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<th>Title</th>
<th>ON EXPONENTIAL CALCULUS OF SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS OF MINIMUM TYPE (Integral representations and twisted cohomology in the theory of differential equations)</th>
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Kyoto University
ON EXPONENTIAL CALCULUS OF SYMBOLS OF
PSEUDODIFFERENTIAL OPERATORS OF MINIMUM
TYPE

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1. INTRODUCTION

Let $X$ and $Y$ be $n$- and $m$- dimensional complex manifolds, respectively.

$$S^*X := (T^*X - X)/\mathbb{R}^+, \ S^*Y := (T^*Y - Y)/\mathbb{R}^+.$$  

We define the mapping $\gamma$ as

$$\gamma: T^\infty(X \times Y) \ni (z, w; \xi, \eta) \mapsto (z; \frac{\xi}{|\xi|}) \times (w; \frac{\eta}{|\eta|}) \in S^*X \times S^*Y,$$

where

$$T^\infty(X \times Y) := T^*(X \times Y) \setminus \{(T^*X \times Y) \cup (X \times T^*Y)\}.$$  

For $d > 0$ and an open subset $U$ of $S^*X \times S^*Y$ we denote

$$\gamma^{-1}(U) \cap \{ |\xi| > d, |\eta| > d \}$$

by $\gamma^{-1}(U; d, d)$.

Hereafter we write $(z, \xi, w, \eta)$ for coordinates $(z, w; \xi, \eta)$.

2. SYMBOLS OF PRODUCT TYPE

Let $K$ be a compact subset of $S^*X \times S^*Y$.

**Definition 2.1.** $P(z, \xi, w, \eta)$ is said to be a symbol of product type on $K$ if the following hold:

(1) There are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ such that $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$.

(2) For each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{\varepsilon(|\xi| + |\eta|)} \quad \text{on} \ \gamma^{-1}(U; d, d).$$
We denote by $S(K)$ the set of all such symbols on $K$. $S(K)$ becomes a commutative ring with the usual sum and product.

**Definition 2.2.** We denote by $R(K)$ the set of all $P(z, \xi, w, \eta) \in S(K)$ satisfying the following;
there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(|\xi|/|\eta|)e^{-\delta \min\{|\xi|,|\eta|\}}$$
on $\gamma^{-1}(U; d, d)$.

We call an element of $R(K)$ a symbol of $0$-class.

**Definition 2.3.** A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$ is called a formal symbol of product type on $K$ if the following hold:
1. There are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.
2. For each $\epsilon > 0$, there is $C_{\epsilon} > 0$ such that

$$|P_{j,k}(z, \xi, w, \eta)| \leq C_{\epsilon}A^{j+k}e^{\epsilon(|\xi|+|\eta|)}$$
on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

We denote by $\hat{S}(K)$ the set of such formal symbols on $K$. We often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$, in indeterminants $t_1$ and $t_2$ for $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$.

We can easily obtain the following.

**Proposition 2.4.** $\hat{S}(K)$ becomes a commutative ring with the sum and the product as formal power series in $t_1$ and $t_2$.

$S(K)$ is identified with a subring of $\hat{S}(K)$ as follows:

$$S(K) \simeq \hat{S}(K)|_{t_1=0, t_2=0} = \{P = \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}; P_{j,k} \equiv 0 \text{ for all } (j,k) \neq (0,0)\}.$$

**Definition 2.5.** We denote by $\hat{R}(K)$ the set of all $P(t_1, t_2; z, \xi, w, \eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$ in $\hat{S}(K)$ such that there are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:
for each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left| \sum_{0 \leq j \leq s \atop 0 \leq k \leq t} P_{j,k}(z, \xi, w, \eta) \right| \leq C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)}$$
onumber

on $\gamma^{-1}(U;(s+1)d, (t+1)d)$ for each $s, t \geq 0$.

We call an element of $\hat{R}(K)$ a formal symbol of zero class.

