<table>
<thead>
<tr>
<th>Title</th>
<th>NONSTANDARD UNIVERSE (Model Theory and Its Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Murakami, Masahiko</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1213: 39-49</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41162">http://hdl.handle.net/2433/41162</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
NONSTANDARD UNIVERSE

MASAHIKO MURAKAMI
DEPARTMENT OF MATHEMATICS
HOSEI UNIVERSITY

ABSTRACT. The nonstandard universes are frameworks of nonstandard analysis. We find sheaf representation for a nonstandard universe. in Theorem 3.7.

1. NONSTANDARD Universe

Definitions 1.1 (superstructure, base set). Given a set \( X \), we define the iterated power set \( V_n(X) \) over \( X \) recursively by
\[
V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).
\]
The superstructure \( V(X) \) is the union \( \bigcup_{n<\omega} V_n(X) \). The set \( X \) is said to be a base set if \( \emptyset \not\in X \) and each element of \( X \) is disjoint from \( V(X) \).

Definition 1.2 (nonstandard universe). A nonstandard universe is a triple \( \langle V(X), V(Y), \star \rangle \) such that:
1. \( X \) and \( Y \) are infinite base sets.
2. (Transfer Principle) The symbol \( \star \) is a map from \( V(X) \) into \( V(Y) \) such that
\[
V(X) \models \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad V(Y) \models \varphi(\star a_1, \ldots, \star a_n)
\]
holds for any bounded formula \( \varphi(x_1, \ldots, x_n) \) and \( a_1, \ldots, a_n \in V(X) \).
3. \( \star X = Y \).
4. For every infinite subset of \( A \) of \( X \), \( \{\star a \mid a \in A\} \) is a proper subset of \( \star A \).

Definitions 1.3 (standard, internal). For \( a \in V(\star X) \), we call a standard if there is an \( x \in V(X) \) such that \( a = \star x \).

For \( a \in V(\star X) \), we call a internal if there is an \( x \in V(X) \) such that \( a \in \star x \). We denote by \( \star V(X) \) the set of all internal elements in \( V(\star X) \).

From now on, we denote a nonstandard universe by single \( \star V(X) \).

Definitions 1.4 (norm, radius). The norm (of standardness) of an internal element \( a \) is a cardinal defined by
\[
\text{nos}(a) = \min \{|x| \mid a \in \star x\}.
\]
The radius of \( \star V(X) \) is a cardinal defined by
\[
\text{rad}(\star V(X)) = \min \{\kappa \mid \forall y \in \star V(X) \text{ nos}(y) < \kappa\}.
\]

Date: November 14, 2000.
Definition 1.5 (covering number). Let \( a \) be an internal element. The local ultra-power at \( a \) is defined by
\[
V(X)[a] = \{ (\star w)(a) \mid w \in V(X) \text{ and } a \in \star(\text{dom}(w)) \}.
\]
For a subset \( E \subseteq \star V(X) \), we denote
\[
V(X)[E] = \bigcup\{ V(X)[s] \mid s \text{ is a finite subset of } E \}.
\]
The covering number of \( \star V(X) \) is defined by
\[
\text{cov}(\star V(X)) = \min \{ |E| \mid E \subseteq \star V(X) \text{ and } V(X)[E] = \star V(X) \}.
\]

2. Locally atomic complete algebra

Definition 2.1 (regular complete subalgebra). Let \( \langle \mathfrak{B}, \wedge, \vee, \neg, 0_{\mathfrak{B}}, 1_{\mathfrak{B}} \rangle \) be a Boolean algebra. A subset \( C \subseteq \mathfrak{B} \) is said be a regular complete subalgebra of \( \mathfrak{B} \) if \( C \) is a complete subalgebra of \( \mathfrak{B} \) and the inclusion map is also complete.

Notation. Let \( \mathfrak{B} \) be a Boolean algebra. For a subset \( S \subseteq \mathfrak{P}(\mathfrak{B}) \), we denote
\[
S^{0} = \{ C \in S \mid C \text{ is a regular complete subalgebra of } \mathfrak{B} \}.
\]

Definition 2.2 (LCA). A locally complete algebra (LCA) is a set \( \Lambda \) of subsets of a Boolean algebra \( \mathfrak{B} \) satisfying the conditions below.

1. \( \bigcup \Lambda = \mathfrak{B} \).
2. If \( S_1, S_2 \in \Lambda \) then \( S_1 \cup S_2 \in \Lambda \).
3. If \( S \in \Lambda \) and \( T \subseteq S \) then \( T \in \Lambda \).
4. For every \( S \in \Lambda \), there is a \( C \in \Lambda^{0} \) containing \( S \).

For an LCA \( \Lambda \), we denote by \( \mathfrak{B}(\Lambda) \) the Boolean algebra \( \bigcup \Lambda \). We call the Boolean algebra \( \mathfrak{B}(\Lambda) \) the base Boolean algebra of \( \Lambda \).

Definition 2.3 (LACA). An LCA \( \Lambda \) is a locally atomic complete algebra (LACA) if every \( C \in \Lambda^{0} \) is atomic. We denote the set of atoms of \( C \in \Lambda^{0} \) by \( \text{Atom}(C) \).

Definition 2.4 (homomorphism). We introduce notation \( R^{u} S = \{ R^{u} S \mid S \in S \} \). Let \( \Lambda \) and \( \Xi \) be LCAs. A Boolean homomorphism \( f : \mathfrak{B}(\Lambda) \rightarrow \mathfrak{B}(\Xi) \) is a pseudo-homomorphism of LCAs if \( f^{u} \Lambda \subseteq \Xi \). We denotes a pseudo-homomorphism by \( f : \Lambda \rightarrow \Xi \). A pseudo-homomorphism \( h : \Lambda \rightarrow \Xi \) of LCAs is a (complete) homomorphism if \( \forall h^{u} S = h(\bigvee S) \) for all \( S \in \Lambda \). An embedding or monomorphism \( j : \Lambda \rightarrow \Xi \) is an injective homomorphism.

