NONSTANDARD UNIVERSE

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ABSTRACT. The nonstandard universes are frameworks of nonstandard analysis. We find sheaf representation for a nonstandard universe in Theorem 3.7.

1. NONSTANDARD Universe

Definitions 1.1 (superstructure, base set). Given a set $X$, we define the iterated power set $V_n(X)$ over $X$ recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The superstructure $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set $X$ is said to be a base set if $\emptyset \not\in X$ and each element of $X$ is disjoint from $V(X)$.

Definition 1.2 (nonstandard universe). A nonstandard universe is a triple $(V(X), V(Y), \ast)$ such that:

1. $X$ and $Y$ are infinite base sets.
2. (Transfer Principle) The symbol $\ast$ is a map from $V(X)$ into $V(Y)$ such that

$$V(X) \models \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad V(Y) \models \varphi(^{\ast}a_1, \ldots, ^{\ast}a_n)$$

holds for any bounded formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in V(X)$.
3. $^\ast X = Y$.
4. For every infinite subset of $A$ of $X$, $\{^\ast a \mid a \in A\}$ is a proper subset of $^\ast A$.

Definitions 1.3 (standard, internal). For $a \in V(^\ast X)$, we call $a$ a standard if there is an $x \in V(X)$ such that $a = ^\ast x$.

For $a \in V(^\ast X)$, we call $a$ internal if there is an $x \in V(X)$ such that $a \in ^\ast x$. We denote by $^\ast V(X)$ the set of all internal elements in $V(^\ast X)$.

From now on, we denote a nonstandard universe by single $^\ast V(X)$.

Definitions 1.4 (norm, radius). The norm (of standardness) of an internal element $a$ is a cardinal defined by

$$\text{nos}(a) = \min \{|x| \mid a \in ^\ast x\}.$$  

The radius of $^\ast V(X)$ is a cardinal defined by

$$\text{rad}(^\ast V(X)) = \min \{\kappa \mid \forall y \in ^\ast V(X) \text{ nos}(y) < \kappa\}.$$
Definition 1.5 (covering number). Let $a$ be an internal element. The local ultrapower at $a$ is defined by

$$V(X)[a] = \{(^\ast w)(a) \mid w \in V(X) \text{ and } a \in ^\ast(\text{dom}(w))\}.$$  

For a subset $E \subseteq ^\ast V(X)$, we denote

$$V(X)[E] = \bigcup\{V(X)[s] \mid s \text{ is a finite subset of } E\}.$$  

The covering number of $^\ast V(X)$ is defined by

$$\text{cov}(^\ast V(X)) = \min \{|E| \mid E \subseteq ^\ast V(X) \text{ and } V(X)[E] = ^\ast V(X)\}.$$  

2. Locally atomic complete algebra

Definition 2.1 (regular complete subalgebra). Let $(\mathcal{B}, \wedge, \vee, \neg, 0_{\mathcal{B}}, 1_{\mathcal{B}})$ be a Boolean algebra. A subset $C \subseteq \mathcal{B}$ is said to be a regular complete subalgebra of $\mathcal{B}$ if $C$ is a complete subalgebra of $\mathcal{B}$ and the inclusion map is also complete.

Notation. Let $\mathcal{B}$ be a Boolean algebra. For a subset $S \subseteq \mathcal{P}(\mathcal{B})$, we denote

$$S^0 = \{C \in S \mid C \text{ is a regular complete subalgebra of } \mathcal{B}\}.$$  

Definition 2.2 (LCA). A locally complete algebra (LCA) is a set $\Lambda$ of subsets of a Boolean algebra $\mathcal{B}$ satisfying the conditions below.

1. $\bigcup \Lambda = \mathcal{B}$.
2. If $S_1, S_2 \in \Lambda$ then $S_1 \cup S_2 \in \Lambda$.
3. If $S \in \Lambda$ and $T \subseteq S$ then $T \in \Lambda$.
4. For every $S \in \Lambda$, there is a $C \in \Lambda^0$ containing $S$.