**Proposition 2.6.** Under the previous identification, $S(K) \cap \hat{R}(K) = R(K)$ holds.

**Proof.** Let $P(z, \xi, w, \eta)$ be in $S(K)$. Then $P(z, \xi, w, \eta) \in \hat{R}(K)$ is equivalent to the following; there exist $d > 0, \delta > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ such that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{|\xi|,|\eta|\} + \varepsilon(|\xi|+|\eta|)}$$
onumber

on $\gamma^{-1}(U;d,d)$.

$(\subset)$ Using the fact that $(0, \infty) = \{t := \frac{|\xi|}{|\eta|}; (z, \xi, w, \eta) \in \gamma^{-1}(U;d,d)\}$, by the hypothesis, we obtain the following;

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta \min\{1, \frac{1}{t}\}|\xi| + \varepsilon(1+\frac{1}{t})|\xi|}$$

for all $t := \frac{|\xi|}{|\eta|} \in (0, \infty)$ and $(z, \xi, w, \eta) \in \gamma^{-1}(U;d, d)$. We fix any $\varepsilon > 0$ such that $0 < \varepsilon < 1$ and $\varepsilon \leq \frac{\delta}{3}$. Then for every $t \in [\frac{3}{5}\varepsilon, 1]$

$$|P(z, \xi, w, \eta)| \leq C_\varepsilon e^{-\delta |\xi| + \varepsilon(1+\frac{1}{t})|\xi|} \leq C_\varepsilon e^{-\delta |\xi| + \varepsilon(1+\frac{1}{\varepsilon})|\xi|}$$

$$= C_\varepsilon e^{(\varepsilon - \frac{2}{3}\delta)|\xi|} \leq C_\varepsilon e^{-\frac{1}{3}\delta|\xi|}.$$ 

On the other hand, for any sequence $\varepsilon_n$ such that $\min\{1, \frac{\delta}{3}\} > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$, we define a function $C(\cdot)$ on $(0, 1]$ as

$$C(t) := \begin{cases} C_{\frac{2}{3}\varepsilon_1}, & \varepsilon_1 < t \leq 1, \\ C_{\frac{2}{3}\varepsilon_{n+1}}, & \varepsilon_{n+1} < t \leq \varepsilon_n. \end{cases}$$

Then $C(\cdot)$ is locally bounded on $(0, 1]$ and

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right)e^{-\frac{1}{3}\delta|\xi|}$$
on $\gamma^{-1}(U; d, d) \cap \{\xi \leq \eta\}$.
In like manners,

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{\eta}{\xi}\right)e^{-\frac{1}{3}\delta|\eta|}$$
on $\gamma^{-1}(U; d, d) \cap \{\xi \geq \eta\}$.
Here, we define a function $C'(\cdot)$ on $(0, \infty)$ as $C'(t) = C(\min\{t, \frac{1}{t}\})$.
Then $C'(t)$ is locally bounded on $(0, \infty)$ and

$$|P(z, \xi, w, \eta)| \leq C'\left(\frac{\xi}{\eta}\right)e^{-\frac{1}{3}\delta\min\{\xi, \eta\}}$$
on $\gamma^{-1}(U; d, d)$.
That is, $P(z, \xi, w, \eta) \in R(K)$.

(\Rightarrow) Let

Then there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(\frac{\eta}{\xi})e^{-\delta\min\{\xi, \eta\}}$$
on $\gamma^{-1}(U; d, d)$. We fix any $\varepsilon$ such that $0 < \varepsilon < 1$. Then,

$$|P(z, \xi, w, \eta)| \leq \max_{\varepsilon \leq t \leq 1} C(t) \cdot e^{-\delta\min\{\xi, \eta\}}$$
on $\gamma^{-1}(U; d, d) \cap \{\varepsilon \leq \frac{\xi}{\eta} =: t \leq 1\}$. We put $C'_{\varepsilon} := \max_{\varepsilon \leq t \leq 1} C(t)$.
On the other hand, since $P(z, \xi, w, \eta) \in \hat{S}(K)$, there exists $C''_{\varepsilon} > 0$

