NONSTANDARD UNIVERSE

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ABSTRACT. The nonstandard universes are frameworks of nonstandard analysis. We find sheaf representation for a nonstandard universe in Theorem 3.7.

1. NONSTANDARD UNIVERSE

Definitions 1.1 (superstructure, base set). Given a set $X$, we define the iterated power set $V_n(X)$ over $X$ recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The superstructure $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set $X$ is said to be a base set if $\emptyset \notin X$ and each element of $X$ is disjoint from $V(X)$.

Definition 1.2 (nonstandard universe). A nonstandard universe is a triple $(V(X), V(Y), \star)$ such that:

1. $X$ and $Y$ are infinite base sets.
2. (Transfer Principle) The symbol $\star$ is a map from $V(X)$ into $V(Y)$ such that $V(X) \models \varphi(a_1, \ldots, a_n)$ if and only if $V(Y) \models \varphi(\star a_1, \ldots, \star a_n)$ holds for any bounded formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in V(X)$.
3. $\star X = Y$.
4. For every infinite subset of $A$ of $X$, $\{ \star a \mid a \in A \}$ is a proper subset of $\star A$.

Definitions 1.3 (standard, internal). For $a \in V(\star X)$, we call $a$ standard if there is an $x \in V(X)$ such that $a = \star x$.

For $a \in V(\star X)$, we call $a$ internal if there is an $x \in V(X)$ such that $a \in \star x$. We denote by $\star V(X)$ the set of all internal elements in $V(\star X)$.

From now on, we denote a nonstandard universe by single $\star V(X)$.

Definitions 1.4 (norm, radius). The norm (of standardness) of an internal element $a$ is a cardinal defined by

$$\text{nos}(a) = \min \{|x| \mid a \in \star x\}.$$

The radius of $\star V(X)$ is a cardinal defined by

$$\text{rad}(\star V(X)) = \min \{ \kappa \mid \forall y \in \star V(X) \text{ nos}(y) < \kappa \}.$$
Definition 1.5 (covering number). Let $a$ be an internal element. The local ultra-power at $a$ is defined by

$$V(X)[a] = \{(*w)(a) \mid w \in V(X) \text{ and } a \in *(\text{dom}(w))\}.$$ 

For a subset $E \subseteq \nu V(X)$, we denote

$$V(X)[E] = \bigcup\{V(X)[s] \mid s \text{ is a finite subset of } E\}.$$ 

The covering number of $\nu V(X)$ is defined by

$$\text{cov}(\nu V(X)) = \min\{|E| \mid E \subseteq \nu V(X) \text{ and } V(X)[E] = \nu V(X)\}.$$ 

2. Locally atomic complete algebra

Definition 2.1 (regular complete subalgebra). Let $\langle \mathcal{B}, \land, \lor, \neg, 0_{\mathcal{B}}, 1_{\mathcal{B}} \rangle$ be a Boolean algebra. A subset $C \subseteq \mathcal{B}$ is said to be a regular complete subalgebra of $\mathcal{B}$ if $C$ is a complete subalgebra of $\mathcal{B}$ and the inclusion map is also complete.

Notation. Let $\mathcal{B}$ be a Boolean algebra. For a subset $S \subseteq \mathcal{P}(\mathcal{B})$, we denote $S^0 = \{C \in S \mid C$ is a regular complete subalgebra of $\mathcal{B}\}$.

Definition 2.2 (LCA). A locally complete algebra (LCA) is a set $\Lambda$ of subsets of a Boolean algebra $\mathcal{B}$ satisfying the conditions below.

1. $\bigcup \Lambda = \mathcal{B}.$
2. If $S_1, S_2 \in \Lambda$ then $S_1 \cup S_2 \in \Lambda$.
3. If $S \in \Lambda$ and $T \subseteq S$ then $T \in \Lambda$.
4. For every $S \in \Lambda$, there is a $C \in \Lambda^0$ containing $S$.

For an LCA $\Lambda$, we denote by $\mathcal{B}(\Lambda)$ the Boolean algebra $\bigcup \Lambda$. We call the Boolean algebra $\mathcal{B}(\Lambda)$ the base Boolean algebra of $\Lambda$.

Definition 2.3 (LACA). An LCA $\Lambda$ is a locally atomic complete algebra (LACA) if every $C \in \Lambda^0$ is atomic. We denote the set of atoms of $C \in \Lambda^0$ by $\text{Atom}(C)$.

Definition 2.4 (homomorphism). We introduce notation $R^\ast S = \{R^\ast S \mid S \in S\}$. Let $\Lambda$ and $\Xi$ be LCAs. A Boolean homomorphism $f : \mathcal{B}(\Lambda) \to \mathcal{B}(\Xi)$ is a pseudo-homomorphism of LCAs if $f^\ast \Lambda \subseteq \Xi$. We denotes a pseudo-homomorphism by $f : \Lambda \to \Xi$. A pseudo-homomorphism $h : \Lambda \to \Xi$ of LCAs is a (complete) homomorphism if $\bigvee h^\ast S = h(\bigvee S)$ for all $S \in \Lambda$. An embedding or monomorphism $j : \Lambda \to \Xi$ is an injective homomorphism.

Definition 2.5 (subLCA). A subLCA of an LCA $\Lambda$ is a nonempty subset of $\Lambda$ which is itself an LCA and the inclusion map is an embedding.