Definition 2.5 (subLCA). A subLCA of an LCA \( \Lambda \) is a nonempty subset of \( \Lambda \) which is itself an LCA and the inclusion map is an embedding.

Definition 2.6 (generator). Let \( \Lambda \) be an LCA. A subset \( \mathcal{G} \subseteq \Lambda^{0} \) is a generator of \( \Lambda \) or \( \mathcal{G} \) generates \( \Lambda \) if \( \Lambda \) is the only subLCA of \( \Lambda \) containing \( \mathcal{G} \).
Definitions 2.7 (radius, covering number, diameter). The radius of an LCA $\Lambda$ is a cardinal defined by

$$\text{rad}(\Lambda) = \min \{ \kappa \mid \forall C \in \Lambda \cap \text{Atom}(C) \mid \kappa \}$$

The covering number of an LCA $\Lambda$ is a cardinal defined by

$$\text{cov}(\Lambda) = \min \{ |S| \mid S \text{ is a generator of } \Lambda \}.$$ 

The diameter of an LCA $\Lambda$ is a cardinal defined by

$$\text{diam}(\Lambda) = \min \left\{ \sum_{C \in S} |\text{Atom}(C)| \mid S \text{ is a generator of } \Lambda \right\}.$$ 

Definition 2.8 (direct product). Let $I$ be an index set. The direct product $\Lambda^{|I|}$ of the LCA $\Lambda$ is defined by:

$$\Lambda^{|I|} = \left\{ S \subseteq \B(\Lambda)^I \mid \bigcup_{g \in S} \text{rng } g \in \Lambda \right\}$$

with the pointwise Boolean operations on $\B(\Lambda^{|I|}) = \bigcup \Lambda^{|I|} \subseteq \B(\Lambda)^I$. Then $\Lambda^{|I|}$ is an LCA. The LCA $\Lambda$ is embedded into $\Lambda^{|I|}$ by the canonical embedding $b \mapsto I \times \{ b \}$.

Definitions 2.9 (embedding system, direct limit). The embedding system of LCAs is a family of embeddings

$$\mathcal{E} = \{ j_{d}^{d'} : \Lambda_{d} \to \Lambda_{d'} \}_{d \leq d', d, d' \in D}$$

satisfying $j_{d}^{d''} \circ j_{d}^{d'} = j_{d}^{d''}$ for all $d \leq d' \leq d''$, where $D$ is an upper direct set. The direct limit of $\mathcal{E}$ is $\bigcup \{ j_{d}^{d' : \B(\Lambda_{d}) \to \B(\Lambda_{d'})} \}_{d \leq d', d', d' \in D}$ as Boolean algebras.

Definition 2.10 (ultrafilter). Let $\Lambda$ be an LCA. A subset $U$ of $\B(\Lambda)$ is an ultrafilter of an LCA $\Lambda$ if it is an ultrafilter of the base Boolean algebra $\B(\Lambda)$.

3. ULTRALIMIT

Definition 3.1 (LACA-valued model). Let $\Lambda$ be an LCA and let $M$ be a model for a language $L$. The $\B(\Lambda)$-valued universe of $M$ is defined by

$$M^{\langle\Lambda\rangle} = \left\{ u : M \to \B(\Lambda) \mid u(x) \wedge u(y) = 0 \text{ for } x \neq y, \text{ rng } u \in \Lambda, \bigvee \text{rng } u = 1 \right\}.$$ 

For $u \in M^{\langle\Lambda\rangle}$, the support of $u$ is a subset of $M$ defined by

$$\text{supp } u = \{ x \in M \mid u(x) \neq 0 \}.$$ 

To each function $F$ of $L(M)$ and each $u_1, \ldots, u_n \in M^{\langle\Lambda\rangle}$, we assign a $\tilde{F}(u_1, \ldots, u_n) \in M^{\langle\Lambda\rangle}$ by:

$$\tilde{F}(u_1, \ldots, u_n)(y) = \bigvee \left\{ \bigwedge_{i=1}^{n} u_i(x_i) \mid M \models y = F(x_1, \ldots, x_n) \right\} \text{ for } y \in M.$$
We regard a constant of $\mathcal{L}(M)$ as a function without any variables. Note that $\wedge_{i=1}^{n} u_{i}(x_{i}) = 1$ if $n = 0$. To each sentence $\varphi$ of $\mathcal{L}(M^{\langle \Lambda \rangle})$ we assign a truth value $[\varphi]_{\Lambda} \in \overline{\mathcal{B}(\Lambda)}$ by following recursive rules:

$$[u = v] = \bigvee \{u(x) \land v(x) \mid x \in M\},$$

$$[R(u_{1}, \ldots, u_{m})] = \bigvee \\left\{ \bigwedge_{i=1}^{m} u_{i}(x_{i}) \mid \mathfrak{M} \models R(x_{1}, \ldots, x_{m}) \right\},$$

$$[-\varphi] = -[\varphi],$$

$$[\varphi_{1} \lor \varphi_{2}] = [\varphi_{1}] \lor [\varphi_{2}],$$

$$[\exists x \varphi(x)] = \bigvee \{[\varphi(u)] \mid u \in M^{\langle \Lambda \rangle}\},$$

where $R$ is any predicate in $\mathcal{L}$.