For an LCA $\Lambda$, we denote by $\mathcal{B}(\Lambda)$ the Boolean algebra $\bigcup \Lambda$. We call the Boolean algebra $\mathcal{B}(\Lambda)$ the base Boolean algebra of $\Lambda$.

Definition 2.3 (LACA). An LCA $\Lambda$ is a locally atomic complete algebra (LACA) if every $C \in \Lambda^0$ is atomic. We denote the set of atoms of $C \in \Lambda^0$ by $\text{Atom}(C)$.

Definition 2.4 (homomorphism). We introduce notation $R\{S \mid S \in S\}$. Let $\Lambda$ and $\Xi$ be LCAs. A Boolean homomorphism $f : \mathcal{B}(\Lambda) \to \mathcal{B}(\Xi)$ is a pseudo-homomorphism of LCAs if $f\{\Lambda \subseteq \Xi \text{ is a } (\text{complete}) \text{ homomorphism if } \bigvee h\{S = h(\bigvee S) \text{ for all } S \in \Lambda\}. An embedding or monomorphism $j : \Lambda \to \Xi$ is an injective homomorphism.

Definition 2.5 (subLCA). A subLCA of an LCA $\Lambda$ is a nonempty subset of $\Lambda$ which is itself an LCA and the inclusion map is an embedding.

Definition 2.6 (generator). Let $\Lambda$ be an LCA. A subset $\mathcal{G} \subseteq \Lambda^0$ is a generator of $\Lambda$ or $\mathcal{G}$ generates $\Lambda$ if $\Lambda$ is the only subLCA of $\Lambda$ containing $\mathcal{G}$.
Definitions 2.7 (radius, covering number, diameter). The radius of an LCA $\Lambda$ is a cardinal defined by
\[
\text{rad}(\Lambda) = \min \{ \kappa \mid \forall C \in \Lambda^\Diamond \mid |\text{Atom}(C)| < \kappa \}
\]
The covering number of an LCA $\Lambda$ is a cardinal defined by
\[
\text{cov}(\Lambda) = \min \{ |\mathcal{S}| \mid \mathcal{S} \text{ is a generator of } \Lambda \}.
\]
The diameter of an LCA $\Lambda$ is a cardinal defined by
\[
\text{diam}(\Lambda) = \min \left\{ \sum_{C \in \mathcal{S}} |\text{Atom}(C)| \mid \mathcal{S} \text{ is a generator of } \Lambda \right\}.
\]

Definition 2.8 (direct product). Let $I$ be an index set. The direct product $\Lambda^{[I]}$ of the LCA $\Lambda$ is defined by:
\[
\Lambda^{[I]} = \left\{ S \subseteq \mathcal{B}(\Lambda)^I \mid \bigcup_{g \in S} \text{rng } g \in \Lambda \right\}
\]
with the pointwise Boolean operations on $\mathcal{B}(\Lambda^{[I]}) = \bigcup \Lambda^{[I]} \subseteq \mathcal{B}(\Lambda)^I$. Then $\Lambda^{[I]}$ is an LCA. The LCA $\Lambda$ is embedded into $\Lambda^{[I]}$ by the canonical embedding $b \mapsto I \times \{b\}$.  

Definitions 2.9 (embedding system, direct limit). The embedding system of LCAs is a family of embeddings
\[
\mathcal{E} = \{ j_d^{d'} : \Lambda_d \rightarrow \Lambda_{d'} \}_{d \leq d', d, d' \in D}
\]
satisfying $j_d^{d''} \circ j_d^{d'} = j_d^{d''}$ for all $d \leq d' \leq d''$, where $D$ is an upper direct set. The direct limit of $\mathcal{E}$ is $\bigcup \{ j_d^{d'} : \mathcal{B}(\Lambda_d) \rightarrow \mathcal{B}(\Lambda_{d'}) \}_{d \leq d', d' \in D}$ as Boolean algebras.