Definition 2.6 (generator). Let $\Lambda$ be an LCA. A subset $\mathcal{G} \subseteq \Lambda^0$ is a generator of $\Lambda$ or $\mathcal{G}$ generates $\Lambda$ if $\Lambda$ is the only subLCA of $\Lambda$ containing $\mathcal{G}$.
Definitions 2.7 (radius, covering number, diameter). The radius of an LCA $\Lambda$ is a cardinal defined by

$$\text{rad}(\Lambda) = \min \{ \kappa \mid \forall C \in \Lambda^\bigcirc | \text{Atom}(C)| < \kappa \}$$

The covering number of an LCA $\Lambda$ is a cardinal defined by

$$\text{cov}(\Lambda) = \min \{|\mathcal{S}| \mid \mathcal{S} \text{ is a generator of } \Lambda \}.$$ 

The diameter of an LCA $\Lambda$ is a cardinal defined by

$$\text{diam}(\Lambda) = \min \left\{ \sum_{C \in \mathcal{S}} |\text{Atom}(C)| \mid \mathcal{S} \text{ is a generator of } \Lambda \right\}.$$ 

Definition 2.8 (direct product). Let $I$ be an index set. The direct product $\Lambda^{|I|}$ of the LCA $\Lambda$ is defined by:

$$\Lambda^{|I|} = \left\{ S \subseteq \mathfrak{B}(\Lambda)^I \mid \bigcup_{g \in S} \text{rng } g \in \Lambda \right\}$$

with the pointwise Boolean operations on $\mathfrak{B}(\Lambda^{|I|}) = \bigcup \Lambda^{|I|} \subseteq \mathfrak{B}(\Lambda)^I$. Then $\Lambda^{|I|}$ is an LCA. The LCA $\Lambda$ is embedded into $\Lambda^{|I|}$ by the canonical embedding $b \mapsto I \times \{b\}$.

Definitions 2.9 (embedding system, direct limit). The embedding system of LCAs is a family of embeddings

$$\mathcal{E} = \{ j_{d}^{d'} : \Lambda_{d} \rightarrow \Lambda_{d'} \}_{d \leq d', d, d' \in D}$$

satisfying $j_{d}^{d'} \circ j_{d'}^{d''} = j_{d}^{d''}$ for all $d \leq d' \leq d''$, where $D$ is an upper direct set. The direct limit of $\mathcal{E}$ is $\bigcup \{ j_{d}^{d'} : \mathfrak{B}(\Lambda_{d}) \rightarrow \mathfrak{B}(\Lambda_{d'}) \}_{d \leq d', d' \in D}$ as Boolean algebras.

Definition 2.10 (ultrafilter). Let $\Lambda$ be an LCA. A subset $\mathcal{U}$ of $\mathfrak{B}(\Lambda)$ is an ultrafilter of an LCA $\Lambda$ if it is an ultrafilter of the base Boolean algebra $\mathfrak{B}(\Lambda)$.

3. ULTRALIMIT

Definition 3.1 (LACA-valued model). Let $\Lambda$ be an LACA and let $M$ be a model for a language $\mathcal{L}$. The $\mathfrak{B}(\Lambda)$-valued universe of $M$ is defined by

$$M^{\langle \Lambda \rangle} = \left\{ u : M \rightarrow \mathfrak{B}(\Lambda) \mid u(x) \land u(y) = 0 \text{ for } x \neq y, \right.$$

$$\left. \text{rng } u \in \Lambda, \lor \text{rng } u = 1 \right\}.$$

For $u \in M^{\langle \Lambda \rangle}$, the support of $u$ is a subset of $M$ defined by

$$\text{supp } u = \{ x \in M \mid u(x) \neq 0 \}.$$ 

To each function $F$ of $\mathcal{L}(M)$ and each $u_1, \ldots, u_n \in M^{\langle \Lambda \rangle}$, we assign a $F(u_1, \ldots, u_n) \in M^{\langle \Lambda \rangle}$ by:

$$F(u_1, \ldots, u_n)(y) = \bigvee \left\{ \bigwedge_{i=1}^{n} u_i(x_i) \mid M \models y = F(x_1, \ldots, x_n) \right\}$$ 

for $y \in M$. 
We regard a constant of $\mathcal{L}(M)$ as a function without any variables. Note that $\bigwedge_{i=1}^{n} u_i(x_i) = 1$ if $n = 0$. To each sentence $\varphi$ of $\mathcal{L}(\mathcal{M}^{\langle\Lambda\rangle})$ we assign a truth value $[\varphi] \in \overline{\mathcal{B}(\Lambda)}$ by following recursive rules:

\[
[u = v] = \bigvee \{u(x) \land v(x) \mid x \in M\},
\]

\[
[R(u_1, \ldots, u_m)] = \bigvee \left\{ \bigwedge_{i=1}^{m} u_i(x_i) \mid \mathfrak{M} \models R(x_1, \ldots, x_m) \right\},
\]

\[
[-\varphi] = \neg\langle[\varphi]\rangle,
\]

\[
[\varphi \lor \varphi_2] = \langle[\varphi]\rangle \lor \langle[\varphi_2]\rangle,
\]

\[
[\exists x \varphi(x)] = \bigvee \{\langle\varphi(u)\rangle \mid u \in M^{\langle\Lambda\rangle}\},
\]

where $R$ is any predicate in $\mathcal{L}$.

**Definition 3.2 (LACA-valued superstructure).** Let $\Lambda$ be an LACA. The $\Lambda$-valued superstructure of $V(X)$ is defined by

\[
\overline{V}(X)^{\langle\Lambda\rangle} = \{ u \in V(X)^{\langle\Lambda\rangle} \mid \text{supp } u \in V(X) \}.
\]

While the truth values range over $\overline{\mathcal{B}(\Lambda)}$ on this definition, we shall see $[\varphi]_{\Lambda} \in \mathcal{B}(\Lambda)$.

