ON EXPONENTIAL CALCULUS OF SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS OF MINIMUM TYPE (Integral representations and twisted cohomology in the theory of differential equations)

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Citation
数理解析研究所講究録 1212: 144-156

Issue Date
2001-06

URL
http://hdl.handle.net/2433/41154

Type
Departmental Bulletin Paper
ON EXPONENTIAL CALCULUS OF SYMBOLS OF PSEUDODIFFERENTIAL OPERATORS OF MINIMUM TYPE

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1. INTRODUCTION

Let $X$ and $Y$ be $n$- and $m$- dimensional complex manifolds, respectively.

$$S^*X := (T^*X - X)/\mathbb{R}^+, S^*Y := (T^*Y - Y)/\mathbb{R}^+. $$

We define the mapping $\gamma$ as

$$\gamma : \mathring{T}^*(X \times Y) \ni (z, w; \xi, \eta) \mapsto (z; \frac{\xi}{|\xi|}) \times (w; \frac{\eta}{|\eta|}) \in S^*X \times S^*Y, $$

where

$$\mathring{T}^*(X \times Y) := T^*(X \times Y) \setminus \{T^*(X \times Y) \cup (X \times T^*Y)\}. $$

For $d > 0$ and an open subset $U$ of $S^*X \times S^*Y$ we denote

$$\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} $$

by $\gamma^{-1}(U; d, d)$.

Hereafter we write $(z, \xi, w, \eta)$ for coordinates $(z, w; \xi, \eta)$.

2. SYMBOLS OF PRODUCT TYPE

Let $K$ be a compact subset of $S^*X \times S^*Y$.

**Definition 2.1.** $P(z, \xi, w, \eta)$ is said to be a symbol of product type on $K$ if the following hold:

1. There are $d > 0$ and $U \supset K$ open in $S^*X \times S^*Y$ such that $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$.

2. For each $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$(2.1) \quad |P(z, \xi, w, \eta)| \leq C_\epsilon e^{\epsilon(|\xi| + |\eta|)} \quad \text{on} \quad \gamma^{-1}(U; d, d). $$
We denote by $S(K)$ the set of all such symbols on $K$. $S(K)$ becomes a commutative ring with the usual sum and product.

**Definition 2.2.** We denote by $R(K)$ the set of all $P(z, \xi, w, \eta) \in S(K)$ satisfying the following; there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(|\xi|/|\eta|)e^{-\delta \min\{|\xi|,|\eta|\}}$$
on $\gamma^{-1}(U; d, d)$.

We call an element of $R(K)$ a symbol of 0-class.

**Definition 2.3.** A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$ is called a formal symbol of product type on $K$ if the following hold:

1. There are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

2. For each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$|P_{j,k}(z, \xi, w, \eta)| \leq C_\epsilon A^{j+k}e^{\epsilon(|\xi|+|\eta|)}$$
on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

We denote by $\hat{S}(K)$ the set of such formal symbols on $K$.

We often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$, in indeterminants $t_1$ and $t_2$ for $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$.

We can easily obtain the following.

**Proposition 2.4.** $\hat{S}(K)$ becomes a commutative ring with the sum and the product as formal power series in $t_1$ and $t_2$.

$S(K)$ is identified with a subring of $\hat{S}(K)$ as follows:

$$S(K) \simeq \hat{S}(K)|_{t_1=0} = \{ P = \sum_{j,k} t_1^j t_2^k P_{j,k}; P_{j,k} \equiv 0 \text{ for all } (j, k) \neq (0, 0) \}.$$

**Definition 2.5.** We denote by $\hat{R}(K)$ the set of all $P(t_1, t_2; z, \xi, w, \eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$ in $\hat{S}(K)$ such that there are $d > 0, 0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following;
for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\left| \sum_{0 \leq j, k \leq s \atop 0 \leq k \leq t} P_{j,k}(z, \xi, w, \eta) \right| \leq C_\epsilon A^{\min\{s,t\}} e^{\epsilon (|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; (s+1)d, (t+1)d)$ for each $s, t \geq 0$.

We call an element of $\hat{R}(K)$ a formal symbol of zero class.

**Proposition 2.6.** Under the previous identification, $S(K) \cap \hat{R}(K) = R(K)$ holds.

**Proof.** Let $P(z, \xi, w, \eta)$ be in $S(K)$. Then $P(z, \xi, w, \eta) \in \hat{R}(K)$ is equivalent to the following:

there exist $d > 0, \delta > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ such that for each $\epsilon > 0$ there is $C_\epsilon > 0$ satisfying

$$|P(z, \xi, w, \eta)| \leq C_\epsilon e^{-\delta \min\{|\xi|, |\eta|\} + \epsilon (|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; d, d)$. 