Definition 2.10 (ultrafilter). Let $\Lambda$ be an LCA. A subset $\mathcal{U}$ of $\mathcal{B}(\Lambda)$ is an ultrafilter of an LCA $\Lambda$ if it is an ultrafilter of the base Boolean algebra $\mathcal{B}(\Lambda)$.

3. ULTRALIMIT

Definition 3.1 (LACA-valued model). Let $\Lambda$ be an LACA and let $M$ be a model for a language $\mathcal{L}$. The $\mathcal{B}(\Lambda)$-valued universe of $M$ is defined by
\[
M^{\langle \Lambda \rangle} = \left\{ u : M \rightarrow \mathcal{B}(\Lambda) \mid u(x) \land u(y) = 0 \text{ for } x \neq y, \text{ rng } u \in \Lambda, \bigvee \text{rng } u = 1 \right\}.
\]
For $u \in M^{\langle \Lambda \rangle}$, the support of $u$ is a subset of $M$ defined by
\[
\text{supp } u = \{ x \in M \mid u(x) \neq 0 \}.
\]
To each function $F$ of $\mathcal{L}(M)$ and each $u_1, \ldots, u_n \in M^{\langle \Lambda \rangle}$, we assign a $\check{F}(u_1, \ldots, u_n) \in M^{\langle \Lambda \rangle}$ by:
\[
\check{F}(u_1, \ldots, u_n)(y) = \bigvee \left\{ \bigwedge_{i=1}^n u_i(x_i) \mid M \models y = F(x_1, \ldots, x_n) \right\} \text{ for } y \in M.
\]
We regard a constant of \( \mathcal{L}(M) \) as a function without any variables. Note that \( \bigwedge_{i=1}^{n} u_{i}(x_{i}) = 1 \) if \( n = 0 \). To each sentence \( \varphi \) of \( \mathcal{L}(M^{\langle \Lambda \rangle}) \) we assign a truth value \( [\varphi] \in \mathcal{B}(\Lambda) \) by following recursive rules:

\[
[u = v] = \bigvee \{ u(x) \land v(x) \mid x \in M \},
\]

\[
[R(u_{1}, \ldots, u_{m})] = \bigvee \{ \bigwedge_{i=1}^{m} u_{i}(x_{i}) \mid \mathcal{M} \models R(x_{1}, \ldots, x_{m}) \},
\]

\[
[\neg \varphi] = \neg [\varphi],
\]

\[
[\varphi_{1} \lor \varphi_{2}] = [\varphi_{1}] \lor [\varphi_{2}],
\]

\[
[\exists x \varphi(x)] = \bigvee \{ [\varphi(u)] \mid u \in M^{\langle \Lambda \rangle} \},
\]

where \( R \) is any predicate in \( \mathcal{L} \).

**Definition 3.2 (LACA-valued superstructure).** Let \( \Lambda \) be an LACA. The \( \Lambda \)-valued superstructure of \( V(X) \) is defined by

\[
\widehat{V}(X)^{\langle \Lambda \rangle} = \{ u \in V(X)^{\langle \Lambda \rangle} \mid \supp u \in V(X) \}.
\]

While the truth values range over \( \mathcal{B}(\Lambda) \) on this definition, we shall see \( [\varphi]_{\Lambda} \in \mathcal{B}(\Lambda) \).