**Theorem 3.1.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in M^{\langle\Lambda\rangle}$,

\[
([\varphi(u_1, \ldots, u_r)]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \mid \mathfrak{M} \models \varphi(x_1, \ldots, x_r) \right\}.
\]

**Proof.** For $\varphi$ either "$x_1 = x_2$" or $R$, (*) holds by definition. If (*) holds for an atomic formula $\varphi(x)$ then, by simple calculus of Boolean algebra, (*) holds for $\varphi(F(x_1, \ldots, x_n))$. Thus, by induction, (*) holds for $\varphi$ atomic. Suppose (*) holds for $\varphi, \varphi_1$ and $\varphi_2$. Since there is an atomic $C \in \Lambda^0$ containing all the ranges of $u_1, \ldots, u_r$, and every range of $u_i$ is a partition of unity except for 0,

\[
[-\varphi]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \mid \mathfrak{M} \models \neg\varphi(x_1, \ldots, x_r) \right\}.
\]

It is easy to see:

\[
[\varphi_1 \lor \varphi_2]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \mid \mathfrak{M} \models \varphi_1(x_1, \ldots, x_r) \lor \varphi_2(x_1, \ldots, x_r) \right\}.
\]

Since $[\varphi(u)]_{\Lambda} = \bigvee_{x \in M} \langle u(x) \land \langle \varphi(x) \rangle \rangle$, we have $[\exists x \varphi(x)]_{\Lambda} = \bigvee_{x \in M} \langle \varphi(x) \rangle_{\Lambda}$. Therefore (*) holds for $\exists x \varphi(x)$. \qed

Similarly, we shall obtain the superstructure version.

**Corollary 3.2.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}_e$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \overline{V}(X)^{\langle\Lambda\rangle}$,

\[
[\varphi(u_1, \ldots, u_r)]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_i(x_i) \mid V(X) \models \varphi(x_1, \ldots, x_r) \right\}.
\]
By the theorem and the corollary above, we have a fundamental property
\[ [u = v] \land [[\varphi(u)] \leq [[\varphi(v)]]. \]
We have just introduced \( B(\Lambda) \)-valued model
\[ M^{\langle\langle \Lambda \rangle\rangle} = (M^{\langle\langle \Lambda \rangle\rangle}, R, F, c) \] and \( B(\Lambda) \)-valued superstructure \( \widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle} \). We say that
a sentence \( \varphi \) of \( \mathcal{L}(M^{\langle\langle \Lambda \rangle\rangle}) \) holds in \( M^{\langle\langle \Lambda \rangle\rangle} \) if \( [[\varphi]]_{\Lambda} = 1 \) and that a sentence \( \psi \) of
\( \mathcal{L}(\widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle}) \) holds in \( \widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle} \) if \( [[\psi]]_{\Lambda} = 1 \). Theorem 3.1 and Corollary 3.2
follow that we consider the values \( u(x) \) only for \( x \in \text{supp}u \). For \( E \subseteq M \), we may
regard \( E^{\langle\langle \Lambda \rangle\rangle} \) as a subset of \( M^{\langle\langle \Lambda \rangle\rangle} \) by extending the domain of \( u \in E^{\langle\langle \Lambda \rangle\rangle} \) to \( M \).
This means that we define for \( u \in E^{\langle\langle \Lambda \rangle\rangle} \)
\[ u(x) = 0 \quad \text{if} \quad x \notin E. \]
In the superstructure version, if \( E \) is a set relative to \( V(X) \) then we may assume
\[ E^{\langle\langle \Lambda \rangle\rangle} = \{ u \in \widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle} \mid u \in \check{E} \text{ holds in } \widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle} \}. \]

**Theorem 3.3 (Maximum principle).** Let \( \varphi(x) \) be a formula of \( \mathcal{L}(M^{\langle\langle \Lambda \rangle\rangle}) \) with only \( x \) free. Then there is \( u \in M^{\langle\langle \Lambda \rangle\rangle} \) such that \( [[\varphi(u)]_{\Lambda} = [[\exists x \varphi(x)]_{\Lambda} \].

**Proof.** Let \( \{ a_\zeta \}_{\zeta < \alpha} \) be a well-ordering for \( M \). By theorem 3.1, there is \( C \in \Lambda^\emptyset \) containing \( \{ [[\varphi(x)] \mid x \in M \} \). Putting \( b_\zeta = [[\varphi(a_\zeta)] \land \neg \exists \xi < \zeta [[\varphi(a_\xi)] \}, \) we have
\( \{ b_\zeta \}_{\zeta < \alpha} \subseteq C. \) Since \( \{ b_\zeta \}_{\zeta < \alpha} \) is a pairwise disjoint family, we can pick \( u \in M^{\langle\langle C \rangle\rangle} \)
with \( u(a_\zeta) \geq b_\zeta. \) Then \( [[\varphi(u)] \geq u(a_\zeta) \land [\varphi(a_\zeta)] \geq b_\zeta \) for any \( \zeta < \alpha. \) Since
\( \exists \exists x \varphi(x) \} = \forall \zeta < \alpha \varphi(a_\zeta) = \forall \zeta < \alpha b_\zeta, \) we have \( [[\varphi(u)] \geq [[\exists x \varphi(x)] \). 