**Definition 3.2** ($\text{LACA}$-valued superstructure). Let $\Lambda$ be an LACA. The $\Lambda$-valued superstructure of $V(X)$ is defined by

$$\hat{V}(X)^{\langle \Lambda \rangle} = \{u \in V(X)^{\langle \Lambda \rangle} \mid \text{supp } u \in V(X)\}.$$

While the truth values range over $\overline{\mathcal{B}(\Lambda)}$ on this definition, we shall see $[\varphi]_{\Lambda} \in \mathcal{B}(\Lambda)$.

**Theorem 3.1.** Let $\varphi(x_{1}, \ldots, x_{r})$ be a formula of $\mathcal{L}$ with only $x_{1}, \ldots, x_{r}$ free. For $u_{1}, \ldots, u_{r} \in M^{\langle \Lambda \rangle},$

$$[\varphi(u_{1}, \ldots, u_{r})]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathfrak{M} \models \varphi(x_{1}, \ldots, x_{r}) \right\}. \quad (*)$$

**Proof.** For $\varphi$ either "$x_{1} = x_{2}$" or $R$, $(*)$ holds by definition. If $(*)$ holds for an atomic formula $\varphi(x)$ then, by simple calculus of Boolean algebra, $(*)$ holds for $\varphi(F(x_{1}, \ldots, x_{n}))$. Thus, by induction, $(*)$ holds for $\varphi$ atomic. Suppose $(*)$ holds for $\varphi$, $\varphi_{1}$ and $\varphi_{2}$. Since there is an atomic $C \in \Lambda^{0}$ containing all the ranges of $u_{1}, \ldots, u_{r}$, and every range of $u_{i}$ is a partition of unity except for 0,

$$[-\varphi]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathfrak{M} \models -\varphi(x_{1}, \ldots, x_{r}) \right\}.$$ 

It is easy to see:

$$[\varphi_{1} \lor \varphi_{2}]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathfrak{M} \models \varphi_{1}(x_{1}, \ldots, x_{r}) \lor \varphi_{2}(x_{1}, \ldots, x_{r}) \right\}.$$ 

Since $[\varphi(u)]_{\Lambda} = \bigvee_{x \in M} (u(x) \land [\varphi(x)])$, we have $[\exists x \varphi(x)]_{\Lambda} = \bigvee_{x \in M} [\varphi(x)]_{\Lambda}$. Therefore $(*)$ holds for $\exists x \varphi(x)$. \hfill $\Box$

Similarly, we shall obtain the superstructure version.

**Corollary 3.2.** Let $\varphi(x_{1}, \ldots, x_{r})$ be a formula of $\mathcal{L}_{\in}$ with only $x_{1}, \ldots, x_{r}$ free. For $u_{1}, \ldots, u_{r} \in \hat{V}(X)^{\langle \Lambda \rangle},$

$$[\varphi(u_{1}, \ldots, u_{r})]_{\Lambda} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid V(X) \models \varphi(x_{1}, \ldots, x_{r}) \right\}.$$
By the theorem and the corollary above, we have a fundamental property
\[ u = v \land \varphi(u) \leq \varphi(v). \] We have just introduced \( B(\Lambda) \)-valued model
\( M^{\langle \Lambda \rangle} = (M^{\langle \Lambda \rangle}, \check{R}, \check{F}, \check{c}) \) and \( B(\Lambda) \)-valued superstructure \( \hat{V}(X)^{\langle \Lambda \rangle} \). We say that
a sentence \( \varphi \) of \( L(M^{\langle \Lambda \rangle}) \) holds in \( M^{\langle \Lambda \rangle} \) if \( \varphi \Lambda = 1 \) and that a sentence \( \psi \) of
\( \mathcal{L}(\hat{V}(X)^{\langle \Lambda \rangle}) \) holds in \( \hat{V}(X)^{\langle \Lambda \rangle} \) if \( \psi \Lambda = 1 \). Theorem 3.1 and Corollary 3.2
follow that we consider the values \( u(x) \) only for \( x \in \text{supp} u \). For \( E \subseteq M \), we may
regard \( E^{\langle \Lambda \rangle} \) as a subset of \( M^{\langle \Lambda \rangle} \) by extending the domain of \( u \in E^{\langle \Lambda \rangle} \) to \( M \).
This means that we define for \( u \in E^{\langle \Lambda \rangle} \)
\[ u(x) = 0 \quad \text{if } x \notin E. \]
In the superstructure version, if \( E \) is a set relative to \( V(X) \) then we may assume
\[ E^{\langle \Lambda \rangle} = \{ u \in \hat{V}(X)^{\langle \Lambda \rangle} \mid u \in \hat{E} \text{ holds in } \hat{V}(X)^{\langle \Lambda \rangle} \}. \]

**Theorem 3.3 (Maximum principle).** Let \( \varphi(x) \) be a formula of \( L(M^{\langle \Lambda \rangle}) \) with only
\( x \) free. Then there is \( u \in M^{\langle \Lambda \rangle} \) such that \( \varphi(u) \Lambda = \exists x \varphi(x) \Lambda \).

**Proof.** Let \( \{ a_{\zeta} \}_{\zeta < \alpha} \) be a well-ordering for \( M \). By theorem 3.1, there is \( C \in \Lambda^{0} \)
containing \( \{ \varphi(x) \mid x \in M \} \). Putting \( b_{\zeta} = [\varphi(a_{\zeta})] \wedge \supseteq [\varphi(a_{\zeta})] \), we have
\( \{ b_{\zeta} \}_{\zeta < \alpha} \subseteq C \). Since \( \{ b_{\zeta} \}_{\zeta < \alpha} \) is a pairwise disjoint family, we can pick \( u \in M^{\langle \zeta \rangle} \)
with \( u(a_{\zeta}) \geq b_{\zeta} \). Then \( \varphi(u) \geq u(a_{\zeta}) \wedge [\varphi(a_{\zeta})] \geq b_{\zeta} \) for any \( \zeta < \alpha \). Since
\( \exists x \varphi(x) = \supseteq \zeta < \alpha \varphi(a_{\zeta}) = \supseteq \zeta < \alpha b_{\zeta} \), we have \( \varphi(u) \geq [\exists x \varphi(x)] \).