**Theorem 3.1.** Let \( \varphi(x_{1}, \ldots, x_{r}) \) be a formula of \( \mathcal{L} \) with only \( x_{1}, \ldots, x_{r} \) free. For \( u_{1}, \ldots, u_{r} \in M^{\langle \Lambda \rangle} \),

\[
[\varphi(u_{1}, \ldots, u_{r})]_{\Lambda} = \bigvee \{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathcal{M} \models \varphi(x_{1}, \ldots, x_{r}) \}.
\]

**Proof.** For \( \varphi \) either \( \text{"} x_{1} = x_{2} \text{"} \) or \( R \), \((*)\) holds by definition. If \((*)\) holds for an atomic formula \( \varphi(x) \) then, by simple calculus of Boolean algebra, \((*)\) holds for \( \varphi(F(x_{1}, \ldots, x_{n})) \). Thus, by induction, \((*)\) holds for \( \varphi \) atomic. Suppose \((*)\) holds for \( \varphi, \varphi_{1} \) and \( \varphi_{2} \). Since there is an atomic \( C \in \Lambda^{0} \) containing all the ranges of \( u_{1}, \ldots, u_{r} \), and every range of \( u_{i} \) is a partition of unity except for \( 0 \),

\[
[-\varphi]_{\Lambda} = \bigvee \{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathcal{M} \models -\varphi(x_{1}, \ldots, x_{r}) \}.
\]

It is easy to see:

\[
[\varphi_{1} \lor \varphi_{2}]_{\Lambda} = \bigvee \{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathcal{M} \models \varphi_{1}(x_{1}, \ldots, x_{r}) \lor \varphi_{2}(x_{1}, \ldots, x_{r}) \}.
\]

Since \([\varphi(u)]_{\Lambda} = \bigvee_{x \in M} (u(x) \land [\varphi(x)]_{\Lambda})\), we have \([\exists x \varphi(x)]_{\Lambda} = \bigvee_{x \in M} [\varphi(x)]_{\Lambda}\).

Therefore \((*)\) holds for \( \exists x \varphi(x) \). \( \square \)

Similarly, we shall obtain the superstructure version.

**Corollary 3.2.** Let \( \varphi(x_{1}, \ldots, x_{r}) \) be a formula of \( \mathcal{L}_{e} \) with only \( x_{1}, \ldots, x_{r} \) free. For \( u_{1}, \ldots, u_{r} \in \widehat{V}(X)^{\langle \Lambda \rangle} \),

\[
[\varphi(u_{1}, \ldots, u_{r})]_{\Lambda} = \bigvee \{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid V(X) \models \varphi(x_{1}, \ldots, x_{r}) \}.
\]
By the theorem and the corollary above, we have a fundamental property $[u = v] \land [\varphi(u)] \leq [\varphi(v)]$. We have just introduced $B(\Lambda)$-valued model $M^{(\Lambda)} = (M^{(\Lambda)}, \hat{R}, \hat{F}, \hat{c})$ and $B(\Lambda)$-valued superstructure $\hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$. We say that a sentence $\varphi$ of $L(M^{(\Lambda)})$ holds in $M^{(\Lambda)}$ if $[\varphi]_{\Lambda} = 1$ and that a sentence $\psi$ of $L(\hat{V}(X)^{\langle\langle\Lambda\rangle\rangle})$ holds in $\hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$ if $[\psi]_{\Lambda} = 1$. Theorem 3.1 and Corollary 3.2 follow that we consider the values $u(x)$ only for $x \in \text{supp} u$. For $E \subseteq M$, we may regard $E^{\langle\langle\Lambda\rangle\rangle}$ as a subset of $M^{(\Lambda)}$ by extending the domain of $u \in E^{(\Lambda)}$ to $M$.

This means that we define for $u \in E^{\langle\langle\Lambda\rangle\rangle}$

$$u(x) = 0 \quad \text{if } x \notin E.$$  

In the superstructure version, if $E$ is a set relative to $V(X)$ then we may assume

$$E^{\langle\langle\Lambda\rangle\rangle} = \{ u \in \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle} \mid u \in \hat{E} \text{ holds in } \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle} \}.$$  

**Theorem 3.3 (Maximum principle).** Let $\varphi(x)$ be a formula of $L(M^{(\Lambda)})$ with only $x$ free. Then there is $u \in M^{(\Lambda)}$ such that $[\varphi(u)]_{\Lambda} = [\exists x \varphi(x)]_{\Lambda}$.