such that

$$|P(z, \xi, w, \eta)| \leq C''_{\varepsilon}e^{\varepsilon(1+\delta)(|\xi|+|\eta|)}$$
on $\gamma^{-1}(U; d, d)$.
Therefore, the following inequalities hold on $\gamma^{-1}(U; d, d) \cap \{\xi \leq \eta\} \leq \varepsilon$.

$$|P(z, \xi, w, \eta)| \leq C''_{\varepsilon}e^{-\delta(1+\delta)(|\xi|+|\eta|)}$$

If we put $C_{\varepsilon} := \max\{C'_{\varepsilon}, C''_{\varepsilon}\},$

$$|P(z, \xi, w, \eta)| \leq C_{\varepsilon}e^{-\delta(1+\delta)(|\xi|+|\eta|)}$$
on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$.
That is, $P(z, \xi, w, \eta) \in \hat{R}(K)$.

**Proposition 2.7.** $R(K)$ is an ideal in $S(K)$.

**Proof.** It is clear by the part ($\subset$) of the proof of Proposition 2.6.

**Proposition 2.8.** $\hat{R}(K)$ is an ideal in $\hat{S}(K)$. 
Proof. Let \( \sum P_{j,k}(z,\xi,w,\eta) \in \hat{R}(K) \) and \( \sum Q_{j,k}(z,\xi,w,\eta) \in \hat{S}(K) \). Then there exist \( d > 0, 0 < A < 1 \), and \( U \supset K \) open in \( S^*X \times S^*Y \) satisfying the following:

For each \( \epsilon > 0 \), we have some \( C_\epsilon > 0 \) such that

a) \[ |P_{s,t}(z,\xi,w,\eta)|, |Q_{s,t}(z,\xi,w,\eta)| \leq C_\epsilon A^{s+t}e^{\epsilon(|\xi|+|\eta|)} \]

b) \[ \sum_{0 \leq j \leq s, 0 \leq k \leq t} P_{j,k}(z,\xi,w,\eta) \leq C_\epsilon A^{\min\{s,t\}}e^{\epsilon(|\xi|+|\eta|)} \]

on \( \gamma^{-1}(U;(s+1)d, (t+1)d) \) for each \( s, t \geq 0 \).

It suffices to show that \( \sum R_{j,k} \in \hat{R}(K) \), where

\[ R_{j,k} := \sum_{j_1+j_2 = j, k_1+k_2 = k} P_{j_1,k_1}Q_{j_2,k_2}. \]

Since we can easily estimate \( \sum R_{j,k} \) for \( st = 0 \), we suppose \( s \geq 1 \) and \( t \geq 1 \).

Then we can obtain the following inequality:

\[ \left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} R_{j,k} \right| = \left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} \sum_{j_1+j_2 = j, k_1+k_2 = k} P_{j_1,k_1}Q_{j_2,k_2} \right| \leq \left( \sum_{0 \leq j_1 \leq s, 0 \leq k_1 \leq t} P_{j_1,k_1} \right) \left( \sum_{0 \leq j_2 \leq s, 0 \leq k_2 \leq t} Q_{j_2,k_2} \right) + \sum_{s+1 \leq j \leq 2s, 0 \leq k \leq t} P_{j,k}Q_{j,k} \]

\[ + \sum_{0 \leq j \leq s, t+1 \leq k \leq 2t} P_{j,k}Q_{j,k} + \sum_{0 \leq j \leq s, 0 \leq k \leq t} P_{j,k}Q_{j,k}. \]

We shall estimate the four terms in the right side of the inequality, respectively.