**Corollary 3.4.** Let \( \varphi(x) \) be a formula of \( L(\hat{V}(X)^{\langle \Lambda \rangle}) \) with only \( x \) free and let \( v \)
be an element of \( \hat{V}(X)^{\langle \Lambda \rangle} \). Then there is \( u \in \hat{V}(X)^{\langle \Lambda \rangle} \) such that \( u \in v \wedge \varphi(u) \Lambda = \exists x \in v \varphi(x) \Lambda \).

**Proof.** Since there is \( n \) such that \( \text{supp} v \subseteq V_{n+1}(X) \),
\[ [x \in v] = \supseteq \{ v(y) \mid x \in y \in \text{supp} v \} = 0 \quad \text{for } x \notin V_{n}(X). \]
Therefore we can choose \( u \) whose support is a subset of \( V_{n}(X) \).

**Definition 3.3 (ultralimit).** We denote by \( u/\mathcal{U} \) the equivalence class of \( u \in M^{\langle \Lambda \rangle} \)
by the equivalence relation
\[ x \sim u y \equiv [x = y]_{\Lambda} \in \mathcal{U}. \]
The ultralimit \( M^{\langle \Lambda \rangle}/\mathcal{U} \) of \( M \) modulo \( \mathcal{U} \) of \( \Lambda \) is defined by:
\begin{align*}
M^{\langle \Lambda \rangle}/\mathcal{U} & = \{ u/\mathcal{U} \mid u \in M^{\langle \Lambda \rangle} \}. \\
\mathcal{U}(u_{1}/\mathcal{U}, \ldots, u_{n}/\mathcal{U}) & = (\mathcal{U}(u_{1}, \ldots, u_{n}))/\mathcal{U}. \\
M^{\langle \Lambda \rangle}/\mathcal{U} & \models \mathcal{R}(u_{1}/\mathcal{U}, \ldots, u_{m}/\mathcal{U}) \iff [\mathcal{R}(u_{1}, \ldots, u_{m})] \in \mathcal{U}.
\end{align*}

**Definition 3.4 (bounded ultralimit).** We denote by \( u/\mathcal{U} \) the equivalence class of
\( u \in \hat{V}(X)^{\langle \Lambda \rangle} \) by the equivalence relation
\[ x \sim u y \equiv [x = y]_{\Lambda} \in \mathcal{U}. \]
The bounded ultralimit $\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ of $V(X)$ modulo $\mathcal{U}$ of $\Lambda$ is defined by:

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} = \{ u/\mathcal{U} \mid u \in \hat{V}(X)^{\langle\Lambda\rangle} \},$$

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models u/\mathcal{U} \in v/\mathcal{U} \text{ iff } [u \in v] \in \mathcal{U}.$$  

**Theorem 3.5 (Łoś Principle of Ultraproducts).** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \mathcal{M}^{\langle\Lambda\rangle}$,

$$\mathcal{M}^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \ldots, u_r/\mathcal{U}) \text{ iff } \llbracket \varphi(u_1, \ldots, u_r) \rrbracket \in \mathcal{U}.$$  

**Proof.** The proof proceeds by induction on the complexity of formulae. The only nontrivial case is the case where $\varphi$ is of the form $\exists x \psi(x)$. Suppose $\llbracket \exists x \psi(x) \rrbracket \in \mathcal{U}$. By the maximal principle (Theorem 3.3), there is $u$ satisfying $\llbracket \psi(u) \rrbracket = \llbracket \exists x \psi(x) \rrbracket$. Then $\mathcal{M}^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U})$ by the induction assumption. We have thus $\mathcal{M}^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Conversely, suppose $\mathcal{M}^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Then there is some $u$ such that $\mathcal{M}^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U})$. By the induction assumption, $\llbracket \exists x \psi(x) \rrbracket \geq \llbracket \psi(u/\mathcal{U}) \rrbracket \in \mathcal{U}$. 

**Corollary 3.6 (Łoś-Mostowski Principle of Bounded Ultraproducts).**

Let $\varphi(x_1, \ldots, x_r)$ be a $\Delta_0$-formula of $\mathcal{L}_\infty$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \hat{V}(X)^{\langle\Lambda\rangle},$

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \ldots, u_r/\mathcal{U}) \text{ iff } \llbracket \varphi(u_1, \ldots, u_r) \rrbracket \in \mathcal{U}.$$  

**Proof.** The proof is similar to that of Theorem 3.5. The only different part is the if-part of the case where $\varphi$ is of the form $\exists x \subset y \psi(x)$. Suppose $\llbracket \exists x \subset u \psi(x) \rrbracket \in \mathcal{U}$. It follows from Corollary 3.4 that there is $u \in \hat{V}(X)^{\langle\Lambda\rangle}$ satisfying $\llbracket u \subset u \land \psi(u) \rrbracket = \llbracket \exists x \subset u \psi(x) \rrbracket$. 

A bounded ultralimit is a pre-nonstandard universe: that satisfies (1),(2) and (3) of Definition 1.2 with Mostowski collapsing.

**Definition 3.5 (atlas).** An atlas is a pair $\langle \Lambda, \mathcal{U} \rangle$ of an LACA $\Lambda$ and an ultrafilter of $\Lambda$ such that $\text{rad}(\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{rad}(\Lambda)$ and $\text{cov}(\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{cov}(\Lambda)$.