**Proof.** Let $\{a_\zeta\}_{\zeta<\alpha}$ be a well-ordering for $M$. By theorem 3.1, there is $C \in \Lambda^0$ containing $[[\varphi(x)]] / x \in M$). Putting $b_\zeta = [[\varphi(a_\zeta)]] \land \forall \zeta' < \zeta [[\varphi(a_{\zeta'})]]$, we have $\{b_\zeta\}_{\zeta<\alpha} \subseteq C$. Since $\{b_\zeta\}_{\zeta<\alpha}$ is a pairwise disjoint family, we can pick $u \in M^{(C)}$ with $u(a_\zeta) \geq b_\zeta$. Then $[[\varphi(u)]] \geq u(a_\zeta) \land [[\varphi(a_\zeta)]] \geq b_\zeta$ for any $\zeta < \alpha$. Since $[[\exists x \varphi(x)]] = \forall_{\zeta<\alpha} [\varphi(a_\zeta)] = \forall_{\zeta<\alpha} b_\zeta$, we have $[[\varphi(u)]] \geq [\exists x \varphi(x)]$.  

**Corollary 3.4.** Let $\varphi(x)$ be a formula of $L(\hat{V}(X)^{\langle\langle\Lambda\rangle\rangle})$ with only $x$ free and let $v$ be an element of $\hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$. Then there is $u \in \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$ such that $[u \in v \land \varphi(u)]_{\Lambda} = [[\exists x \in v \varphi(x)]_{\Lambda}$.

**Proof.** Since there is $n$ such that $\text{supp} v \subseteq V_{n+1}(X)$,

$$[[x \in v]] = \bigvee \{ v(y) \mid x \in y \in \text{supp} v \} = 0 \quad \text{for } x \notin V_n(X).$$

Therefore we can choose $u$ whose support is a subset of $V_n(X)$.  

**Definition 3.3 (ultralimit).** We denote by $u/U$ the equivalence class of $u \in M^{(\Lambda)}$ by the equivalence relation

$$x \sim u y \iff [x = y]_{\Lambda} \in U.$$  

The *ultralimit* $M^{(\Lambda)} / U$ of $M$ modulo $U$ of $\Lambda$ is defined by:

$$M^{(\Lambda)} / U = \{ u/U \mid u \in M^{(\Lambda)} \}.$$  

$$\mathfrak{F}/U(u_1/U, \ldots, u_n/U) = (\mathfrak{F}(u_1, \ldots, u_n))/U.$$  

$$M^{(\Lambda)} / U \models R(u_1/U, \ldots, u_m/U) \iff [R(u_1, \ldots, u_m)] \in U.$$  

**Definition 3.4 (bounded ultralimit).** We denote by $u/U$ the equivalence class of $u \in \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$ by the equivalence relation

$$x \sim u y \iff [x = y]_{\Lambda} \in U.$$
The bounded ultralimit \( \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \) of \( V(X) \) modulo \( \mathcal{U} \) of \( \Lambda \) is defined by:
\[
\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} = \{ u/\mathcal{U} \mid u \in \hat{V}(X)^{\langle\Lambda\rangle} \},
\]
\[
\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models u/\mathcal{U} \in v/\mathcal{U} \iff [u \in v] \in \mathcal{U}.
\]

**Theorem 3.5 (Łoś Principle of Ultralimits).** Let \( \varphi(x_1, \ldots, x_r) \) be a formula of \( \mathbb{L} \) with only \( x_1, \ldots, x_r \) free. For \( u_1, \ldots, u_r \in M^{\langle\Lambda\rangle} \),
\[
M^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \ldots, u_r/\mathcal{U}) \iff [\varphi(u_1, \ldots, u_r)] \in \mathcal{U}.
\]