the first term \( \leq C_\epsilon A^{\min\{s,t\}}e^{\epsilon(|\xi|+|\eta|)} \cdot \sum_{0 \leq j \leq s, 0 \leq k \leq t} C_\epsilon A^{j+k}e^{\epsilon(|\xi|+|\eta|)} \]

\[ \leq C_\epsilon \cdot C_\epsilon \cdot A^{\min\{s,t\}} e^{2\epsilon(|\xi|+|\eta|)} \cdot \frac{1}{1-A} \cdot \frac{1}{1-A} \]
on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the 2nd term

$$\leq \sum_{s+1\leq j \leq 2s} \sum_{j_1+j_2=j} C_e A^{j_1+k_1} e^{\epsilon(|\xi|+|\eta|)} \cdot C_e A^{j_2+k_2} e^{\epsilon(|\xi|+|\eta|)}$$

$$= C_e \cdot C_e \cdot e^{2\epsilon(|\xi|+|\eta|)} \cdot (\sum_{s+1\leq j \leq 2s} \sum_{j_1+j_2=j} A^j)(\sum_{t+1\leq k \leq 2t} \sum_{k_1+k_2=k} A^k).$$

If we choose any $B$ and $C$ such that $0 < B < 1$, $0 < C < 1$, and $BC \geq A$, we can get the following inequality:

$$\sum_{s+1\leq j \leq 2s} \sum_{j_1+j_2=j} A^j \leq C^{s+1}(B^0 + B^1 + B^2 + \cdots)^2 = C^{s+1} \left(\frac{1}{1-B}\right)^2.$$

Then,

the second term $\leq C_e \cdot C_e \cdot e^{2\epsilon(|\xi|+|\eta|)} \cdot C^{s+1} \left(\frac{1}{1-B}\right)^2 \cdot C^{t+1} \left(\frac{1}{1-B}\right)^2$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

the third term

$$\leq \sum_{0\leq j \leq s} \sum_{j_1+j_2=j} C_e A^{j_1+k_1} e^{\epsilon(|\xi|+|\eta|)} \cdot C_e A^{j_2+k_2} e^{\epsilon(|\xi|+|\eta|)}$$

$$= C_e \cdot C_e \cdot e^{2\epsilon(|\xi|+|\eta|)} \cdot (\sum_{0\leq j \leq s} \sum_{j_1+j_2=j} A^j)(\sum_{t+1\leq k \leq 2t} \sum_{k_1+k_2=k} A^k)$$

$$\leq C_e \cdot C_e \cdot e^{2\epsilon(|\xi|+|\eta|)} \cdot \left(\frac{1}{1-A}\right)^2 \cdot C^{t+1} \left(\frac{1}{1-B}\right)^2$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

In like manners,

the fourth term $\leq C_e \cdot C_e \cdot e^{2\epsilon(|\xi|+|\eta|)} \cdot C^{s+1} \cdot (\frac{1}{1-A})^2 \cdot (\frac{1}{1-B})^2$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 1$.

Hence, we conclude that $\sum R_{j,k} \in \hat{R}(k)$. 
\( \hat{S}(K)/\hat{R}(K) \) becomes a commutative ring by Proposition 2.8. By Propositions 2.6 and 2.7, the inclusion \( S(K) \to \hat{S}(K) \) induces the injective ring homomorphism

\[ \iota_K : S(K)/R(K) \to \hat{S}(K)/\hat{R}(K). \]

Conversely, we obtain the following.

**Theorem 2.9.** If \( \sum P_{j,k}(z, \xi, w, \eta) \in \hat{S}(K) \), there exists \( P(z, \xi, w, \eta) \in S(K) \) such that \( P - \sum P_{j,k} \in \hat{R}(K) \).

Thus, \( S(K)/R(K) \) is isomorphic to \( \hat{S}(K)/\hat{R}(K) \) in the sense of commutative rings.