**Theorem 3.7 (Sheaf representation Theorem for Nonstandard Universes).** For any nonstandard universe $\mathcal{V}(X)$, there is an atlas $\langle \Lambda, \mathcal{U} \rangle$ such that $\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ is isomorphic to $\mathcal{V}(X)$.

We prove the theorem in the next section.

**4. LOCAL ULTRALIMITS**

We shall see that a homomorphism of LACAs induces an elementary embedding of ultralimits and a bounded elementary embedding of bounded ultralimits. Let $h : \Lambda \rightarrow \Xi$ be a homomorphism. The induced map $h_* : \mathcal{M}^{\langle\Lambda\rangle} \rightarrow \mathcal{M}^{\langle\Xi\rangle}$ is defined by $h_*(u) = h \circ u$. Then we have the lemma below.

**Lemma 4.1.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \mathcal{M}^{\langle\Lambda\rangle}$

$$\llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_{\Xi} = h(\llbracket \varphi(u_1, \ldots, u_r) \rrbracket_{\Lambda}).$$
Proof. There is $C \in \Lambda^0$ containing all the ranges of $u_k$. Since $h\upharpoonright C$ is complete, we have from Theorem 3.1
\[
\bigvee \{ \bigwedge_{i=1}^{r} h(u_i(x_i)) \mid \mathfrak{M} \models \varphi(x_1, \ldots, x_r) \} = h(\bigvee \{ \bigwedge_{i=1}^{r} u_i(x_i) \mid \mathfrak{M} \models \varphi(x_1, \ldots, x_r) \})
\]

\[
\llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_{\Xi} = h(\llbracket \varphi(u_1, \ldots, u_r) \rrbracket_{\Lambda}^\Lambda).
\]

We have thus proved the lemma.

For $u \in \hat{V}(X)^{\llangle \Lambda \rrangle}$, since $\text{supp}(h \circ u) \subseteq \text{supp} u$, we can define the induced map $h_* : \hat{V}(X)^{\llangle \Lambda \rrangle} \to \hat{V}(X)^{\llangle \Lambda \rrangle}$ similarly.

**Corollary 4.2.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}_{\Xi}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \hat{V}(X)^{\llangle \Lambda \rrangle}$
\[
\llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_{\Xi} = h(\llbracket \varphi(u_1, \ldots, u_r) \rrbracket_{\Lambda}^\Lambda).
\]

**Proof.** Using Corollary 3.2, we see the proof is similar to that of Lemma 4.1.

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters of $\Lambda$ and $\Xi$, respectively. Suppose $h^{-1}u \mathcal{V} = \mathcal{U}$. Then we have from Lemma 4.1 or from Corollary 4.2
\[
[u = u']^\Lambda \in \mathcal{U} \quad \text{iff} \quad [h_*(u) = h_*(u')]^\Xi \in \mathcal{V}.
\]

Therefore we can define the injection $h_* : M^{\llangle \Lambda \rrangle}/\mathcal{U} \to M^{\llangle \Xi \rrangle}/\mathcal{V}$, denoted by same $h_*$, by $h_*(u/\mathcal{U}) = h_*(u)/\mathcal{V}$. Since $\text{supp} h_*(u) \subseteq \text{supp} u$, we can define the injection $h_* : \hat{V}(X)^{\llangle \Lambda \rrangle}/\mathcal{U} \to \hat{V}(X)^{\llangle \Xi \rrangle}/\mathcal{V}$ similarly.

**Lemma 4.3.** The injection $h_*$ is an elementary embedding of $M^{\llangle \Lambda \rrangle}/\mathcal{U}$ into $M^{\llangle \Xi \rrangle}/\mathcal{V}$.

**Proof.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. From Theorem 3.5, we have for $u_1, \ldots, u_r \in M^{\llangle \Lambda \rrangle}$
\[
h(\llbracket \varphi(u_1, \ldots, u_r) \rrbracket_{\Lambda}^\Lambda) \in \mathcal{V} \quad \text{iff} \quad \llbracket \varphi(u_1, \ldots, u_r) \rrbracket_{\Lambda}^\Lambda \in h^{-1}u \mathcal{V}
\]
\[
\llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_{\Xi} \in \mathcal{V} \quad \text{iff} \quad \llbracket \varphi(u_1, \ldots, u_r) \rrbracket_{\Lambda}^\Lambda \in \mathcal{U}
\]
\[
\mathcal{M}^{\llangle \Xi \rrangle}/\mathcal{V} \models \varphi(h_*(u_1/\mathcal{U}), \ldots, h_*(u_r/\mathcal{U})) \quad \text{iff} \quad \mathcal{M}^{\llangle \Lambda \rrangle}/\mathcal{U} \models \varphi(u_1, \ldots, u_r).
\]

**Corollary 4.4.** The injection $h_*$ is a bounded elementary embedding of $\hat{V}(X)^{\llangle \Lambda \rrangle}/\mathcal{U}$ into $\hat{V}(X)^{\llangle \Xi \rrangle}/\mathcal{V}$.