**Proof.** The proof proceeds by induction on the complexity of formulæ. The only nontrivial step is the case where \( \varphi \) is of the form \( \exists x \psi(x) \). Suppose \( [\exists x \psi(x)] \in \mathcal{U} \). By the maximal principle (Theorem 3.3), there is \( u \) satisfying \( [\psi(u)] = [\exists x \psi(x)] \). Then \( M^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U}) \) by the induction assumption. We have thus \( M^{\langle\Lambda\rangle} \models \exists x \psi(x) \). Conversely, suppose \( M^{\langle\Lambda\rangle} \models \exists x \psi(x) \). Then there is some \( u \) such that \( M^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U}) \). By the induction assumption, \( [\exists x \psi(x)] \geq [\psi(u/\mathcal{U})] \in \mathcal{U} \). □

**Corollary 3.6 (Łoś-Mostowski Principle of Bounded Ultralimits).**

Let \( \varphi(x_1, \ldots, x_r) \) be a \( \Delta_0 \)-formula of \( \mathbb{L} \) with only \( x_1, \ldots, x_r \) free. For \( u_1, \ldots, u_r \in \hat{V}(X)^{\langle\Lambda\rangle} \),
\[
\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \ldots, u_r/\mathcal{U}) \iff [\varphi(u_1, \ldots, u_r)] \in \mathcal{U}.
\]

**Proof.** The proof is similar to that of Theorem 3.5. The only different part is the if-part of the case where \( \varphi \) is of the form \( \exists x \in y \psi(x) \). Suppose \( [\exists x \in u_k \psi(x)] \in \mathcal{U} \). It follows from Corollary 3.4 that there is \( u \in \hat{V}(X)^{\langle\Lambda\rangle} \) satisfying \( [u \in u_k \land \psi(u)] = [\exists x \in u_k \psi(x)] \). □

A bounded ultralimit is a pre-nonstandard universe: that satisfies (1),(2) and (3) of Definition 1.2 with Mostowski collapsing.

**Definition 3.5 (atlas).** An atlas is a pair \( \langle \Lambda, \mathcal{U} \rangle \) of an LACA \( \Lambda \) and an ultrafilter of \( \Lambda \) such that \( \text{rad}(\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{rad}(\Lambda) \) and \( \text{cov}(\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{cov}(\Lambda) \).

**Theorem 3.7 (Sheaf representation Theorem for Nonstandard Universes).** For any nonstandard universe \( \hat{V}(X) \), there is an atlas \( \langle \Lambda, \mathcal{U} \rangle \) such that \( \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \) is isomorphic to \( \hat{V}(X) \).

We prove the theorem in the next section.

## 4. Local Ultralimits

We shall see that a homomorphism of LACAs induces an elementary embedding of ultralimits and a bounded elementary embedding of bounded ultralimits. Let \( h : \Lambda \rightarrow \Xi \) be a homomorphism. The induced map \( h_* : M^{\langle\Lambda\rangle} \rightarrow M^{\langle\Xi\rangle} \) is defined by \( h_*(u) = h \circ u \). Then we have the lemma below.

**Lemma 4.1.** Let \( \varphi(x_1, \ldots, x_r) \) be a formula of \( \mathbb{L} \) with only \( x_1, \ldots, x_r \) free. For \( u_1, \ldots, u_r \in M^{\langle\Lambda\rangle} \)
\[
[\varphi(h_*(u_1), \ldots, h_*(u_r))]_{\Xi} = h([\varphi(u_1, \ldots, u_r)]_{\Lambda}).
\]
Proof. There is $C \in \Lambda^0$ containing all the ranges of $u_k$. Since $h|C$ is complete, we have from Theorem 3.1
\[
\bigvee \left\{ \bigwedge_{i=1}^r h(u_i(x_i)) \big| \mathcal{M} \models \varphi(x_1, \ldots, x_r) \right\} = h\left( \bigvee \left\{ \bigwedge_{i=1}^r u_i(x_i) \big| \mathcal{M} \models \varphi(x_1, \ldots, x_r) \right\} \right)
\]
\[
\|\varphi(h_*(u_1), \ldots, h_*(u_r))\|_\Xi = h(\|\varphi(u_1, \ldots, u_r)\|_\Lambda).
\]
We have thus proved the lemma. \qed