($\subset$) Using the fact that $(0, \infty) = \{t := \frac{|\xi|}{|\eta|}; (z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)\}$, by the hypothesis, we obtain the following;

$$|P(z, \xi, w, \eta)| \leq C e^{-\delta \min\{1, \frac{1}{t}\} |\xi| + \epsilon (1 + \frac{1}{t}) |\xi|}$$

for all $t := \frac{|\xi|}{|\eta|} \in (0, \infty)$ and $(z, \xi, w, \eta) \in \gamma^{-1}(U; d, d)$.

We fix any $\epsilon > 0$ such that $0 < \epsilon < 1$ and $\epsilon \leq \frac{\delta}{3}$.

Then for every $t \in [\frac{\delta}{3} \epsilon, 1]$

$$|P(z, \xi, w, \eta)| \leq C e^{-\delta \min\{1, \frac{1}{t}\} |\xi| + \epsilon (1 + \frac{1}{t}) |\xi|} \leq C e^{-\delta |\xi| + \epsilon (1 + \frac{1}{t}) |\xi|}$$

On the other hand, for any sequence $\epsilon_n$ such that $\min\{1, \frac{\delta}{3}\} > \epsilon_1 > \epsilon_2 > \cdots \to 0$, we define a function $C(\cdot)$ on $(0, 1]$ as

$$C(t) := \begin{cases} C_{\epsilon_1}, & \epsilon_1 < t \leq 1, \\ C_{\epsilon_{n+1}}, & \epsilon_n < t \leq \epsilon_n. \end{cases}$$

Then $C(\cdot)$ is locally bounded on $(0, 1]$ and

$$|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|}) e^{-\frac{3}{2} \delta |\xi|}$$
on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$.
In like manners,

$$|P(z, \xi, w, \eta)| \leq C\left(\frac{|\eta|}{|\xi|}\right)e^{-\frac{1}{2}\delta|\eta|}$$

on $\gamma^{-1}(U; d, d) \cap \{|\xi| \geq |\eta|\}$.

Here, we define a function $C'(\cdot)$ on $(0, \infty)$ as $C'(t) = C(\min\{t, \frac{1}{t}\})$.
Then $C'(t)$ is locally bounded on $(0, \infty)$ and $|P(z, \xi, w, \eta)| \leq C'(\frac{|\xi|}{|\eta|})e^{-\frac{1}{2}\delta\min\{|\xi|, |\eta|\}}$ on $\gamma^{-1}(U; d, d)$.
That is, $P(z, \xi, w, \eta) \in R(K)$.

(⇒) Let $P(z, \xi, w, \eta) \in R(K)$.
Then there are $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that

$$|P(z, \xi, w, \eta)| \leq C(\frac{|\xi|}{|\eta|})e^{-\delta\min\{|\xi|, |\eta|\}}$$

on $\gamma^{-1}(U; d, d)$. We fix any $\epsilon$ such that $0 < \epsilon < 1$. Then,

$$|P(z, \xi, w, \eta)| \leq \max_{\epsilon \leq t \leq 1} C(t) \cdot e^{-\delta\min\{|\xi|, |\eta|\}}$$

on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$. We put $C'_\epsilon := \max_{\epsilon \leq t \leq 1} C(t)$.
On the other hand, since $P(z, \xi, w, \eta) \in S(K)$, there exists $C''_\epsilon > 0$ such that

$$|P(z, \xi, w, \eta)| \leq C''_\epsilon e^{\epsilon(|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; d, d)$.

Therefore, the following inequalities hold on $\gamma^{-1}(U; d, d) \cap \{\frac{|\xi|}{|\eta|} \leq \epsilon\}$.

$$|P(z, \xi, w, \eta)| \leq C''_\epsilon e^{-\delta\min\{|\xi|, |\eta|\} + \delta\min\{|\xi|, |\eta|\} + \epsilon(|\xi| + |\eta|)}$$

$$\leq C''_\epsilon e^{-\delta\min\{|\xi|, |\eta|\} + \epsilon(1 + \delta)(|\xi| + |\eta|)}.$$  

If we put $C_\epsilon := \max\{C'_\epsilon, C''_\epsilon\}$,

$$|P(z, \xi, w, \eta)| \leq C_\epsilon e^{-\delta\min\{|\xi|, |\eta|\} + \epsilon(1 + \delta)(|\xi| + |\eta|)}$$

on $\gamma^{-1}(U; d, d) \cap \{|\xi| \leq |\eta|\}$.
That is, $P(z, \xi, w, \eta) \in \hat{R}(K)$.