**Definition 2.10.** We call an element in the ring \( \hat{S}(K)/\hat{R}(K) \) a pseudo-differential operator of the product type on \( K \). We write : \( \sum P_{j,k} : \) for the associated pseudo-differential operator of the product type on \( K \) using an element \( \sum P_{j,k} \) in \( \hat{S}(K) \).

The mapping \( \gamma \) is the composition of the following \( \gamma_1 \) and \( \gamma_2 \).

\[ \gamma : \tilde{S}^*_{\circ}(X \times Y) \ni (z, w; \xi, \eta) \mapsto (z, w; \frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}) \in \tilde{S}^*(X \times Y), \]

\[ \gamma : \tilde{S}^*_{\circ}(X \times Y) \ni (z, w; \frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}) \mapsto (z, \frac{\xi}{|\xi|}) \times (w, \frac{\eta}{|\eta|}) \in S^*X \times S^*Y, \]

where \( \tilde{S}^*_\circ(X \times Y) := S^*(X \times Y) \backslash \{(S^*X \times Y) \cup (X \times S^*Y)\} \).

**Proposition 2.11.** If \( P(z, \xi, w, \eta) \) is a symbol of product type on \( K \), \( P \) is a symbol on \( \gamma_1^{-1}(K) \) in the sense of AOKI's symbol.

**Proof.** By the hypothesis, there are \( d > 0 \) and \( U \supset K \) open in \( S^*X \times S^*Y \) satisfying the following:

a) \( P(z, \xi, w, \eta) \) is holomorphic in \( \gamma^{-1}(U; d, d) \), and

b) for each \( \epsilon > 0 \) there is \( C_\epsilon > 0 \) such that \( |P(z, \xi, w, \eta)| \leq C_\epsilon e^{\epsilon(|\xi| + |\eta|)} \) on \( \gamma^{-1}(U; d, d) \).

Let \( K' \) be compact in \( \tilde{S}^*_{\circ}(X \times Y) \) and \( \gamma_1^{-1}(K) \supset K' \).

Then there exist \( d' > 0 \) and \( U' \supset K' \) open in \( \tilde{S}^*_{\circ}(X \times Y) \) such that

\[ \gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}. \]

In fact, for each \( (\check{z}, \check{w}; \check{\xi}, \check{\eta}) \in \gamma_1^{-1}(K) \) we can choose \( d' > 0 \) such that

\[ d' > \frac{d}{\min\{|\check{\xi}|, |\check{\eta}|\}}. \]
Then there exists a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta}) \in \gamma_1^{-1}(K)$ in $S^*(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.$$ 

By the compactness of $K'$, the proof is completed.

**Proposition 2.12.** If $P(z, \xi, w, \eta)$ is a symbol of product type of 0-class on $K$, that is, $P \in R(K)$, $P$ is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.

**Proof.** Let $K'$ be compact in $S^*(X \times Y)$ and $\gamma_1^{-1}(K) \supset K'$.

It suffices to show that $P$ is a zero symbol on $K'$ in the sense of AOKI's symbol. By the hypothesis, there exist $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|})e^{-\delta\min\{|\xi|, |\eta|\}}$$

on $\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$.

Let $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ be any point of $\gamma_1^{-1}(K)$. By Proposition 2.11, there exist $d' > 0$ and a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ in $S^*(X \times Y)$ such that

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\},$$

and that there exists $\delta' > 0$ satisfying

$$\min\{\frac{|\xi|}{|\xi| + |\eta|}, \frac{|\eta|}{|\xi| + |\eta|}\} > \delta'$$

on $\gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}$. Hence,

$$|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|})e^{-\delta'(|\xi| + |\eta|)}$$

on $\gamma_2^{-1}(U') \cap \{||\xi| + |\eta| > d'\}$.

Since $K$ is compact, $P$ is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.