For $u \in \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}$, since $\text{supp}(h \circ u) \subseteq \text{supp} u$, we can define the \textit{induced map} $h_*: \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U} \rightarrow \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$ similarly.

**Corollary 4.2.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}_\Xi$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$
\[
\|\varphi(h_*(u_1), \ldots, h_*(u_r))\|_\Xi = h(\|\varphi(u_1, \ldots, u_r)\|_\Lambda).
\]

**Proof.** Using Corollary 3.2, we see the proof is similar to that of Lemma 4.1. \qed

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters of $\Lambda$ and $\Xi$, respectively. Suppose $h^{-1}_u \mathcal{V} = \mathcal{U}$. Then we have from Lemma 4.1 or from Corollary 4.2
\[
[u = u']_\Lambda \in \mathcal{U} \iff [h_*(u) = h_*(u')]_\Xi \in \mathcal{V}.
\]
Therefore we can define the injection $h_*: M^{\langle\langle\Lambda\rangle\rangle}/\mathcal{U} \rightarrow M^{\langle\langle\Xi\rangle\rangle}/\mathcal{V}$, denoted by the same $h_*$, by $h_*(u/\mathcal{U}) = h_*(u)/\mathcal{V}$. Since $\text{supp} h_*(u) \subseteq \text{supp} u$, we can define the injection $h_*: \hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathcal{U} \rightarrow \hat{V}(X)^{\langle\langle\Xi\rangle\rangle}/\mathcal{V}$ similarly.

**Lemma 4.3.** The injection $h_*$ is an elementary embedding of $M^{\langle\langle\Lambda\rangle\rangle}/\mathcal{U}$ into $M^{\langle\langle\Xi\rangle\rangle}/\mathcal{V}$.

**Proof.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. From Theorem 3.5, we have for $u_1, \ldots, u_r \in M^{\langle\langle\Lambda\rangle\rangle}$
\[
h(\|\varphi(u_1, \ldots, u_r)\|_\Lambda) \in \mathcal{V} \iff \|\varphi(u_1, \ldots, u_r)\|_\Lambda \in h^{-1}_u \mathcal{V}.
\]
\[
\|\varphi(h_*(u_1), \ldots, h_*(u_r))\|_\Xi \in \mathcal{V} \iff \|\varphi(h_*(u_1), \ldots, h_*(u_r))\|_\Xi \in \mathcal{V}.
\]
\[
\mathcal{M}^{\langle\langle\Xi\rangle\rangle}/\mathcal{V} \models \varphi(h_*(u_1/\mathcal{U}), \ldots, h_*(u_r/\mathcal{U})) \iff \mathcal{M}^{\langle\langle\Lambda\rangle\rangle}/\mathcal{U} \models \varphi(u_1, \ldots, u_r).
\]
\qed

**Corollary 4.4.** The injection $h_*$ is a bounded elementary embedding of $\hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathcal{U}$ into $\hat{V}(X)^{\langle\langle\Xi\rangle\rangle}/\mathcal{V}$.