**Proof.** Using Corollary 3.6, we see the proof is similar to that of Lemma 4.3.

Let $I$ be a set relative to $V(X)$. We shall find a one-to-one correspondence between $\mathcal{P}(I)^{\llangle \Lambda \rrangle}$ and $\mathcal{B}(\Lambda[I])$. Note that $\mathcal{P}(I)^{\llangle \Lambda \rrangle}$ is the set of "the subsets of $I$ in $\hat{V}(X)^{\llangle \Lambda \rrangle}"$. For $A \in \mathcal{P}(I)^{\llangle \Lambda \rrangle}$, there is $C \in \Lambda^0$ such that $\text{rng} A \subseteq C$. Define $g : I \to \mathcal{B}(\Lambda)$ by $g(i) = [i \in A]^\Lambda$. Then we have $\text{rng} g \subseteq C$ and $g \in \mathcal{B}(\Lambda[I])$. Conversely, for $g \in \mathcal{B}(\Lambda[I])$, there is $C \in \Lambda^0$ such that $\text{rng} g \subseteq C$. Define $A : \mathcal{P}(I) \to C$ by
\[
A(x) = \bigwedge_{i \in I} \text{sg}_x(i, g(i)), \quad \text{where} \quad \text{sg}_x(i, b) = \begin{cases} 
1 & \text{if } i \in x, \\
0 & \text{if } i \in I \setminus x.
\end{cases}
\]
Since $C$ is completely distributive, we have $A \in \mathcal{P}(I)^{\langle\Lambda\rangle}$. Suppose $g(i) = [i \in A]_\Lambda$ and $g'(i) = [i \in A']_\Lambda$. Then we see $(g \land g')(i) = [i \in A \cap A']_\Lambda$ and $\neg g(i) = [i \in I \setminus A]_\Lambda$. In the context above, the relation $g(i) = [i \in A]_\Lambda$ sets up a one-to-one correspondence between $\mathcal{P}(I)^{\langle\Lambda\rangle}$ and $\mathcal{B}(\Lambda^{[I]})$ as Boolean algebras. From now on, we identify $\mathcal{P}(I)^{\langle\Lambda\rangle}$ with $\mathcal{B}(\Lambda^{[I]})$.

We shall define the special element $\delta \in I^{\langle\Lambda\rangle} \subseteq \hat{V}(X)^{\langle\Lambda\rangle}$ by

$$\delta(x)(i) = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{if } x \neq i. \end{cases}$$

We call the $\delta$ diagonal element of $I$ on $\Lambda$. Let $j : \Lambda \to \Lambda^{[I]}$ be the canonical embedding. Then $j$ is also a Boolean monomorphism of $\mathcal{B}(\Lambda)$ into $\mathcal{P}(I)^{\langle\Lambda\rangle}$. The diagonal element $\delta$ has following properties.

**Lemma 4.5.** The following statements hold.

1. $[j(b) = \check{1}]_\Lambda = b$ for every $b \in \mathcal{B}(\Lambda)$.
2. $[\delta \in j_* (g)]_{\Lambda^{[I]}} = g$ for every $g \in \mathcal{P}(I)^{\langle\Lambda\rangle}$.

**Proof.** Since $[i \in j(b)]_\Lambda = j(b)(i) = b$ for all $i \in I$, $[j(b) = \check{1}]_\Lambda = [j(b) \supseteq \check{1}]_\Lambda = \bigwedge_{i \in I} [i \in j(b)]_\Lambda = b$. From the definition of $\delta$, it is clear that $[\delta = [i]_{\Lambda^{[I]}}(i) = \delta(i)(i) = 1$. Then we have $[\delta \in j_* (g)]_{\Lambda^{[I]}} = [i \in j_* (g)]_{\Lambda^{[I]}}(i) = [i \in g]_\Lambda = g(i)$. $\square$

**Theorem 4.6.** For any $v \in \hat{V}(X)^{\langle\Lambda\rangle}$, there is a map $w : \check{1} \to (\text{supp } v)^\vee$ in $\hat{V}(X)^{\langle\Lambda\rangle}$ such that $v = j_*(w)(\delta)$ holds in $\hat{V}(X)^{\langle\Lambda\rangle}$.

**Proof.** Since $\text{rng } v \in \Lambda^{[I]}$, $\bigcup_{g \in \text{rng } v} \text{rng } g \in \Lambda$. Therefore we can define $w : (\text{supp } v)^I \to \mathcal{B}(\Lambda)$ by

$$w(s) = \bigwedge_{i \in I} v(s(i))(i).$$

Then we get $w$ as required. First, we show $w \in \hat{V}(X)^{\langle\Lambda\rangle}$. If $s \neq s'$, then there is $i_0 \in I$ such that $s(i_0) \neq s'(i_0)$. Since $\text{rng } v$ is pairwise disjoint, we have

$$w(s) \land w(s') \leq v(s(i_0))(i_0) \land v(s'(i_0))(i_0) = 0.$$

There is $C \in \Lambda^\circ$ such that $\bigcup_{g \in \text{rng } v} \text{rng } g \subseteq C$. Then we have $\text{rng } w \subseteq C$ and then

$$\bigvee_{s \in (\text{supp } v)^I} w(s) = \bigvee_{s \in (\text{supp } v)^I} \bigwedge_{i \in I} v(s(i))(i) = \bigwedge_{i \in I} \bigvee_{y \in \text{supp } v} v(y)(i) = 1.$$
We have thus shown $w \in \hat{V}(X)^{\langle\Lambda\rangle}$. For each $i \in I$, since $w(s) \leq v(s(i))(i)$ holds for every $s \in (\text{supp } v)^I$, we have
\[
\begin{align*}
\ll v = j_*(w)(\delta) \rr_{\Lambda^I}(i) &= \ll v = j_*(w)(\check{i}) \rr_{\Lambda^I}(i) \\
&= \left( \bigvee \{ v(y) \land j(w(s)) \land \check{i}(x) \mid y = s(x) \} \right)(i) \\
&= \bigvee_{s \in (\text{supp } v)^I} (v(s(i))(i) \land w(s)) \\
&= \bigvee_{s \in (\text{supp } v)^I} w(s) = 1.
\end{align*}
\]
We have thus proved the theorem. \hfill \square

Let $\mathcal{U}$ be an ultrafilter of an LACA $\Lambda$. A local ultralimit $\rho : \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \to ^*V(X)$ is a bounded elementary embedding satisfying $\rho(\hat{x}/\mathcal{U}) = ^*x$ for every $x \in V(X)$.

