 1$. Then,

$$|P(z, \xi, w, \eta)| \leq \max_{\epsilon \leq t \leq 1} C(t)\cdot e^{-\frac{1}{3}\delta\min\{|\xi|,|\eta|\}}$$
on $\gamma^{-1}(U; d, d) \cap \{t \leq \frac{|\xi|}{|\eta|} \leq \epsilon\}$. We put $C'_{\epsilon} := \max_{\epsilon \leq t \leq 1} C(t)$.

On the other hand, since $P(z, \xi, w, \eta) \in S(K)$, there exists $C''_{\epsilon} > 0$ such that

$$|P(z, \xi, w, \eta)| \leq C''_{\epsilon} \cdot e^{\epsilon(|\xi| + |\eta|)}$$
on $\gamma^{-1}(U; d, d)$.

Therefore, the following inequalities hold on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq \epsilon\}$.

$$|P(z, \xi, w, \eta)| \leq C''_{\epsilon} e^{-\frac{1}{3}\delta\min\{|\xi|,|\eta|\} + \frac{1}{3}\delta\min\{|\xi|,|\eta|\} + \epsilon(|\xi| + |\eta|)}$$

$$\leq C''_{\epsilon} e^{-\frac{1}{3}\delta\min\{|\xi|,|\eta|\} + \epsilon(1+\delta)(|\xi| + |\eta|)}.$$

If we put $C_{\epsilon} := \max\{C''_{\epsilon}, C''_{\epsilon}\}$,

$$|P(z, \xi, w, \eta)| \leq C_{\epsilon} e^{-\frac{1}{3}\delta\min\{|\xi|,|\eta|\} + \epsilon(1+\delta)(|\xi| + |\eta|)}$$
on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$.

That is, $P(z, \xi, w, \eta) \in \widehat{R}(K)$.

**Proposition 2.7.** $R(K)$ is an ideal in $S(K)$.

**Proof.** It is clear by the part (C) of the proof of Proposition 2.6.

**Proposition 2.8.** $\widehat{R}(K)$ is an ideal in $\widehat{S}(K)$.
Proof. Let \( \sum P_{j,k}(z, \xi, w, \eta) \in \hat{R}(K) \) and \( \sum Q_{j,k}(z, \xi, w, \eta) \in \hat{S}(K) \).

Then there exist \( d > 0, 0 < A < 1 \), and \( U \supset K \) open in \( S^*X \times S^*Y \) satisfying the following:

For each \( \epsilon > 0 \), we have some \( C_\epsilon > 0 \) such that

\[
\begin{align*}
a) & \quad |P_{s,t}(z, \xi, w, \eta)|, |Q_{s,t}(z, \xi, w, \eta)| \leq C_\epsilon A^{s+t} e^{\epsilon(|\xi|+|\eta|)} \\
b) & \quad \sum_{0 \leq j \leq s, 0 \leq k \leq t} P_{j,k}(z, \xi, w, \eta) \leq C_\epsilon A^{\min\{s,t\}} e^{\epsilon(|\xi|+|\eta|)}
\end{align*}
\]

on \( \gamma^{-1}(U; (s + 1)d, (t + 1)d) \) for each \( s, t \geq 0 \).

It suffices to show that \( \sum R_{j,k} \in \hat{R}(K) \), where

\[
R_{j,k} := \sum_{j_1+j_2=j, k_1+k_2=k} P_{j_1,k_1} Q_{j_2,k_2}.
\]

Since we can easily estimate \( \sum R_{j,k} \) for \( st = 0 \),

we suppose \( s \geq 1 \) and \( t \geq 1 \).