**Proposition 2.7.** $R(K)$ is an ideal in $S(K)$.

**Proof.** It is clear by the part (C) of the proof of Proposition 2.6.

**Proposition 2.8.** $\hat{R}(K)$ is an ideal in $\hat{S}(K)$.
Proof. Let \( \sum P_{j,k}(z, \xi, w, \eta) \in \hat{R}(K) \) and \( \sum Q_{j,k}(z, \xi, w, \eta) \in \hat{S}(K) \). Then there exist \( d > 0, 0 < A < 1 \), and \( U \supset K \) open in \( S^*X \times S^*Y \) satisfying the following:

For each \( \varepsilon > 0 \), we have some \( C_\varepsilon > 0 \) such that

\[
\begin{align*}
\text{a)} & \quad |P_{s,t}(z, \xi, w, \eta)|, |Q_{s,t}(z, \xi, w, \eta)| \leq C_\varepsilon A^{s+t} e^{\varepsilon(|\xi|+|\eta|)} \\
\text{b)} & \quad \left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} P_{j,k}(z, \xi, w, \eta) \right| \leq C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)}
\end{align*}
\]

on \( \gamma^{-1}(U; (s+1)d, (t+1)d) \) for each \( s, t \geq 0 \).

It suffices to show that \( \sum R_{j,k} \in \hat{R}(K) \), where

\[
R_{j,k} := \sum_{j_1+j_2=j, 0 \leq k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2}.
\]

Since we can easily estimate \( \sum R_{j,k} \) for \( st = 0 \),

we suppose \( s \geq 1 \) and \( t \geq 1 \).

Then we can obtain the following inequality:

\[
\left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} R_{j,k} \right| = \left| \sum_{0 \leq j \leq s, 0 \leq k \leq t} \sum_{j_1+j_2=j, 0 \leq k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2} \right|
\]

\[
\leq \left( \sum_{0 \leq j_1 \leq s, 0 \leq k_1 \leq t} \sum_{0 \leq j_2 \leq s, 0 \leq k_2 \leq t} P_{j_1,k_1}Q_{j_2,k_2} \right) + \left( \sum_{s+1 \leq j_1 \leq 2s, 0 \leq k_1 \leq t} \sum_{0 \leq j_2 \leq s, 0 \leq k_2 \leq t} P_{j_1,k_1}Q_{j_2,k_2} \right)
\]

\[
+ \left( \sum_{0 \leq j \leq s, j_1+j_2=j, t+1 \leq k \leq 2t} \sum_{0 \leq k_1 \leq k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2} \right) + \left( \sum_{0 \leq j \leq s, j_1+j_2=j, 0 \leq k_1 \leq k_1+k_2=k} P_{j_1,k_1}Q_{j_2,k_2} \right).
\]

We shall estimate the four terms in the right side of the inequality, respectively.

the first term \( \leq C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)} \cdot \sum_{0 \leq j \leq s, 0 \leq k_1 \leq t} C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)} \)

\[
\leq C_\varepsilon \cdot C_\varepsilon \cdot A^{\min\{s,t\}} e^{2\varepsilon(|\xi|+|\eta|)} \cdot \frac{1}{1-A} \cdot \frac{1}{1-A}
\]
on $\gamma^{-1}(U; (s + 1)d, (t + 1)d)$ for each $s, t \geq 1$.

the 2nd term

$$
\leq \sum_{s+1 \leq j \leq 2s} \sum_{j_1 + j_2 = j} C_{e} A^{j_1 + k_1} e^{e(|\xi| + |\eta|)} \cdot C_{e} A^{j_2 + k_2} e^{e(|\xi| + |\eta|)}
$$

$$
= C_{e} \cdot C_{e} \cdot e^{2e(|\xi| + |\eta|)} \cdot (\sum_{s+1 \leq j \leq 2s} \sum_{j_1 + j_2 = j} A^{j_1})(\sum_{t+1 \leq k \leq 2t} \sum_{k_1 + k_2 = k} A^{k}).
$$