**Theorem 4.7 (Local Ultralimit Theorem).** Let $\rho : \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \to ^*V(X)$ be a local ultralimit and let $p$ be an internal element of $^*V(X)$. Then there is a local ultralimit $\tau : \hat{V}(X)^{\langle\Lambda^I\rangle}/\mathcal{V} \to ^*V(X)$ such that the following conditions hold.

(i) The index set $I$ is a set relative to $V(X)$ and $|I| = \text{nos}(p)$.

(ii) Let $j : \Lambda \to \Lambda^I$ be the canonical embedding. Then $\mathcal{U} = j^{-1}\mathcal{V}$ and $\rho = \tau \circ j_*$.

(iii) The submodel $\text{rng} \tau$ of $^*V(X)$ is the minimal bounded elementary submodel of $^*V(X)$ that contains $\{p\} \cup \text{rng } \rho$.

\[
\begin{array}{c}
\hat{V}(X)^{\langle\Lambda^I\rangle}/\mathcal{V}  \\
\text{-----} \rho  \\
V(X) \quad \quad j_* \quad \quad \tau  \\
\end{array}
\]

$$g \in \mathcal{V} \iff p \in \rho(g/\mathcal{U}).$$

Then $\mathcal{V}$ is an ultrafilter of $\Lambda^I$. Let $b$ be an element of $\mathcal{U}$. From (1) of Lemma 4.5, $\rho(j(b)/\mathcal{U})$ coincides $^*b$. Then we have $j(b) \in \mathcal{V}$ from the definition of $\mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ are maximal filters, we obtain $j^{-1}\mathcal{V} = \mathcal{U}$. Let $\varphi(x_1, \ldots, x_r)$ be a $\Delta_0$-formula of $L_{\mathcal{E}}$ with only $x_1, \ldots, x_r$ free. Let $v_1, \ldots, v_r$ be elements of $\hat{V}(X)^{\langle\Lambda\rangle}$. By Theorem 4.6, there are maps $w_1, \ldots, w_r$ from $\check{I}$ in $\hat{V}(X)^{\langle\Lambda\rangle}$ such that $v_k = j_*(w_k)(\delta)$ hold, where $\delta$ is the diagonal element of $I$ on $\Lambda$. Putting $g_0 = \{i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i))\}$ in $\hat{V}(X)^{\langle\Lambda\rangle}$, we have from (2) of Lemma 4.5
\[
\begin{align*}
g_0 &= \ll \delta \in j_*(g_0) \rr_{\Lambda^I} \\
&= \ll \delta \in \{i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i))\} \rr_{\Lambda^I} \\
&= \ll \delta \in \{i \in \check{I} \mid \varphi(j_*(w_1)(i), \ldots, j_*(w_r)(i))\} \rr_{\Lambda^I} \\
&= \ll \varphi(j_*(w_1)(\delta), \ldots, j_*(w_r)(\delta)) \rr_{\Lambda^I} \\
&= \ll \varphi(v_1, \ldots, v_r) \rr_{\Lambda^I}
\end{align*}
\]
and we have
\[ \hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \models \varphi(g_0/\mathfrak{U}) = \{ i \in \mathcal{I} : \varphi((w_1/\mathfrak{U})(i), \ldots, (w_r/\mathfrak{U})(i)) \} \]
\[ \rho(g_0/\mathfrak{U}) = \{ i \in \mathcal{I} : \varphi(\rho(w_1/\mathfrak{U})(i), \ldots, \rho(w_r/\mathfrak{U})(i)) \}. \]

By the definition of \( \mathcal{V} \), we obtain
\[ g_0 \in \mathcal{V} \iff p \in \rho(g_0/\mathfrak{U}) \]
\[ [\varphi(v_1, \ldots, v_r)]_{\lambda[I]} \in \mathcal{V} \iff \varphi(\rho(w_1/\mathfrak{U})(p), \ldots, \rho(w_r/\mathfrak{U})(p)). \]

The case \( \varphi(x_1, x_2) \equiv "x_1 = x_2" \) enables us to define the operation \( v/\mathcal{V} = \rho(w/\mathcal{V})(p) \) where \( v = j_*= (w)(\delta) \) holds in \( \hat{V}(X)^{\langle\Lambda[I]\rangle}/\mathfrak{V} \). Thus, defining \( \tau: \hat{V}(X)^{\langle\Lambda[I]\rangle}/\mathfrak{V} \to \mathcal{V}(X) \) by \( \tau(v/\mathcal{V}) = \rho(w/\mathcal{V})(p) \) where \( v = j_*(w)(\delta) \) holds in \( \hat{V}(X)^{\langle\Lambda[I]\rangle}/\mathfrak{V} \), we get \( \tau \) as required. In fact, it is clear in the preceding context that \( \tau \) is a bounded elementary embedding of \( \hat{V}(X)^{\langle\Lambda[I]\rangle}/\mathfrak{V} \) into \( \hat{V}(X)^{\langle\Lambda[I]\rangle}/\mathfrak{V} \). Let \( \check{u} \) be the identity map on \( I \), then we see \( \check{u}(i/\mathfrak{U})(\delta/\mathcal{V}) = \rho(i/\mathfrak{U})(p) = u(p) = p \). For \( u/\mathfrak{U} \in \hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \), let \( \tilde{u} \) be the constant map from \( \mathcal{I} \) onto \( \{ u \} \) in \( \hat{V}(X)^{\langle\Lambda\rangle} \), then we have \( \check{u}(u/\mathfrak{U})(\delta) = \rho(\tilde{u}/\mathfrak{U})(p) = \rho(u/\mathfrak{U}) \).