**Corollary 3.4.** Let \( \varphi(x) \) be a formula of \( \mathcal{L}(\widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle}) \) with only \( x \) free and let \( v \) be an element of \( \widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle} \). Then there is \( u \in \widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle} \) such that \( [[u \in v \land \varphi(u)]_{\Lambda} = [[\exists x \in v \varphi(x)]_{\Lambda} \]

**Proof.** Since there is \( n \) such that \( \text{supp}v \subseteq V_{n+1}(X) \),
\[ [[x \in v] = \bigvee \{ v(y) \mid x \in y \in \text{supp}v \} = 0 \quad \text{for} \quad x \notin V_n(X). \]
Therefore we can choose \( u \) whose support is a subset of \( V_n(X). \)

**Definition 3.3 (ultralimit).** We denote by \( u/\mathcal{U} \) the equivalence class of \( u \in M^{\langle\langle \Lambda \rangle\rangle} \)
by the equivalence relation
\[ x \sim_u y \equiv [x = y]_{\Lambda} \in \mathcal{U}. \]
The ultralimit \( M^{\langle\langle \Lambda \rangle\rangle}/\mathcal{U} \) of \( M \) modulo \( \mathcal{U} \) of \( \Lambda \) is defined by:
\[ M^{\langle\langle \Lambda \rangle\rangle}/\mathcal{U} = \{ u/\mathcal{U} \mid u \in M^{\langle\langle \Lambda \rangle\rangle} \}. \]
\[ \check{F}/\mathcal{U}(u_1/\mathcal{U}, \ldots, u_n/\mathcal{U}) = (\check{F}(u_1, \ldots, u_n))/\mathcal{U}. \]
\[ M^{\langle\langle \Lambda \rangle\rangle}/\mathcal{U} \models R(u_1/\mathcal{U}, \ldots, u_m/\mathcal{U}) \quad \text{iff} \quad [[R(u_1, \ldots, u_m)] \in \mathcal{U}. \]

**Definition 3.4 (bounded ultralimit).** We denote by \( u/\mathcal{U} \) the equivalence class of \( u \in \widehat{V}(X)^{\langle\langle \Lambda \rangle\rangle} \) by the equivalence relation
\[ x \sim_u y \equiv [x = y]_{\Lambda} \in \mathcal{U}. \]
The bounded ultralimit $\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ of $V(X)$ modulo $\mathcal{U}$ of $\Lambda$ is defined by:

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} = \{ u/\mathcal{U} | u \in \hat{V}(X)^{\langle\Lambda\rangle}\},$$

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models u/\mathcal{U} \in v/\mathcal{U} \iff [u \in v] \in \mathcal{U}.$$

**Theorem 3.5 (Łoś Principle of Ultralimits).** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in M^{\langle\Lambda\rangle}$,

$$M^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \ldots, u_r/\mathcal{U}) \iff \langle\varphi(u_1, \ldots, u_r)\rangle \in \mathcal{U}.$$

**Proof.** The proof proceeds by induction on the complexity of formulae. The only nontrivial step is the case where $\varphi$ is of the form $\exists x \psi(x)$. Suppose $[\exists x \psi(x)] \in \mathcal{U}$. By the maximal principle (Theorem 3.3), there is $u$ satisfying $[\psi(u)] = [\exists x \psi(x)]$. Then $M^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U})$ by the induction assumption. We have thus $M^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Conversely, suppose $M^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Then there is some $u$ such that $M^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U})$. By the induction assumption, $[\exists x \psi(x)] \geq [\psi(u/\mathcal{U})] \in \mathcal{U}$. \hfill \Box

**Corollary 3.6 (Łoś-Mostowski Principle of Bounded Ultralimits).**

Let $\varphi(x_1, \ldots, x_r)$ be a $\Delta_0$-formula of $\mathcal{L}_E$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \hat{V}(X)^{\langle\Lambda\rangle}$,

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \ldots, u_r/\mathcal{U}) \iff \langle\varphi(u_1, \ldots, u_r)\rangle \in \mathcal{U}.$$

**Proof.** The proof is similar to that of Theorem 3.5. The only different part is the if-part of the case where $\varphi$ is of the form $\exists x \in y \psi(x)$. Suppose $[\exists x \in u_k \psi(x)] \in \mathcal{U}$. It follows from Corollary 3.4 that there is $u \in \hat{V}(X)^{\langle\Lambda\rangle}$ satisfying $[u \in u_k \wedge \psi(u)] = [\exists x \in u_k \psi(x)]$. \hfill \Box

A bounded ultralimit is a pre-nonstandard universe: that satisfies (1),(2) and (3) of Definition 1.2 with Mostowski collapsing.

**Definition 3.5 (atlas).** An atlas is a pair $\langle \Lambda, \mathcal{U} \rangle$ of an LACA $\Lambda$ and an ultrafilter of $\Lambda$ such that $\text{rad} (\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{rad}(\Lambda)$ and $\text{cov}(\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{cov}(\Lambda)$.

**Theorem 3.7 (Sheaf representation Theorem for Nonstandard Universes).** For any nonstandard universe $^*V(X)$, there is an atlas $\langle \Lambda, \mathcal{U} \rangle$ such that $\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ is isomorphic to $^*V(X)$.

We prove the theorem in the next section.

**4. LOCAL ULTRALIMITS**

We shall see that a homomorphism of LACAs induces an elementary embedding of ultralimits and a bounded elementary embedding of bounded ultralimits. Let $h : \Lambda \to \Xi$ be a homomorphism. The induced map $h_* : M^{\langle\Lambda\rangle} \to M^{\langle\Xi\rangle}$ is defined by $h_*(u) = h \circ u$. Then we have the lemma below.