**Definition 2.13.** The canonical mapping $H_K$ is defined as follows;

$$S(K)/R(K) \ni P : \xrightarrow{H_K} [P] \in \lim_{U \supset \gamma_1^{-1}(K)} \mathcal{E}^R(U).$$

**Proposition 2.14.** Suppose $K_1$ and $K_2$ are compact in $S^*X \times S^*Y$, respectively, and $K_1 \supset K_2$. Then, $H_{K_1}(\cdot) |_{\gamma_1^{-1}(K_2)} = H_{K_2}(\cdot)|_{K_2 :}$ for all $P \in S(K)/R(K)$. 

Definition 2.15. We define the product $*$ of two elements of $\hat{S}(K)$ as follows:

\[
\left(\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)\right) * \left(\sum_{j,k=0}^{\infty} Q_{j,k}(z, \xi, w, \eta)\right) = \sum_{j,k=0}^{\infty} R_{j,k}(z, \xi, w, \eta),
\]

where

\[
\sum_{j,k=0}^{\infty} t_1^j t_2^k R_{j,k}(z, \xi, w, \eta) := e^{t_1 \langle \partial_{\xi}, \partial_z \rangle + t_2 \langle \partial_{\eta}, \partial_w \rangle} \left(\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)\right) \times \left(\sum_{j,k=0}^{\infty} Q_{j,k}(z^*, \xi^*, w^*, \eta^*)\right) \bigg|_{w^* = w, \eta^* = \eta} z^* = z, \xi^* = \xi.
\]

That is,

\[
R_{j,k}(z, \xi, w, \eta) := \sum_{j_1+j_2+|\alpha|=j, k_1+k_2+|\beta|=k} \frac{1}{\alpha!\beta!} \partial_\xi^\alpha \partial_\eta^\beta P_{j_1,k_1}(z, \xi, w, \eta) \times \partial_z^\alpha \partial_w^\beta Q_{j_2,k_2}(z, \xi, w, \eta).
\]

Then we obtain the following.

Lemma 2.16. If $\sum P_{j,k}$ and $\sum Q_{j,k}$ are formal symbols of product type on $K$, then $\sum R_{j,k}$ is also a formal symbol of product type on $K$.

Proposition 2.17. If $\sum P_{j,k} \in \hat{S}(K)$ and $\sum Q_{j,k} \in \hat{R}(K)$, otherwise $\sum P_{j,k} \in \hat{R}(K)$ and $\sum Q_{j,k} \in \hat{S}(K)$, $\sum R_{j,k}$ is also in $\hat{R}(K)$.

By Lemma 2.16 and Proposition 2.17, the following composition of two elements in $\hat{S}(K)/\hat{R}(K)$ is well-defined;

\[
: \sum P_{j,k} : \circ : \sum Q_{j,k} ::= (\sum P_{j,k}) * (\sum Q_{j,k}).
\]

We can easily verify the associativity about the operation $\circ$. That is, $\hat{S}(K)/\hat{R}(K)$ becomes an associative $\mathbb{C}$ algebra. Hence the mapping $H_K$ is a homomorphism about the operation $\circ$, $+$, and $\cdot$, where

\[
\mathcal{E}^{\text{prod}}_{X \times Y}(K) \equiv \hat{S}(K)/\hat{R}(K) \xrightarrow{H_K} \mathcal{E}^R_{X \times Y}(\gamma^{-1}(K)).
\]

Definition 2.18. The reverse of $\sum P_{j,k}$ in $\hat{S}(K)$ is defined as

\[
(\sum t_1^j t_2^k P_{j,k})^R := e^{t_1 \langle \partial_{\xi}, \partial_z \rangle + t_2 \langle \partial_{\eta}, \partial_w \rangle} \left(\sum t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)\right).
\]

We can verify that if $\sum P_{j,k}$ is in $\hat{S}(K)$ ($\hat{R}(K)$), then $(\sum P_{j,k})^R$ is in $\hat{S}(K)$ ($\hat{R}(K)$), respectively.
3. EXPONENTIAL CALCULUS OF SYMBOLS OF MINIMUM TYPE

**Definition 3.1.** A function $\Lambda : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following hold;

1. $\Lambda$ is continuous,
2. for each $\alpha > 1$, $\Lambda(\alpha t) \leq \alpha \Lambda(t)$ on $(0, \infty)$,
3. $\Lambda$ is increasing,
4. $\lim_{t \to \infty} \frac{\Lambda(t)}{t} = 0$.