If we choose any $B$ and $C$ such that $0 < B < 1$, $0 < C < 1$, and $BC \geq A$, we can get the following inequality:

$$
\sum_{s+1 \leq j \leq 2s} \sum_{j_1 + j_2 = j} A^{j} \leq C^{s+1}(B^0 + B^1 + B^2 + \ldots)^2 = C^{s+1}\left(\frac{1}{1-B}\right)^2.
$$

Then,

the second term \leq C_{e} \cdot C_{e} \cdot e^{2e(|\xi| + |\eta|)} \cdot C^{s+1}\left(\frac{1}{1-B}\right)^2 \cdot C^{t+1}\left(\frac{1}{1-B}\right)^2

on $\gamma^{-1}(U; (s + 1)d, (t + 1)d)$ for each $s, t \geq 1$.

the third term

$$
\leq \sum_{0 \leq j \leq s} \sum_{j_1 + j_2 = j} C_{e} A^{j_1 + k_1} e^{e(|\xi| + |\eta|)} \cdot C_{e} A^{j_2 + k_2} e^{e(|\xi| + |\eta|)}
$$

$$
= C_{e} \cdot C_{e} \cdot e^{2e(|\xi| + |\eta|)}(\sum_{0 \leq j \leq s} \sum_{j_1 + j_2 = j} A^{j_1})(\sum_{t+1 \leq k \leq 2t} \sum_{k_1 + k_2 = k} A^{k})
$$

\leq C_{e} \cdot C_{e} \cdot e^{2e(|\xi| + |\eta|)} \cdot (\frac{1}{1-A})^2 \cdot C^{t+1}\left(\frac{1}{1-B}\right)^2
$$

on $\gamma^{-1}(U; (s + 1)d, (t + 1)d)$ for each $s, t \geq 1$.

In like manners,

the fourth term \leq C_{e} \cdot C_{e} \cdot e^{2e(|\xi| + |\eta|)} \cdot C^{s+1} \cdot \left(\frac{1}{1-B}\right)^2 \cdot \left(\frac{1}{1-A}\right)^2

on $\gamma^{-1}(U; (s + 1)d, (t + 1)d)$ for each $s, t \geq 1$.

Hence, we conclude that $\sum R_{j,k} \in \hat{R}(k)$. 
\( \hat{S}(K)/\hat{R}(K) \) becomes a commutative ring by Proposition 2.8. By Propositions 2.6 and 2.7, the inclusion \( S(K) \rightarrow \hat{S}(K) \) induces the injective ring homomorphism

\[ \iota_K : S(K)/R(K) \longrightarrow \hat{S}(K)/\hat{R}(K). \]

Conversely, we obtain the following.

**Theorem 2.9.** If \( \sum P_{j,k}(z, \xi, w, \eta) \in \hat{S}(K) \), there exists \( P(z, \xi, w, \eta) \in S(K) \) such that \( P - \sum P_{j,k} \in \hat{R}(K) \).

Thus, \( S(K)/R(K) \) is isomorphic to \( \hat{S}(K)/\hat{R}(K) \) in the sense of commutative rings.

**Definition 2.10.** We call an element in the ring \( \hat{S}(K)/\hat{R}(K) \) a pseudo-differential operator of the product type on \( K \). We write : \( \sum P_{j,k} \) : for the associated pseudo-differential operator of the product type on \( K \) using an element \( \sum P_{j,k} \) in \( \hat{S}(K) \).

The mapping \( \gamma \) is the composition of the following \( \gamma_1 \) and \( \gamma_2 \).

\[
\begin{align*}
\tilde{T}^\circ(X \times Y) & \ni (z, w; \xi, \eta) \longmapsto (z, w; \frac{\xi}{|\xi, \eta|}, \frac{\eta}{|\xi, \eta|}) \in \tilde{S}^\circ(X \times Y), \\
\tilde{S}^\circ(X \times Y) & \ni (z, w; \frac{\xi}{|\xi, \eta|}, \frac{\eta}{|\xi, \eta|}) \longmapsto (z, \frac{\xi}{|\xi|} \times (w, \frac{\eta}{|\eta|}) \in S^\circ X \times S^\circ Y,
\end{align*}
\]

where \( \tilde{S}^\circ(X \times Y) := S^\circ(X \times Y) \setminus \{(S^\circ X \times Y) \cup (X \times S^\circ Y)\} \).