**Lemma 4.1.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in M^{\langle\Lambda\rangle}$

$$\langle\varphi(h_*(u_1), \ldots, h_*(u_r))\rangle_\Xi = h(\langle\varphi(u_1, \ldots, u_r)\rangle_\Lambda).$$
Proof. There is $C \in \Lambda^\delta$ containing all the ranges of $u_k$. Since $h \models C$ is complete, we have from Theorem 3.1
\[
\bigvee_{i=1}^{r} h(u_i(x_i)) \models \varphi(x_1, \ldots, x_r) = h\left(\bigvee_{i=1}^{r} u_i(x_i) \models \varphi(x_1, \ldots, x_r)\right)
\]
We have thus proved the lemma. \qed

For $u \in \hat{V}(X)^{\langle\Lambda\rangle}$, since $\text{supp}(h \circ u) \subseteq \text{supp} u$, we can define the induced map $h_* : \hat{V}(X)^{\langle\Lambda\rangle} \to \hat{V}(X)^{\langle\Lambda\rangle}$ similarly.

Corollary 4.2. Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}_\in$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \hat{V}(X)^{\langle\Lambda\rangle}$
\[
\varphi(h_*(u_1), \ldots, h_*(u_r)) \models \varphi(u_1, \ldots, u_r).
\]
Proof. Using Corollary 3.2, we see the proof is similar to that of Lemma 4.1. \qed

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters of $\Lambda$ and $\Xi$, respectively. Suppose $h^{-1}\mathcal{V} = \mathcal{U}$. Then we have from Lemma 4.1 or from Corollary 4.2
\[
[u = u']_{\Lambda} \in \mathcal{U} \iff [h_*(u) = h_*(u')]_{\Xi} \in \mathcal{V}
\]
Therefore we can define the injection $h_* : M^{\langle\Lambda\rangle}/\mathcal{U} \to M^{\langle\Xi\rangle}/\mathcal{V}$, denoted by the same $h_*$, by $h_*(u/\mathcal{U}) = h_*(u)/\mathcal{V}$. Since $\text{supp} h_*(u) \subseteq \text{supp} u$, we can define the injection $h_* : \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \to \hat{V}(X)^{\langle\Xi\rangle}/\mathcal{V}$ similarly.

Lemma 4.3. The injection $h_*$ is an elementary embedding of $M^{\langle\Lambda\rangle}/\mathcal{U}$ into $M^{\langle\Xi\rangle}/\mathcal{V}$.
Proof. Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. From Theorem 3.5, we have for $u_1, \ldots, u_r \in M^{\langle\Lambda\rangle}$
\[
h(\varphi(u_1, \ldots, u_r)_{\Lambda}) \in \mathcal{V} \iff \varphi(u_1, \ldots, u_r)_{\Lambda} \in h^{-1}\mathcal{V}
\]
\[
\varphi(h_*(u_1), \ldots, h_*(u_r))_{\Xi} \in \mathcal{V} \iff \varphi(u_1, \ldots, u_r)_{\Lambda} \in \mathcal{U}
\]
\[
M^{\langle\Xi\rangle}/\mathcal{V} \models \varphi(h_*(u_1/\mathcal{U}), \ldots, h_*(u_r/\mathcal{U})) \iff M^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1, \ldots, u_r).
\]
\qed

Corollary 4.4. The injection $h_*$ is a bounded elementary embedding of $\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ into $\hat{V}(X)^{\langle\Xi\rangle}/\mathcal{V}$.
Proof. Using Corollary 3.6, we see the proof is similar to that of Lemma 4.3. \qed

Let $I$ be a set relative to $V(X)$. We shall find a one-to-one correspondence between $\mathcal{P}(I)^{\langle\Lambda\rangle}$ and $\mathcal{B}(\Lambda[I])$. Note that $\mathcal{P}(I)^{\langle\Lambda\rangle}$ is the set of "the subsets of $I$ in $\hat{V}(X)^{\langle\Lambda\rangle}"$. For $A \in \mathcal{P}(I)^{\langle\Lambda\rangle}$, there is $C \in \Lambda^\delta$ such that $\text{rng} A \subseteq C$. Define $g : I \to \mathcal{B}(\Lambda)$ by $g(i) = [i \in A]_{\Lambda}$. Then we have $\text{rng} g \subseteq C$ and $g \in \mathcal{B}(\Lambda[I])$. Conversely, for $g \in \mathcal{B}(\Lambda[I])$, there is $C \in \Lambda^\delta$ such that $\text{rng} g \subseteq C$. Define $A : \mathcal{P}(I) \to C$ by
\[
A(x) = \bigwedge_{i \in I} \mathcal{S}_g(i, g(i)), \text{ where } \mathcal{S}_g(i, b) = \begin{cases} b & \text{if } i \in x, \\ \neg b & \text{if } i \in I \setminus x. \end{cases}
\]
Since $C$ is completely distributive, we have $A \in \mathcal{P}(I)^{\langle\Lambda\rangle}$. Suppose $g(i) = \llbracket i \in A \rrbracket_{\Lambda}$ and $g'(i) = \llbracket i \in A' \rrbracket_{\Lambda}$. Then we see $(g \wedge g')(i) = \llbracket i \in A \cap A' \rrbracket_{\Lambda}$ and $(\neg g)(i) = \llbracket i \in I \setminus A \rrbracket_{\Lambda}$. In the context above, the relation $g(i) = \llbracket i \in A \rrbracket_{\Lambda}$ sets up a one-to-one correspondence between $\mathcal{P}(I)^{\langle\Lambda\rangle}$ and $\mathcal{B}(\Lambda^{|I|})$ as Boolean algebras. From now on, we identify $\mathcal{P}(I)^{\langle\Lambda\rangle}$ with $\mathcal{B}(\Lambda^{|I|})$.