**Definition 3.2.** $P(z, \xi, w, \eta) \in S(K)$ is called a symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

1. $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$, and
2. $|P(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$ on $\gamma^{-1}(U; d, d)$.

**Example 3.3.** (by K. Kataoka)

$\Omega = \Omega' := \mathbb{C} \times \{\xi \in \mathbb{C}; |\arg \xi| < \delta, \xi \neq 0\} (0 < \delta < \frac{\pi}{2})$.

Let $K$ be any compact subset of $S^*\mathbb{C}_z \times S^*\mathbb{C}_w$ such that $\gamma^{-1}(K) \subset \Omega \times \Omega'$.

$P(z, \xi, w, \eta) := (\xi \eta)^{(1+\sigma)/2}/(\xi + \eta)$,

$\Lambda_1(t) = \Lambda_2(t) := t^\sigma$ with $0 < \sigma < 1$.

**Remark 3.4.** If $P$ is a symbol of minimum type on $K$, $e^P$ is a symbol of product type on $K$.

**Definition 3.5.** $\sum P_{j,k}$ in $\hat{S}(K)$ is called a formal symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, $0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;

1. $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$,
2. $|P_{j,k}(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\} \cdot A^{j+k}$ on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

**Remark 3.6.** If $\sum P_{j,k}$ is a formal symbol of minimum type on $K$, $e^{\sum P_{j,k}}$ is a formal symbol of product type on $K$. 
Proposition 3.7. If $P$ and $Q$ are in $S(K)$, then

$$P(z, \xi, w, \eta) \ast (Q(z, \xi, w, \eta))^R = e^{t_1(\partial_{\xi}, \partial_{z}^*) + t_2(\partial_{\eta}, \partial_{w}^*)} P(z, \xi, w, \eta)Q(z, \xi, w, \eta) \bigg|_{z^{*}=z}.$$  

Theorem 3.8. If $P$ and $Q$ are symbols of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$, there exists a formal symbol, $\sum R_{j,k}$, of minimum type on $K$ satisfying $e^P \ast e^Q = e^{\sum t_1^j t_2^k R_{j,k}}$.

Proof. $W(s, t; z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) := e^{s(\partial_{\xi}, \partial_{z}^*) + t(\partial_{\eta}, \partial_{w}^*)} \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*))$ is the unique formal series solution to the following system of partial differential equations:

$$\partial_s W = (\partial_{\xi}, \partial_{z}^*) W, \quad \partial_t W = (\partial_{\eta}, \partial_{w}^*) W,$$

$$W_{s=t=0} = \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*)).$$  

If we put $W = \exp(\sum_{j,k}^\infty s^{j}t^{k}W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*))$, by the above system, we obtain the following recursive formulas about $\{W_{j,k}\}_{j,k \geq 0}$:

$$W_{j+1,k}=\frac{1}{j+1}\{(\partial_{\xi}, \partial_{z}^*)W_{j,k} + \sum_{j_1+j_2=j, k_1+k_2=k} \langle \partial_{\xi}W_{j_1,k_1}, \partial_{z}^*W_{j_2,k_2} \rangle \}.$$

Then $R_{j,k} = W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) \bigg|_{z^{*}=z} = \xi^{*} = \eta^{*}$.  