**Proposition 2.11.** If \( P(z, \xi, w, \eta) \) is a symbol of product type on \( K \), \( P \) is a symbol on \( \gamma_1^{-1}(K) \) in the sense of AOKI's symbol.

**Proof.** By the hypothesis, there are \( d > 0 \) and \( U \supset K \) open in \( S^\circ X \times S^\circ Y \) satisfying the following:

a) \( P(z, \xi, w, \eta) \) is holomorphic in \( \gamma^{-1}(U; d, d) \), and

b) for each \( \epsilon > 0 \) there is \( C_\epsilon > 0 \) such that \(|P(z, \xi, w, \eta)| \leq C_\epsilon e^{\epsilon(|\xi| + |\eta|)}\) on \( \gamma^{-1}(U; d, d) \).

Let \( K' \) be compact in \( \tilde{S}^\circ(X \times Y) \) and \( \gamma_1^{-1}(K) \supset K' \).

Then there exist \( d' > 0 \) and \( U' \supset K' \) open in \( \tilde{S}^\circ(X \times Y) \) such that

\[
\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \subset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.
\]

In fact, for each \( (\check{z}, \check{w}; \check{\xi}, \check{\eta}) \in \gamma_1^{-1}(K) \) we can choose \( d' > 0 \) such that

\[
d' > \frac{d}{\min\{|\xi|, |\eta|\}}.
\]
Then there exists a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta}) \in \gamma_1^{-1}(K)$ in $\hat{\mathcal{S}}^\circ(X \times Y)$ such that
\[
\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}.
\]
By the compactness of $K'$, the proof is completed.

**Proposition 2.12.** If $P(z, \xi, w, \eta)$ is a symbol of product type of 0-class on $K$, that is, $P \in R(K)$, $P$ is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.

**Proof.** Let $K'$ be compact in $\hat{\mathcal{S}}^\circ(X \times Y)$ and $\gamma_1^{-1}(K) \supset K'$. It suffices to show that $P$ is a zero symbol on $K'$ in the sense of AOKI's symbol. By the hypothesis, there exist $d > 0, \delta > 0, U \supset K$ open in $S^*X \times S^*Y$, and a locally bounded function $C(\cdot)$ on $(0, \infty)$ such that
\[
|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right)e^{-\delta \min\{|\xi|, |\eta|\}}
\]
on $\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\}$.

Let $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ be any point of $\gamma_1^{-1}(K)$. By Proposition 2.11, there exist $d' > 0$ and a neighborhood $U'$ of $(\hat{z}, \hat{w}; \hat{\xi}, \hat{\eta})$ in $\hat{\mathcal{S}}^\circ(X \times Y)$ such that
\[
\gamma^{-1}(U) \cap \{|\xi| > d, |\eta| > d\} \supset \gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\},
\]
and that there exists $\delta' > 0$ satisfying
\[
\min\left\{\frac{|\xi|}{|\xi| + |\eta|}, \frac{|\eta|}{|\xi| + |\eta|}\right\} > \delta' \text{ on } \gamma_2^{-1}(U').
\]
Hence,
\[
|P(z, \xi, w, \eta)| \leq C\left(\frac{|\xi|}{|\eta|}\right)e^{-\delta'(|\xi| + |\eta|)}
\]
on $\gamma_2^{-1}(U') \cap \{|\xi| + |\eta| > d'\}$.

Since $K$ is compact, $P$ is a zero symbol on $\gamma_1^{-1}(K)$ in the sense of AOKI's symbol.

**Definition 2.13.** The canonical mapping $H_K$ is defined as follows;
\[
\mathcal{S}(K)/R(K) \ni P : H_K(P) = \lim_{U \supset \gamma_1^{-1}(K)} \mathcal{E}_1^\mathbb{R}(U).
\]

**Proposition 2.14.** Suppose $K_1$ and $K_2$ are compact in $S^*X \times S^*Y$, respectively, and $K_1 \supset K_2$. Then, $H_{K_1}(P_P : H_{K_2}(P|_{K_2})$ for all $P \in \mathcal{S}(K)/R(K)$. 