We shall define the special element $\delta \in I^{\langle\Lambda^{|I|}\rangle} \subseteq \hat{V}(X)^{\langle\Lambda^{|I|}\rangle}$ by

$$\delta(x)(i) = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{if } x \neq i. \end{cases}$$

We call the $\delta$ diagonal element of $I$ on $\Lambda$. Let $j : \Lambda \rightarrow \Lambda^{|I|}$ be the canonical embedding. Then $j$ is also a Boolean monomorphism of $\mathcal{B}(\Lambda)$ into $\mathcal{P}(I)^{\langle\Lambda\rangle}$. The diagonal element $\delta$ has following properties.

**Lemma 4.5.** The following statements hold.

1. $[j(b) = \check{I}]_{\Lambda} = b$ for every $b \in \mathcal{B}(\Lambda)$.
2. $[\delta \in j_{*}(g)]_{\Lambda^{|I|}} = g$ for every $g \in \mathcal{P}(I)^{\langle\Lambda\rangle}$.

**Proof.** Since $[i \in j(b)]_{\Lambda} = j(b)(i) = b$ for all $i \in I$, $[j(b) = \check{I}]_{\Lambda} = [j(b) \supseteq \check{I}]_{\Lambda} = \bigwedge_{i \in I} \mathbb{I} [i \in j(b)] = b$. From the definition of $\delta$, it is clear that $[\delta = i]_{\Lambda^{|I|}}(i) = \delta(i)(i) = 1$. Then we have $[\delta \in j_{*}(g)]_{\Lambda^{|I|}}(i) = [i \in j_{*}(g)]_{\Lambda^{|I|}}(i) = [i \in g]_{\Lambda} = g(i)$. \hfill $\Box$

**Theorem 4.6.** For any $v \in \hat{V}(X)^{\langle\Lambda^{|I|}\rangle}$, there is a map $w : \check{I} \rightarrow (\text{supp } v)^{\check{I}}$ in $\hat{V}(X)^{\langle\Lambda\rangle}$ such that $v = j_{*}(w)(\delta)$ holds in $\hat{V}(X)^{\langle\Lambda\rangle}$.

**Proof.** Since $\text{rng } v \in \Lambda^{|I|}$, $\bigcup_{g \in \text{rng } v} \text{rng } g \in \Lambda$. Therefore we can define $w : (\text{supp } v)^{\check{I}} \rightarrow \mathcal{B}(\Lambda)$ by

$$w(s) = \bigwedge_{i \in I} v(s(i))(i).$$

Then we get $w$ as required. First, we show $w \in \hat{V}(X)^{\langle\Lambda\rangle}$. If $s \neq s'$, then there is $i_0 \in I$ such that $s(i_0) \neq s'(i_0)$. Since $\text{rng } v$ is pairwise disjoint, we have

$$w(s) \wedge w(s') \leq v(s(i_0))(i_0) \wedge v(s'(i_0))(i_0) = 0.$$}

There is $C \in \Lambda^{|0}$ such that $\bigcup_{g \in \text{rng } v} \text{rng } g \subseteq C$. Then we have $\text{rng } w \subseteq C$ and then

$$\bigvee_{s \in (\text{supp } v)^{\check{I}}} w(s) = \bigvee_{s \in (\text{supp } v)^{\check{I}}} \bigwedge_{i \in I} v(s(i))(i) = \bigwedge_{i \in I} \bigvee_{y \in \text{supp } v} v(y)(i) = 1.$$
We have thus shown \( w \in \hat{V}(X)^{\Lambda} \). For each \( i \in I \), since \( w(s) \leq v(s(i))(i) \) holds for every \( s \in (\text{supp} v)^I \), we have
\[
\begin{align*}
[v = j_* (w)(\delta)]_{\Lambda[I]}(i) &= [v = j_* (w)(\tilde{i})]_{\Lambda[I]}(i) \\
&= \left( \bigvee \{v(y) \land j(w(s)) \land \tilde{i}(x) \mid y = s(x) \} \right)(i) \\
&= \bigvee_{s \in (\text{supp} v)^I} (v(s(i))(i) \land w(s)) \\
&= \bigvee_{s \in (\text{supp} v)^I} w(s) = 1.
\end{align*}
\]
We have thus proved the theorem. \( \square \)

Let \( \mathcal{U} \) be an ultrafilter of an LACA \( \Lambda \). A local ultralimit \( \rho: \hat{V}(X)^{\Lambda}/\mathcal{U} \rightarrow \star V(X) \) is a bounded elementary embedding satisfying \( \rho(x/\mathcal{U}) = x \) for every \( x \in V(X) \).

**Theorem 4.7** (Local Ultralimit Theorem). Let \( \rho: \hat{V}(X)^{\Lambda}/\mathcal{U} \rightarrow \star V(X) \) be a local ultralimit and let \( p \) be an internal element of \( \star V(X) \). Then there is a local ultralimit \( \tau: \hat{V}(X)^{\Lambda[I]}/\mathcal{V} \rightarrow \star V(X) \) such that the following conditions hold.

(i) The index set \( I \) is a set relative to \( V(X) \) and \( |I| = \text{nos}(p) \).

(ii) Let \( j: \Lambda \rightarrow \Lambda[I] \) be the canonical embedding. Then \( \mathcal{U} = j^{-1} \mathcal{U} \mathcal{V} \) and \( \rho = \tau \circ j_* \).

(iii) The submodel \( \text{rng} \tau \) of \( \star V(X) \) is the minimal bounded elementary submodel of \( \star V(X) \) that contains \( \{p\} \cup \text{rng} \rho \).

\[
\begin{array}{ccc}
\hat{V}(X)^{\Lambda[I]}/\mathcal{V} & \longrightarrow & \star V(X) \\
\downarrow \tau & & \downarrow \rho \\
V(X) & \longrightarrow & \hat{V}(X)^{\Lambda}/\mathcal{U}
\end{array}
\]

\( p \in \text{rng} \tau \).