Suppose there exist $C_P (= C_Q) > 0$, $d > 0$, and an open subset $U \supset K$ of $S^*X \times S^*Y$ satisfying the following:

1. $P$ and $Q$ are holomorphic in $\gamma^{-1}(U; d, d)$,
2. $|P(z, \xi, w, \eta)|$, and $|Q(z, \xi, w, \eta)| \leq C_P \cdot \Lambda(|\xi|, |\eta|)$ on $\gamma^{-1}(U; d, d)$,

where $\Lambda(|\xi|, |\eta|) := \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$.  

Then we obtain the following lemma.
Lemma 3.9. Suppose \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfy the following conditions:

1. \[ C_{j+1,k}^{(\mu,\nu)} \geq \frac{9ne^{10}}{j+1} \{ C_{j,k}^{(\mu,\nu)} (j+1)^2 + \sum (j_1+1)(j_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)} \}, \]

2. \[ C_{j,k+1}^{(\mu,\nu)} \geq \frac{9me^{10}}{k+1} \{ C_{j,k}^{(\mu,\nu)} (k+1)^2 + \sum (k_1+1)(k_2+1)C_{j_1,k_1}^{(\mu_1,\nu_1)}C_{j_2,k_2}^{(\mu_2,\nu_2)} \}, \]

3. \[ C_{0,0}^{0,0} \leq C_P + C_Q \]

4. \[ C_{j,k}^{(\mu,\nu)} \geq 0 \quad (j, k \geq 0, 0 \leq \mu \leq j, 0 \leq \nu \leq k) \]

5. \[ C_{j,k}^{(\mu,\nu)} = 0 \quad (\text{otherwise}). \]

Here, the sum \( \sum^* \), \( \sum^{**} \) mean \( \sum j_1 + j_2 = j, \sum k_1 + k_2 = k \), respectively.

Then for each \( \varepsilon_1 \) and \( \varepsilon_2 \) such that \( 0 < \varepsilon_1 \ll 1 \) and \( 0 < \varepsilon_2 \ll 1 \), the following hold:

\[ |W_{j,k}| \leq \sum_{0 \leq \mu \leq j, 0 \leq \nu \leq k} \frac{C_{j,k}^{(\mu,\nu)}}{\varepsilon_1^{2j} \varepsilon_2^{2k} |\xi|^j |\eta|^k (\tilde{\Lambda}(|\xi|, |\eta|) + \tilde{\Lambda}(|\xi^*|, |\eta^*|))} \]

\[ \times (\Lambda_1(|\xi|) + \Lambda_1(|\xi^*|))\mu (\Lambda_2(|\eta|) + \Lambda_2(|\eta^*|))\nu \]

on \( V^{\varepsilon_1,\varepsilon_2} \) for all \( j, k \geq 0 \).

The following lemma guarantees the existence of \( C_{j,k}^{(\mu,\nu)} \) satisfying the conditions from (1) through (5) of the previous lemma.

Lemma 3.10. The following sequence \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfies the conditions from (1) through (5) of the previous lemma.

\[ C_{j,k}^{(\mu,\nu)} := \begin{cases} 
1B^{j+k}(j+1)^{-\mu-3}(k+1)^{-\nu-3}, & (0 \leq \mu \leq j, \ 0 \leq \nu \leq k), \\
0, & (\text{otherwise})
\end{cases} \]

where \( l \) and \( B \) are constants satisfying \( l \geq \max\{C_P + C_Q, 1\} \) and \( B \geq 72l \cdot \max\{m, n\} \cdot e^{10} \cdot (c^2 + 1) \) and \( c \) is a constant satisfying the following \( T.Aoki's \) inequality:

\[ \frac{1}{j+1} \sum_{\mu=0}^{\nu-1} \sum_{k=\mu}^{(k+1)^{k-\nu-2}(j-k+1)^{j-k-\nu+1}} \leq c(j+1)^{j-\nu-2} \]
for all $j, \nu$ such that $0 \leq \nu - 1 \leq j$.

(continued) We can prove the theorem using the above two lemmas.

REFERENCES