Definition 2.15. We define the product $*$ of two elements of $\hat{S}(K)$ as follows:

$$(\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)) * (\sum_{j,k=0}^{\infty} Q_{j,k}(z, \xi, w, \eta)) = \sum_{j,k=0}^{\infty} R_{j,k}(z, \xi, w, \eta),$$

where

$$\sum_{j,k=0}^{\infty} t_1^j t_2^k R_{j,k}(z, \xi, w, \eta) := e^{t_1(\partial_{\xi}, \partial_{z}) + t_2(\partial_{\eta}, \partial_{w})} \left( (\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)) \right. \left. \times (\sum_{j,k=0}^{\infty} Q_{j,k}(z^*, \xi^*, w^*, \eta^*)) \right|_{z^*=z, \xi^*=\xi, \eta^*=\eta}.$$

That is,

$$R_{j,k}(z, \xi, w, \eta) := \sum_{j_1+j_2+|\alpha|=j, k_1+k_2+|\beta|=k} \frac{1}{\alpha! \beta!} \partial_{\xi}^\alpha \partial_{\eta}^\beta P_{j_1,k_1}(z, \xi, w, \eta) \times \partial_{\xi}^\alpha \partial_{w}^\beta Q_{j_2,k_2}(z, \xi, w, \eta).$$

Then we obtain the following.

Lemma 2.16. If $\sum P_{j,k}$ and $\sum Q_{j,k}$ are formal symbols of product type on $K$, then $\sum R_{j,k}$ is also a formal symbol of product type on $K$.

Proposition 2.17. If $\sum P_{j,k} \in \hat{S}(K)$ and $\sum Q_{j,k} \in \hat{R}(K)$, otherwise $\sum P_{j,k} \in \hat{R}(K)$ and $\sum Q_{j,k} \in \hat{S}(K)$, $\sum R_{j,k}$ is also in $\hat{R}(K)$.

By Lemma 2.16 and Proposition 2.17, the following composition of two elements in $\hat{S}(K)/\hat{R}(K)$ is well-defined;

$$\sum P_{j,k} : o : \sum Q_{j,k} := (\sum P_{j,k}) * (\sum Q_{j,k}).$$

We can easily verify the associativity about the operation $o$. That is, $\hat{S}(K)/\hat{R}(K)$ becomes an associative $\mathbb{C}$ algebra. Hence the mapping $H_K$ is a homomorphism about the operation $o$, $+$, and $:$, where

$$\mathcal{E}_{X \times Y}^R(\gamma^{-1}(K)) \equiv \hat{S}(K)/\hat{R}(K) \xrightarrow{H_K} \mathcal{E}_{X \times Y}^R(\gamma^{-1}(K)).$$

Definition 2.18. The reverse of $\sum P_{j,k}$ in $\hat{S}(K)$ is defined as

$$(\sum t_1^j t_2^k P_{j,k})^R := e^{t_1(\partial_{\xi}, \partial_{z}) + t_2(\partial_{\eta}, \partial_{w})} (\sum t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)).$$

We can verify that if $\sum P_{j,k}$ is in $\hat{S}(K)$ ($\hat{R}(K)$) then $(\sum P_{j,k})^R$ is in $\hat{S}(K)$ ($\hat{R}(K)$), respectively.
3. EXPONENTIAL CALCULUS OF SYMBOLS OF MINIMUM TYPE

**Definition 3.1.** A function $\Lambda : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following hold:

1. $\Lambda$ is continuous,
2. for each $\alpha > 1$, $\Lambda(\alpha t) \leq \alpha \Lambda(t)$ on $(0, \infty)$,
3. $\Lambda$ is increasing,
4. $\lim_{t \to \infty} \frac{\Lambda(t)}{t} = 0$.

**Definition 3.2.** $P(z, \xi, w, \eta) \in S(K)$ is called a symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:

1. $P(z, \xi, w, \eta)$ is holomorphic in $\gamma^{-1}(U; d, d)$, and
2. $|P(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$ on $\gamma^{-1}(U; d, d)$.

**Example 3.3.** (by K. Kataoka)

Let $\Omega = \Omega' := \mathbb{C} \times \{\xi \in \mathbb{C} ; \arg \xi < \delta, \xi \neq 0\} (0 < \delta < \frac{\pi}{2})$.

Let $K$ be any compact subset of $S^*\mathbb{C}_z \times S^*\mathbb{C}_w$ such that $\gamma^{-1}(K) \subset \Omega \times \Omega'$.

$P(z, \xi, w, \eta) := (\xi \eta)^{(1+\sigma)/2} / (\xi + \eta)$,

$\Lambda_1(t) = \Lambda_2(t) := t^\sigma$ with $0 < \sigma < 1$.