Suppose a bounded elementary submodel \( W \) of \( \mathcal{V}(X) \) contains \( \{ p \} \cup \text{rng} \rho \). From the definition of \( \tau, \tau(v/\mathcal{V}) = \rho(w/\mathcal{V})(p) \in W \) for some \( w/\mathfrak{U} \in \hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \). Therefore \( \text{rng} \tau \) is the minimum. We have completed the proof of Theorem 4.7.

Let \( \{ j_{d^d}^d : \Lambda_d \to \Lambda_{d^d} \}_{d \leq d^d, d, d^d \in D} \) be an embedding system of LACAs with direct limit \( \{ j_d : \Lambda_d \to \Lambda \}_{d \in D} \). Let \( \mathfrak{U} \) be an ultrafilter of \( \Lambda \), then each \( \mathfrak{U}_d = j_{d}^{-1}\text{rngr} \mathfrak{U} \) is an ultrafilter of \( \Lambda_d \).

**Theorem 4.8 (Elementary Net Theorem of Ultralimits).** Let \( \mathfrak{M} \) and \( \mathfrak{N} \) be models for \( \mathcal{L} \). Suppose there are elementary embeddings \( \tau_d: \mathfrak{M}^{\langle\Lambda_d\rangle}/\mathfrak{U}_d \to \mathfrak{N} \) satisfying the condition \( \tau_d = \tau_{d^d} \circ j_{d^d}^d \) for \( d \leq d^d \). Then there is an elementary embedding \( \tau: \mathfrak{M}^{\langle\Lambda\rangle}/\mathfrak{U} \to \mathfrak{N} \) such that \( \tau_d = \tau \circ j_{d^d} \) for \( d \in D \).

**Proof.** Let \( v \) be an element of \( \mathfrak{M}^{\langle\Lambda\rangle} \). Since \( \text{rng} v \in \Lambda = \{ j_d^d S \mid d \in D \text{ and } S \in \Lambda_d \} \) from the definition of direct limits, there is \( u \in \mathfrak{M}^{\langle\Lambda_d\rangle} \) such that \( v = j_{d^d}^d(u) \). Therefore defining \( \tau(v/\mathfrak{U}) = \tau_d(u/\mathfrak{U}_d) \) where \( v = j_{d^d}^d(u) \), we get \( \tau \) as required.

Let \( \varphi(x_1, \ldots, x_r) \) be a formula of \( \mathcal{L} \) with only \( x_1, \ldots, x_r \) free and let \( v_1, \ldots, v_r \) be elements \( \mathfrak{M}^{\langle\Lambda\rangle} \). Then there are \( d \in D \) and \( u_1, \ldots, u_r \in \mathfrak{M}^{\langle\Lambda_d\rangle} \) such that \( v_k = j_{d^d}^d(u_k) \). We conclude as below.

\[ \mathfrak{M}^{\langle\Lambda_d\rangle}/\mathfrak{U}_d \models \varphi(u_1/\mathfrak{U}_d, \ldots, u_r/\mathfrak{U}_d) \iff \mathfrak{M} \models \varphi(\tau_d(u_1/\mathfrak{U}), \ldots, \tau_d(u_r/\mathfrak{U})) \]
\[ \mathfrak{M}^{\langle\Lambda\rangle}/\mathfrak{U} \models \varphi(v_1/\mathfrak{U}, \ldots, v_r/\mathfrak{U}) \iff \mathfrak{M} \models \varphi(\tau(v_1/\mathfrak{U}), \ldots, \tau(v_r/\mathfrak{U})). \]
Theorem 4.9 (Bounded Elementary Net Theorem of Bounded Ultralimits).
Suppose there are local ultralimits $\tau_d: \hat{V}(X)^{\langle\Lambda_d\rangle}/\mathfrak{U}_d \to \ast V(X)$ satisfying the condition $\tau_d = \tau_{d'} \circ j_{d*}^{d'}$ for $d \leq d'$. Then there is a local ultralimit $\tau: \hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \to \ast V(X)$ such that $\tau_d = \tau \circ j_{d*}$ for $d \in D$.

Proof. Similar to the proof of Theorem 4.8. \qed

We call the pair $\langle \Lambda, \mathfrak{U} \rangle$ in Theorem 4.8 or Theorem 4.7 the direct limit of $\{(\Lambda_d, \mathfrak{U}_d)\}_{d \leq d', d, d' \in D}$.

Proof of Theorem 3.7 Let $\{p_\zeta\}_{\zeta < \kappa}$ be a sequence in $\ast V(X)$ with $\kappa = \text{cov}(\ast V(X))$. We define local ultralimits $\{\rho_\zeta: \hat{V}^{\langle\Lambda_\zeta\rangle}/\mathfrak{U}_\zeta \to \ast V(X)\}_{\zeta < \kappa}$ of $\ast V(X)$ by:

$\Lambda_0 = \mathcal{P}(\{0, 1\})$, $\mathfrak{U}_0 = \{1\}$.
$\Lambda_{\zeta+1} = \Lambda^{[I_\zeta]}$, where $|I_\zeta| = \text{nos}(p_\zeta)$, $p_\zeta \in \text{rng} \rho_{\zeta+1}$ in Theorem 4.7.
$\langle \Lambda_\lambda, \mathfrak{U}_\lambda \rangle$ is the direct limit of $\{(\Lambda_\zeta, \mathfrak{U}_\zeta)_{\zeta < \lambda}\}$ in Theorem 4.9.

Then the direct limit $\langle \Lambda, \mathfrak{U} \rangle$ of $\{(\Lambda_\zeta, \mathfrak{U}_\zeta)\}_{\zeta < \kappa}$ is an atlas of $\ast V(X)$. \qed

References


E-mail address: muramasa@ms.u-tokyo.ac.jp