**Proof.** Let \( I \) be a set relative to \( V(X) \) such that \( p \in \star I \) and \( |I| = \text{nos}(p) \). We have identified \( \mathcal{B}(\Lambda[I]) \) with \( \mathcal{P}(I)^{\Lambda} \). Define \( \mathcal{V} \subseteq \mathcal{B}(\Lambda[I]) \) by
\[
g \in \mathcal{V} \iff p \in \rho(g/\mathcal{U}).
\]
Then \( \mathcal{V} \) is an ultrafilter of \( \Lambda[I] \). Let \( b \) be an element of \( \mathcal{U} \). From (1) of Lemma 4.5, \( \rho(j(b)/\mathcal{U}) \) coincides \( *I \). Then we have \( j(b) \in \mathcal{V} \) from the definition of \( \mathcal{V} \). Since \( \mathcal{U} \) and \( \mathcal{V} \) are maximal filters, we obtain \( j^{-1} \mathcal{U} \mathcal{V} = \mathcal{U} \). Let \( \varphi(x_1, \ldots, x_r) \) be a \( \Delta_0 \)-formula of \( L_\mathcal{E} \) with only \( x_1, \ldots, x_r \) free. Let \( v_1, \ldots, v_r \) be elements of \( \hat{V}(X)^{\Lambda[I]} \). By Theorem 4.6, there are maps \( w_1, \ldots, w_r \) from \( \check{I} \) in \( \hat{V}(X)^{\Lambda[I]} \) such that \( v_k = j_*(w_k)(\delta) \) hold, where \( \delta \) is the diagonal element of \( I \) on \( \Lambda \). Putting \( g_0 = \{i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i)) \} \) in \( \hat{V}(X)^{\Lambda[I]} \), we have from (2) of Lemma 4.5
\[
g_0 = [\delta \in j_*(g_0)]_{\Lambda[I]}
\]
\[
= [\delta \in j_*(\{i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i))\})]_{\Lambda[I]}
\]
\[
= [\delta \in \{i \in \check{I} \mid \varphi(j_*(w_1)(i), \ldots, j_*(w_r)(i))\}]_{\Lambda[I]}
\]
\[
= [\varphi(j_*(w_1)(\delta), \ldots, j_*(w_r)(\delta))]_{\Lambda[I]}
\]
\[
= [\varphi(v_1, \ldots, v_r)]_{\Lambda[I]}
\]
and we have
\[ \hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \models \varphi_{\mathfrak{U}}(x_{1}, \ldots, x_{r}) \iff p \in \rho(g_{0}/\mathfrak{U}) \]
where
\[ p \in \rho(g_{0}/\mathfrak{U}) \iff \varphi(\rho(w_{1}/\mathfrak{U})(p), \ldots, \rho(w_{r}/\mathfrak{U})(p)) \subseteq \hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \models \varphi_{\mathfrak{U}}(x_{1}, \ldots, x_{r}) \]

By the definition of \( \varphi \), we obtain
\[ g_{0} \in \mathcal{V} \iff p \in \rho(g_{0}/\mathfrak{U}) \]

The case \( \varphi(x_{1}, x_{2}) \equiv "x_{1} = x_{2}" \) enables us to define the operation \( v/\mathcal{V} \mapsto \rho(w/\mathfrak{U})(p) \) where \( v = j_{*}(w)(\delta) \) holds in \( \hat{V}(X)^{\langle\Lambda\rangle} \). Thus, defining \( \tau: \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{V} \mapsto \hat{V}(X)^{\langle\Lambda\rangle} \)
by \( \tau(v/\mathcal{V}) = \rho(w/\mathfrak{U})(p) \) where \( v = j_{*}(w)(\delta) \) holds in \( \hat{V}(X)^{\langle\Lambda\rangle} \), we get \( \tau \) as required. In fact, it is clear in the preceding context that \( \tau \) is a bounded elementary embedding of \( \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{V} \) into \( \hat{V}(X)^{\langle\Lambda\rangle} \). Let \( u \) be the identity map on \( I \), then we see \( \tau(j_{*}(u/\mathfrak{U})(\delta/\mathcal{V})) = \rho(u/\mathfrak{U})(p) \).

The case \( \varphi(x_{1}, x_{2}) \equiv "x_{1} = x_{2}" \) enables us to define the operation \( v/\mathcal{V} \mapsto \rho(w/\mathfrak{U})(p) \) where \( v = j_{*}(w)(\delta) \) holds in \( \hat{V}(X)^{\langle\Lambda\rangle} \). Thus, defining \( \tau: \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{V} \mapsto \hat{V}(X)^{\langle\Lambda\rangle} \)
by \( \tau(v/\mathcal{V}) = \rho(w/\mathfrak{U})(p) \) where \( v = j_{*}(w)(\delta) \) holds in \( \hat{V}(X)^{\langle\Lambda\rangle} \), we get \( \tau \) as required. In fact, it is clear in the preceding context that \( \tau \) is a bounded elementary embedding of \( \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{V} \) into \( \hat{V}(X)^{\langle\Lambda\rangle} \). Let \( u \) be the identity map on \( I \), then we see \( \tau(j_{*}(u/\mathfrak{U})(\delta/\mathcal{V})) = \rho(u/\mathfrak{U})(p) \).