**Remark 3.4.** If $P$ is a symbol of minimum type on $K$, $e^P$ is a symbol of product type on $K$.

**Definition 3.5.** $\sum P_{j,k}$ in $\hat{S}(K)$ is called a formal symbol of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$ if there exist constants $C > 0$, $d > 0$, $0 < A < 1$, and $U \supset K$ open in $S^*X \times S^*Y$ satisfying the following:

1. $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$,
2. $|P_{j,k}(z, \xi, w, \eta)| \leq C \cdot \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\} \cdot A^{j+k}$ on $\gamma^{-1}(U; (j+1)d, (k+1)d)$ for each $j, k \geq 0$.

**Remark 3.6.** If $\sum P_{j,k}$ is a formal symbol of minimum type on $K$, $e^{\sum P_{j,k}}$ is a formal symbol of product type on $K$. 


Proposition 3.7. If $P$ and $Q$ are in $S(K)$, then

\[
P(z, \xi, w, \eta) \ast (Q(z, \xi, w, \eta))^R
= e^{t_1(\partial_\xi, \partial_z*) + t_2(\partial_\eta, \partial_w*)} P(, \xi, w, \eta) Q(, \xi, w, \eta) \big|_{z^* = z, \eta^* = \eta}.
\]

Theorem 3.8. If $P$ and $Q$ are symbols of minimum type of growth order $(\Lambda_1, \Lambda_2)$ on $K$, there exists a formal symbol, $\sum R_{j,k}$, of minimum type on $K$ satisfying $e^P \ast e^Q = e^{\sum_{j,k} t^j t^k R_{j,k}}$.

Proof. $W(s, t; z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) := e^{s(\partial_\xi, \partial_z*) + t(\partial_\eta, \partial_w*)} \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*))$ is the unique formal series solution to the following system of partial differential equations:

\[
\left\{ \begin{array}{l}
\partial_s W = (\partial_\xi, \partial_z*) W, \\
\partial_t W = (\partial_\eta, \partial_w*) W, \\
W_{s=t=0} = \exp(P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*)).
\end{array} \right.
\]

If we put $W = \exp(\sum_{j,k}^\infty s^j t^k W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*))$, by the above system, we obtain the following recursive formulas about $\{W_{j,k}\}_{j,k\geq 0}$:

\[
\left\{ \begin{array}{l}
W_{0,0} = P(z, \xi, w, \eta) + Q(z^*, \xi^*, w^*, \eta^*), \\
W_{j+1,k} = \frac{1}{j+1} \{ (\partial_\xi, \partial_z*) W_{j,k} + \sum_{\begin{array}{l} j_1 + j_2 = j \\
k_1 + k_2 = k \end{array}} \langle \partial_\xi W_{j_1,k_1}, \partial_z* W_{j_2,k_2} \rangle \}, \\
W_{j,k+1} = \frac{1}{k+1} \{ (\partial_\eta, \partial_w*) W_{j,k} + \sum_{\begin{array}{l} j_1 + j_2 = j \\
k_1 + k_2 = k \end{array}} \langle \partial_\eta W_{j_1,k_1}, \partial_w* W_{j_2,k_2} \rangle \}.
\end{array} \right.
\]

Then $R_{j,k} = W_{j,k}(z, \xi, w, \eta, z^*, \xi^*, w^*, \eta^*) \big|_{z^* = z, \xi^* = \xi}$.

Suppose there exist $C_P (= C_Q) > 0$, $d > 0$, and an open subset $U(\supset K)$ of $S^*X \times S^*Y$ satisfying the following:

1. $P$ and $Q$ are holomorphic in $\gamma^{-1}(U; d, d)$,
2. $|P(z, \xi, w, \eta)|$, and $|Q(z, \xi, w, \eta)| \leq C_P \cdot \Lambda(|\xi|, |\eta|)$ on $\gamma^{-1}(U; d, d)$, where $\Lambda(|\xi|, |\eta|) := \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}$.