Then we can obtain the following inequality:

\[
\begin{align*}
\left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} R_{j,k} \right| &= \left| \sum_{0 \leq j \leq s} \sum_{0 \leq k \leq t} P_{j_1,k_1} Q_{j_2,k_2} \right| \\
&\leq \left( \sum_{0 \leq j \leq s, 0 \leq k \leq t} P_{j_1,k_1} \right) \left( \sum_{0 \leq j \leq s, 0 \leq k \leq t} Q_{j_2,k_2} \right) + \left| \sum_{s+1 \leq j \leq 2s} \sum_{0 \leq k \leq t} P_{j_1,k_1} Q_{j_2,k_2} \right| \\
&\quad + \left| \sum_{t+1 \leq k \leq 2t} \sum_{j \leq s, 0 \leq k \leq t} P_{j_1,k_1} Q_{j_2,k_2} \right| \\
&\quad + \left| \sum_{0 \leq s, t+1 \leq k \leq 2t} P_{j_1,k_1} Q_{j_2,k_2} \right|
\end{align*}
\]

We shall estimate the four terms in the right side of the inequality, respectively.

The first term \( \leq C_\epsilon A^{\min\{s,t\}} e^{\epsilon(|\xi|+|\eta|)} \cdot \sum_{0 \leq j \leq s, 0 \leq k \leq t} C_\epsilon A^{j+k} e^{\epsilon(|\xi|+|\eta|)} \)

\[
\leq C_\epsilon \cdot C_\epsilon \cdot A^{\min\{s,t\}} e^{2\epsilon(|\xi|+|\eta|)} \cdot \frac{1}{1-A} \cdot \frac{1}{1-A}
\]
on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the 2nd term

$$ \leq \sum_{s+1 \leq j \leq 2s} \sum_{t+1 \leq k} C_e A^{j_1+k_1} e^{c(|\xi|+|\eta|)} C_e A^{j_2+k_2} e^{c(|\xi|+|\eta|)} $$

$$ = C_e \cdot C_e \cdot e^{2c(|\xi|+|\eta|)} \cdot \left( \sum_{s+1 \leq j \leq 2s} \sum_{j_1+j_2=j} A^j \right) \left( \sum_{t+1 \leq k \leq 2t} \sum_{k_1+k_2=k} A^k \right). $$

If we choose any $B$ and $C$ such that $0 < B < 1$, $0 < C < 1$, and $BC \geq A$, we can get the following inequality:

$$ \sum_{s+1 \leq j \leq 2s} \sum_{j_1+j_2=j} A^j \leq C^{s+1}(B^0 + B^1 + B^2 + \cdots)^2 = C^{s+1}(\frac{1}{1-B})^2. $$

Then,

the second term $\leq C_e \cdot C_e \cdot e^{2c(|\xi|+|\eta|)} \cdot C^{s+1}(\frac{1}{1-B})^2 \cdot C^{t+1}(\frac{1}{1-B})^2$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the third term

$$ \leq \sum_{0 \leq j \leq s} \sum_{t+1 \leq k} C_e A^{j_1+k_1} e^{c(|\xi|+|\eta|)} C_e A^{j_2+k_2} e^{c(|\xi|+|\eta|)} $$

$$ = C_e \cdot C_e \cdot e^{2c(|\xi|+|\eta|)} \cdot \left( \sum_{0 \leq j \leq s} \sum_{j_1+j_2=j} A^j \right) \left( \sum_{t+1 \leq k \leq 2t} \sum_{k_1+k_2=k} A^k \right) $$

$$ \leq C_e \cdot C_e \cdot e^{2c(|\xi|+|\eta|)} \cdot \left( \frac{1}{1-A} \right)^2 \cdot C^{t+1}(\frac{1}{1-B})^2 $$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

In like manners,

the fourth term $\leq C_e \cdot C_e \cdot e^{2c(|\xi|+|\eta|)} \cdot C^{s+1} \cdot \left( \frac{1}{1-B} \right)^2 \cdot \left( \frac{1}{1-A} \right)^2$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

Hence, we conclude that $\sum R_{j,k} \in \hat{R}(k)$. 
\( \hat{S}(K)/\hat{R}(K) \) becomes a commutative ring by Proposition 2.8. By Propositions 2.6 and 2.7, the inclusion \( S(K) \hookrightarrow \hat{S}(K) \) induces the injective ring homomorphism

\[ \iota_K : S(K)/R(K) \longrightarrow \hat{S}(K)/\hat{R}(K) \]

Conversely, we obtain the following.

**Theorem 2.9.** If \( \sum P_{j,k}(z, \xi, w, \eta) \in \hat{S}(K) \), there exists \( P(z, \xi, w, \eta) \in S(K) \) such that \( P - \sum P_{j,k} \in \hat{R}(K) \).