Therefore \( \tau \) is the minimum. We have completed the proof of Theorem 4.7. \( \square \)

Let \( \{j_{d}': \Lambda_{d} \rightarrow \Lambda_{d'}\}_{d \leq d', d, d' \in D} \) be an embedding system of LACAs with direct limit \( \{j_{d}: \Lambda_{d} \rightarrow \Lambda\}_{d \in D} \). Let \( \mathcal{U} \) be an ultrafilter of \( \Lambda \), then each \( \mathcal{U}_{d} = j_{d}^{-1}\mathcal{U} \) is an ultrafilter of \( \Lambda_{d} \).

**Theorem 4.8 (Elementary Net Theorem of Ultralimits).** Let \( \mathfrak{M} \) and \( \mathfrak{N} \) be models for \( \mathcal{L} \). Suppose there are elementary embeddings \( \tau_{d}: \mathfrak{M}^{\langle\Lambda\rangle}_{d \leftarrow \mathfrak{U}_{d}} \rightarrow \mathfrak{N} \) satisfying the condition \( \tau_{d} = \tau_{d'} \circ j_{d'}^{d} \) for \( d \leq d' \). Then there is an elementary embedding \( \tau: \mathfrak{M}^{\langle\Lambda\rangle}_{d \leftarrow \mathfrak{U}} \rightarrow \mathfrak{N} \) such that \( \tau_{d} = \tau \circ j_{d}^{d} \) for \( d \in D \).

**Proof.** Let \( v \) be an element of \( \mathfrak{M}^{\langle\Lambda\rangle} \). Since \( \text{rng} \, v \in \Lambda = \{j_{d}^{d} \in S \mid d \in D \text{ and } S \in \Lambda_{d}\} \)
from the definition of direct limits, there is \( u \in \mathfrak{M}^{\langle\Lambda_{d}\rangle} \) such that \( v = j_{d_{*}}^{d}(u) \).
Therefore defining \( \tau(v/\mathfrak{U}) = \tau_{d}(u/\mathfrak{U}_{d}) \) where \( v = j_{d_{*}}^{d}(u) \), we get \( \tau \) as required.

Let \( \varphi(x_{1}, \ldots, x_{r}) \) be a formula of \( \mathcal{L} \) with only \( x_{1}, \ldots, x_{r} \) free and let \( v_{1}, \ldots, v_{r} \)
be elements \( \mathfrak{M}^{\langle\Lambda\rangle} \). Then there are \( d \in D \) and \( u_{1}, \ldots, u_{r} \in \mathfrak{M}^{\langle\Lambda_{d}\rangle} \) such that \( v_{k} = j_{d_{*}}^{d}(u_{k}) \). We conclude as below.

\[
\mathfrak{M}^{\langle\Lambda_{d}\rangle}_{d \leftarrow \mathfrak{U}_{d}} \models \varphi(u_{1}/\mathfrak{U}_{d}, \ldots, u_{r}/\mathfrak{U}_{d}) \iff \mathfrak{N} \models \varphi(\tau_{d}(u_{1}/\mathfrak{U}), \ldots, \tau_{d}(u_{r}/\mathfrak{U}))
\]

\[
\mathfrak{M}^{\langle\Lambda\rangle}_{d \leftarrow \mathfrak{U}} \models \varphi(v_{1}/\mathfrak{U}, \ldots, v_{r}/\mathfrak{U}) \iff \mathfrak{N} \models \varphi(\tau(v_{1}/\mathfrak{U}), \ldots, \tau(v_{r}/\mathfrak{U})).
\]
Theorem 4.9 (Bounded Elementary Net Theorem of Bounded Ultralimits).
Suppose there are local ultralimits $\tau_d: \hat{V}(X)^{\langle\Lambda_d\rangle}/\mathfrak{U}_d \to \star V(X)$ satisfying the condition $\tau_d = \tau_{d'} \circ j_{d*}^{d'}$ for $d \leq d'$. Then there is a local ultralimit $\tau: \hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \to \star V(X)$ such that $\tau_d = \tau \circ j_{d*}$ for $d \in D$.

Proof. Similar to the proof of Theorem 4.8. \qed

We call the pair $\langle \Lambda, \mathfrak{U}\rangle$ in Theorem 4.8 or Theorem 4.7 the direct limit of $\{\langle \Lambda_d, \mathfrak{U}_d\rangle\}_{d \leq d', d, d' \in D}$.

Proof of Theorem 3.7 Let $\{p_\zeta\}_{\zeta < \kappa}$ be a sequence in $\star V(X)$ with $\kappa = \text{cov}(\star V(X))$.

We define local ultralimits $\{\rho_\zeta: \hat{V}^{\langle\Lambda_\zeta\rangle}/\mathfrak{U}_\zeta \to \star V(X)\}_{\zeta < \kappa}$ of $\star V(X)$ by:

$\Lambda_0 = \mathcal{P} \{0, 1\}$, $\mathfrak{U}_0 = \{1\}$.

$\Lambda_{\zeta+1} = \Lambda^{[I_\zeta]}$, where $|I_\zeta| = \text{nos}(p_\zeta)$, $p_\zeta \in \text{rng} \rho_{\zeta+1}$ in Theorem 4.7.

$\langle \Lambda_\lambda, \mathfrak{U}_\lambda \rangle$ is the direct limit of $\{\langle \Lambda_\zeta, \mathfrak{U}_\zeta\rangle_{\zeta < \lambda}\}$ in Theorem 4.9.

Then the direct limit $\langle \Lambda, \mathfrak{U}\rangle$ of $\{\langle \Lambda_\zeta, \mathfrak{U}_\zeta\rangle\}_{\zeta < \kappa}$ is an atlas of $\star V(X)$. \qed

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