$V := \gamma^{-1}(U; d, d) \times \gamma^{-1}(U; d, d)$,

$V^{\varepsilon_1, \varepsilon_2} := \{(z, w, \xi, \eta, z^*, \xi^*, \eta^*) \in V; |\xi' - \xi| \leq \varepsilon_1 |\xi|, |z^{*'} - z^*| \leq \varepsilon_1, |\eta' - \eta| \leq \varepsilon_2 |\eta|, |w^{*'} - w^*| \leq \varepsilon_2 \Rightarrow (z, w, \xi', \eta', z^{*'}, \xi^*, \eta^*) \in V\}$. Then we obtain the following lemma.
Lemma 3.9. Suppose \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfy the following conditions:

(1) 
\[
C_{j+1,k}^{(\mu,\nu)} \geq \frac{9ne^{10}}{j+1} \{C_{j,k}^{(\mu,\nu)}(j+1)^2 + \sum^{*}(j_{1}+1)(j_{2}+1)C_{j_{1},k_{1}}^{(\mu_{1},\nu_{1})}C_{j_{2},k_{2}}^{(\mu_{2},\eta)}\},
\]

(2) 
\[
C_{j,k+1}^{(\mu,\nu)} \geq \frac{9me^{10}}{k+1} \{C_{j,k}^{(\mu,\nu)}(k+1)^2 + \sum^{**}(k_{1}+1)(k_{2}+1)C_{j_{1},k_{1}}^{(\mu_{1},\nu_{1})}C_{j_{2},k_{2}}^{(\mu_{2},\eta)}\},
\]

(3) 
\[
C_{0,0}^{0,0} \leq C_{P} + C_{Q},
\]

(4) 
\[
C_{j,k}^{(\mu,\nu)} \geq 0 \ (j, k \geq 0, \ 0 \leq \mu \leq j, \ 0 \leq \nu \leq k),
\]

(5) 
\[
C_{j,k}^{(\mu,\nu)} = 0 \ \text{(otherwise)}.
\]

Here, the sum \( \sum^{*}, \sum^{**} \) mean \( \sum j_{1}+j_{2}=j \), \( \sum k_{1}+k_{2}=k \), \( \mu_{1}+\mu_{2}=\mu-1 \), \( \mu_{1}+\mu_{2}=\mu \), \( \nu_{1}+\nu_{2}=\nu \), \( \nu_{1}+\nu_{2}=\nu-1 \), respectively.

Then for each \( \epsilon_{1} \) and \( \epsilon_{2} \) such that \( 0 < \epsilon_{1} \ll 1 \) and \( 0 < \epsilon_{2} \ll 1 \), the following hold:

(3.4) 
\[
|W_{j,k}| \leq \sum_{0 \leq \mu \leq j, \ 0 \leq \nu \leq k} C_{j,k}^{(\mu,\nu)} \frac{(\tilde{A}(\epsilon_{1}|\xi|,|\eta|) + \tilde{A}(\epsilon_{2}|\xi^{*}|,|\eta^{*}|))}{\epsilon_{1}^{2j}\epsilon_{2}^{2k}|\xi|^{j}|\eta|^{k}} \times (\Lambda_{1}(\epsilon_{1}|\xi|) + \Lambda_{1}(\epsilon_{2}|\xi^{*}|))^{\mu}(\Lambda_{2}(\epsilon_{1}|\eta|) + \Lambda_{2}(\epsilon_{2}|\eta^{*}|))^{\nu}
\]

on \( V^{\epsilon_{1},\epsilon_{2}} \) for all \( j, k \geq 0 \).

The following lemma guarantees the existence of \( C_{j,k}^{(\mu,\nu)} \) satisfying the conditions from (1) through (5) of the previous lemma.

Lemma 3.10. The following sequence \( \{C_{j,k}^{(\mu,\nu)}\}_{j,k,\mu,\nu \geq 0} \) satisfies the conditions from (1) through (5) of the previous lemma.

\[
C_{j,k}^{(\mu,\nu)} := \begin{cases} 
 lB^{j+k}(j+1)^{j-\mu-3}(k+1)^{k-\nu-3}, & (0 \leq \mu \leq j, \ 0 \leq \nu \leq k) \\
 0, & \text{(otherwise)}
\end{cases}
\]

where \( l \) and \( B \) are constants satisfying \( l \geq \max\{C_{P} + C_{Q}, 1\} \) and \( B \geq 72l \cdot \max\{m, n\} \cdot e^{10} \cdot (c^{2} + 1) \) and \( c \) is a constant satisfying the following T.Aoki's inequality:

(3.5) 
\[
\frac{1}{j+1} \sum_{\mu=0}^{\nu-1} \sum_{k=\mu}^{j-\nu+\mu+1} (k+1)^{k-\mu-2}(j-k+1)^{j-k-\nu+\mu-1} \leq c(j+1)^{j-\nu-2}
\]
for all $j, \nu$ such that $0 \leq \nu - 1 \leq j$.

(continued) We can prove the theorem using the above two lemmas.

REFERENCES