Thus, \( S(K)/R(K) \) is isomorphic to \( \hat{S}(K)/\hat{R}(K) \) in the sense of commutative rings.

**Definition 2.10.** We call an element in the ring \( \hat{S}(K)/\hat{R}(K) \) a pseudodifferential operator of the product type on \( K \). We write \( \sum P_{j,k} : \) for the associated pseudodifferential operator of the product type on \( K \) using an element \( \sum P_{j,k} \) in \( \hat{S}(K) \).

The mapping \( \gamma \) is the composition of the following \( \gamma_1 \) and \( \gamma_2 \).

\[
\gamma_1 : (X \times Y) \ni (z, w; \xi, \eta) \longrightarrow \left( z, \frac{\xi}{|\xi|}, w, \frac{\eta}{|\eta|} \right) \in S^*(X \times Y),
\]

\[
\gamma_2 : (X \times Y) \ni (z, \frac{\xi}{|\xi|}, w, \frac{\eta}{|\eta|}) \longrightarrow (z, w; \xi, \eta) \in S^*X \times S^*Y,
\]

where \( S^*(X \times Y) := S^*(X \times Y) \setminus \{(S^*X \times Y) \cup (X \times S^*Y)\} \).

**Proposition 2.11.** If \( P(z, \xi, w, \eta) \) is a symbol of product type on \( K \), \( P \) is a symbol on \( \gamma_1^{-1}(K) \) in the sense of AOKI's symbol.

**Proof.** By the hypothesis, there are \( d > 0 \) and \( U \supset K \) open in \( S^*X \times S^*Y \) satisfying the following:

a) \( P(z, \xi, w, \eta) \) is holomorphic in \( \gamma^{-1}(U; d, d) \), and

b) for each \( \epsilon > 0 \) there is \( C\epsilon > 0 \) such that \(|P(z, \xi, w, \eta)| \leq C\epsilon e^{\epsilon(|\xi| + |\eta|)}\) on \( \gamma^{-1}(U; d, d) \).

Let \( K' \) be compact in \( S^*(X \times Y) \) and \( \gamma_1^{-1}(K) \supset K' \).

Then there exist \( d' > 0 \) and \( U' \supset K' \) open in \( S^*(X \times Y) \) such that

\[
\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.
\]

In fact, for each \( \hat{(z, w; \xi, \eta)} \in \gamma_1^{-1}(K) \) we can choose \( d' > 0 \) such that

\[
d' > \frac{d}{\min\{|\xi|, |\eta|\}}.
\]
Then there exists a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta}) \in \gamma^{-1}_1(K)$ in $\mathcal{O}(X \times Y)$ such that
\[
\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma^{-1}_2(U') \cap \{|\xi| + |\eta| > d'\}.
\]
By the compactness of $K'$, the proof is completed.

**Proposition 2.12.** If $P(z, \xi, w, \eta)$ is a symbol of product type of $0$-class on $K$, that is, $P \in R(K)$, $P$ is a zero symbol on $\gamma^{-1}_1(K)$ in the sense of AOKI’s symbol.

**Proof.** Let $K'$ be compact in $\mathcal{O}(X \times Y)$ and $\gamma^{-1}_1(K) \supset K'$.

It suffices to show that $P$ is a zero symbol on $K'$ in the sense of AOKI’s symbol. By the hypothesis, there exist $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that
\[
|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|})e^{-\delta \min\{|\xi|, |\eta|\}}
\]
on $\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$.

Let $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ be any point of $\gamma^{-1}_1(K)$. By Proposition 2.11, there exist $d' > 0$ and a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ in $\mathcal{O}(X \times Y)$ such that
\[
\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma^{-1}_2(U') \cap \{|\xi| + |\eta| > d'\},
\]
and that there exists $\delta' > 0$ satisfying
\[
\min\{\frac{|\xi|}{|\xi| + |\eta|}, \frac{|\eta|}{|\xi| + |\eta|}\} > \delta' \text{ on } \gamma^{-1}_2(U').
\]
Hence,
\[
|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|})e^{-\delta'(|\xi| + |\eta|)}
\]
on $\gamma^{-1}_2(U') \cap \{|\xi| + |\eta| > d'\}$.

Since $K$ is compact, $P$ is a zero symbol on $\gamma^{-1}_1(K)$ in the sense of AOKI’s symbol.

**Definition 2.13.** The canonical mapping $H_K$ is defined as follows;
\[
S(K)/R(K) \ni P : H_K[P] \in \lim_\to U \supset \gamma^{-1}_1(K)
\]

**Proposition 2.14.** Suppose $K_1$ and $K_2$ are compact in $S^*X \times S^*Y$, respectively, and $K_1 \supset K_2$. Then, $H_{K_1}(P) : \gamma^{-1}_1(K_2) = H_{K_2}(P|_{K_2})$ for all $P \in S(K)/R(K)$. 

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Definition 2.15. We define the product \(*\) of two elements of \(\hat{S}(K)\) as follows:

\[
(\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)) \ast (\sum_{j,k=0}^{\infty} Q_{j,k}(z, \xi, w, \eta)) = \sum_{j,k=0}^{\infty} R_{j,k}(z, \xi, w, \eta),
\]

where

\[
\sum_{j,k=0}^{\infty} t_{1}^{j} t_{2}^{k} R_{j,k}(z, \xi, w, \eta) := e^{t_{1}(\partial_{\xi}, \partial_{z}) + t_{2}(\partial_{\eta}, \partial_{w})} \left( (\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)) \ast (\sum_{j,k=0}^{\infty} Q_{j,k}(z^{*}, \xi^{*}, w^{*}, \eta^{*})) \right) |_{w^{*}=w, \eta^{*} = \eta} z^{*}=z, \xi^{*}=\xi.
\]

That is,

\[
R_{j,k}(z, \xi, w, \eta) := \sum_{j_{1}+j_{2}+|\alpha|=j, k_{1}+k_{2}+|\beta|=k} \frac{1}{\alpha!\beta!} \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} P_{j_{1}k_{1}}(z, \xi, w, \eta) \ast \partial_{z}^{|\alpha|} \partial_{w}^{|\beta|} Q_{j_{2}k_{2}}(z, \xi, w, \eta).
\]

Then we obtain the following.

Lemma 2.16. If \(\sum P_{j,k}\) and \(\sum Q_{j,k}\) are formal symbols of product type on \(K\), then \(\sum R_{j,k}\) is also a formal symbol of product type on \(K\).

Proposition 2.17. If \(\sum P_{j,k} \in \hat{S}(K)\) and \(\sum Q_{j,k} \in \hat{R}(K)\), otherwise \(\sum P_{j,k} \in \hat{R}(K)\) and \(\sum Q_{j,k} \in \hat{S}(K)\), \(\sum R_{j,k}\) is also in \(\hat{R}(K)\).

By Lemma 2.16 and Proposition 2.17, the following composition of two elements in \(\hat{S}(K)/\hat{R}(K)\) is well-defined:

\[
: \sum P_{j,k} \circ : \sum Q_{j,k} := (\sum P_{j,k}) \ast (\sum Q_{j,k}) :
\]

We can easily verify the associativity about the operation \(\circ\). That is, 

\(\hat{S}(K)/\hat{R}(K)\) becomes an associative \(\mathbb{C}\) algebra. Hence the mapping \(H_K\) is a homomorphism about the operation \(\circ\), +, and \(\cdot\), where

\[
\mathcal{E}_{X \times Y}^{R, \text{prod}}(K) \equiv \hat{S}(K)/\hat{R}(K) \xrightarrow{H_K} \mathcal{E}_{X \times Y}^{\mathbb{R}}(\gamma^{-1}(K)).
\]

Definition 2.18. The reverse of \(\sum P_{j,k}\) in \(\hat{S}(K)\) is defined as

\[
(\sum t_{1}^{j} t_{2}^{k} P_{j,k})^{R} := e^{t_{1}(\partial_{\xi}, \partial_{z}) + t_{2}(\partial_{\eta}, \partial_{w})} (\sum t_{1}^{j} t_{2}^{k} P_{j,k}(z, \xi, w, \eta)).
\]

We can verify that if \(\sum P_{j,k}\) is in \(\hat{S}(K)\) \((\hat{R}(K))\) then \((\sum P_{j,k})^{R}\) is in \(\hat{S}(K)\) \((\hat{R}(K))\) , respectively.
3. EXPONENTIAL CALCULUS OF SYMBOLS OF MINIMUM TYPE

Definition 3.1. A function $\Lambda : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following hold;

1. $\Lambda$ is continuous,
2. for each $\alpha > 1$, $\Lambda(\alpha t) \leq \alpha \Lambda(t)$ on $(0, \infty)$,
3. $\Lambda$ is increasing,
4. $\lim_{t \rightarrow \infty} \frac{\Lambda(t)}{t} = 0$.

Definition 3.2. $P(z, \xi, w, \eta) \in S(K)$ is called a symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

1. $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$, and
2. $|P(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$ on $\gamma^{-1}(U; d, d)$.

Example 3.3. (by K. Kataoka)
Let $K$ be any compact subset of $S^*\mathbb{C}_z \times S^*\mathbb{C}_w$ such that $\gamma^{-1}(K) \subset \Omega \times \Omega'$.

$P(z, \xi, w, \eta) := (\xi \eta)^{(1+\sigma)/2}/(\xi + \eta)$,
$\Lambda_1(t) = \Lambda_2(t) := t^\sigma$ with $0 < \sigma < 1$.

Remark 3.4. If $P$ is a symbol of minimum type on $K$, $e^P$ is a symbol of product type on $K$.

Definition 3.5. $\sum P_{j,k}$ in $\hat{S}(K)$ is called a formal symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, $0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

1. $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$,
2. $|P_{j,k}(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\} \cdot A^{j+k}$ on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

Remark 3.6. If $\sum P_{j,k}$ is a formal symbol of minimum type on $K$, $e^{\sum P_{j,k}}$ is a formal symbol of product type on $K$. 
Proposition 3.7. If $P$ and $Q$ are in $S(K)$, then

\begin{equation}
(3.1) \quad P(z, \xi, w, \eta) \ast (Q(z, \xi, w, \eta))^R \quad = e^{t_1(\partial_{\xi}, \partial_{z^*}) + t_2(\partial_{\eta}, \partial_{w^*})} P(\xi, w, \eta) Q(z^*, \xi, w^*, \eta) \bigg|_{z^*=z, w^*=w}.
\end{equation}

Theorem 3.8. If $P$ and $Q$ are symbols of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$, there exists a formal symbol, $\sum R_{j,k}$, of minimum type on $K$ satisfying $e^P \ast e^Q = e^{\sum t_1^j t_2^k R_{j,k}}$.

Proof. $W(s, t; z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) := e^{s(\partial_{\xi}, \partial_{z^*}) + t(\partial_{\eta}, \partial_{w^*})} \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*))$ is the unique formal series solution to the following system of partial differential equations:

\begin{equation}
(3.2) \quad \begin{cases}
\partial_s W = (\partial_{\xi}, \partial_{z^*}) W, \\
\partial_t W = (\partial_{\eta}, \partial_{w^*}) W, \\
W_{s=t=0} = \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*)).
\end{cases}
\end{equation}

If we put $W = \exp(\sum_{j,k}^{\infty} s^j t^k W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*))$, by the above system, we obtain the following recursive formulas about $\{W_{j,k}\}_{j,k \geq 0}$:

\begin{equation}
(3.3) \quad \begin{cases}
W_{0,0} = P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*),
W_{j+1,k} = \frac{1}{j+1} \{ (\partial_{\xi}, \partial_{z^*}) W_{j,k} + \sum_{j_1+j_2=j, k_1+k_2=k} \langle \partial_{\xi} W_{j_1,k_1}, \partial_{z^*} W_{j_2,k_2} \rangle \},
W_{j,k+1} = \frac{1}{k+1} \{ (\partial_{\eta}, \partial_{w^*}) W_{j,k} + \sum_{j_1+j_2=j, k_1+k_2=k} \langle \partial_{\eta} W_{j_1,k_1}, \partial_{w^*} W_{j_2,k_2} \rangle \}.
\end{cases}
\end{equation}

Then $R_{j,k} = W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) \bigg|_{z^*=z, \xi^* = \xi, w^* = w, \eta^* = \eta}$.

Suppose there exist $C_P(=C_Q) > 0$, $d > 0$, and an open subset $U(\supset K)$ of $S^*X \times S^*Y$ satisfying the following:

1. $P$ and $Q$ are holomorphic in $\gamma^{-1}(U; d, d)$,
2. $|P(z, \xi, w, \eta)|$, and $|Q(z, \xi, w, \eta)| \leq C_P \cdot \Lambda(|\xi|, |\eta|)$ on $\gamma^{-1}(U; d, d)$, where $\Lambda(|\xi|, |\eta|) := \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$.

Then we obtain the following lemma.
Lemma 3.9. Suppose \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfy the following conditions:

1. 
\[
C_{j+1,k}^{(\mu,\nu)} \geq \frac{9ne^{10}}{j+1} \left\{ C_{j,k}^{(\mu,\nu)}(j+1)^2 + \sum^* (j_1+1)(j_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)} \right\},
\]

2. 
\[
C_{j,k+1}^{(\mu,\nu)} \geq \frac{9me^{10}}{k+1} \left\{ C_{j,k}^{(\mu,\nu)}(k+1)^2 + \sum^{**} (k_1+1)(k_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)} \right\},
\]

3. 
\[
C_{0,0}^{0,0} \leq C_P + C_Q,
\]

4. 
\[
C_{j,k}^{(\mu,\nu)} \geq 0 \quad (j, k \geq 0, 0 \leq \mu \leq j, 0 \leq \nu \leq k),
\]

5. 
\[
C_{j,k}^{(\mu,\nu)} = 0 \quad (otherwise).
\]

Here, the sum \( \sum^* \), \( \sum^{**} \) mean \( \sum j_1+j_2=j, \sum \prod j_1+j_2=j \), respectively.

Then for each \( \epsilon_1 \) and \( \epsilon_2 \) such that \( 0 < \epsilon_1 \ll 1 \) and \( 0 < \epsilon_2 \ll 1 \), the following hold:

\[
|W_{j,k}| \leq \sum_{0 \leq \mu \leq j, 0 \leq \nu \leq k} \frac{C_{j,k}^{(\mu,\nu)}}{\epsilon_1^{2j}\epsilon_2^{2k}|\xi|^j|\eta|^k}(\widetilde{\Lambda}(|\xi|, |\eta|) + \tilde{\Lambda}(|\xi^*|, |\eta^*|))
\]

\[
\times (\Lambda_1(|\xi|) + \Lambda_1(|\xi^*|))\mu(\Lambda_2(|\eta|) + \Lambda_2(|\eta^*|))\nu
\]
on \( V^{\epsilon_1,\epsilon_2} \) for all \( j, k \geq 0 \).

The following lemma guarantees the existence of \( C_{j,k}^{(\mu,\nu)} \) satisfying the conditions from (1) through (5) of the previous lemma.

Lemma 3.10. The following sequence \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfies the conditions from (1) through (5) of the previous lemma.

\[
C_{j,k}^{(\mu,\nu)} := \begin{cases} 
\lfloor B^{j+k}(j+1)^{j-\mu-3}(k+1)^{k-\nu-3}, & (0 \leq \mu \leq j, 0 \leq \nu \leq k) \\
0, & (otherwise)
\end{cases}
\]

where \( l \) and \( B \) are constants satisfying \( l \geq \max\{C_P + C_Q, 1\} \) and \( B \geq 72l \cdot \max\{m, n\} \cdot e^{10} \cdot (c^2 + 1) \) and \( c \) is a constant satisfying the following T.Aoki's inequality:

\[
\sum_{\mu=0}^{\nu-1} \sum_{k=\mu}^{j-\nu+\mu+1} (k+1)^{k-\mu-2}(j-k+1)^{j-k-\nu+\mu-1} \leq c(j+1)^{j-\nu-2}.
\]
for all $j, \nu$ such that $0 \leq \nu - 1 \leq j$.

(continued) We can prove the theorem using the above two lemmas.

**References**