**Proof.** Using Corollary 3.6, we see the proof is similar to that of Lemma 4.3. \qed

Let $I$ be a set relative to $V(X)$. We shall find a one-to-one correspondence between $\mathcal{P}(I)^{\langle\langle\Lambda\rangle\rangle}$ and $\mathcal{B}(\Lambda^{[I]})$. Note that $\mathcal{P}(I)^{\langle\langle\Lambda\rangle\rangle}$ is the set of "the subsets of $I$ in $\hat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$". For $A \in \mathcal{P}(I)^{\langle\langle\Lambda\rangle\rangle}$, there is $C \in \Lambda^0$ such that $\text{rng} A \subseteq C$. Define $g: I \rightarrow \mathcal{B}(\Lambda)$ by $g(i) = \llbracket i \in A \rrbracket_\Lambda$. Then we have $\text{rng} g \subseteq C$ and $g \in \mathcal{B}(\Lambda^{[I]})$. Conversely, for $g \in \mathcal{B}(\Lambda^{[I]})$, there is $C \in \Lambda^0$ such that $\text{rng} g \subseteq C$. Define $A: \mathcal{P}(I) \rightarrow C$ by
\[
A(x) = \bigwedge_{i \in I} \text{sg}(i, g(i)), \text{ where } \text{sg}(i, b) = \begin{cases} b & \text{if } i \in x, \\ -b & \text{if } i \in I \setminus x. \end{cases}
\]
Since \( C \) is completely distributive, we have \( A \in \mathcal{P}(I)^{\langle\Lambda\rangle} \). Suppose \( g(i) = [\check{\ i} \in A]_\Lambda \) and \( g'(i) = [\check{\ i} \in A']_\Lambda \). Then we see \((g \land g')(i) = [\check{\ i} \in A \cap A']_\Lambda\) and \((\neg g)(i) = [\check{\ i} \in I \setminus A]_\Lambda\). In the context above, the relation \( g(i) = [\check{\ i} \in A]_\Lambda \) sets up a one-to-one correspondence between \( \mathcal{P}(I)^{\langle\Lambda\rangle} \) and \( \mathcal{B}(\Lambda^{|I|}) \) as Boolean algebras. From now on, we identify \( \mathcal{P}(I)^{\langle\Lambda\rangle} \) with \( \mathcal{B}(\Lambda^{|I|}) \).

We shall define the special element \( \delta \in I^{\langle\Lambda^{|I|}\rangle} \subseteq \hat{V}(X)^{\langle\Lambda^{|I|}\rangle} \) by

\[
\delta(x)(i) = \begin{cases} 
1 & \text{if } x = i, \\
0 & \text{if } x \neq i. 
\end{cases}
\]

We call the \( \delta \) diagonal element of \( I \) on \( \Lambda \). Let \( j : \Lambda \rightarrow \Lambda^{|I|} \) be the canonical embedding. Then \( j \) is also a Boolean monomorphism of \( \mathcal{B}(\Lambda) \) into \( \mathcal{P}(I)^{\langle\Lambda\rangle} \). The diagonal element \( \delta \) has following properties.

**Lemma 4.5.** The following statements hold.

1. \( [j(b)]_\Lambda = b \) for every \( b \in \mathcal{B}(\Lambda) \).
2. \( [\delta \in j_*(g)]_{\Lambda^{|I|}} = g \) for every \( g \in \mathcal{P}(I)^{\langle\Lambda\rangle} \).

**Proof.** Since \( [\check{\ i} \in j(b)]_\Lambda = j(b)(i) = b \) for all \( i \in I \), \( [j(b)]_\Lambda = [j(b) \supseteq \check{I}]_\Lambda = \wedge_{i \in I} [\check{\ i} \in j(b)]_\Lambda = b \). From the definition of \( \delta \), it is clear that \( [\delta = \check{I}]_{\Lambda^{|I|}}(i) = \delta(i)(i) = 1 \). Then we have \( [\delta \in j_*(g)]_{\Lambda^{|I|}}(i) = [\check{\ i} \in j_*(g)]_{\Lambda^{|I|}}(i) = [\check{\ i} \in g]_\Lambda = g(i) \).

**Theorem 4.6.** For any \( v \in \hat{V}(X)^{\langle\Lambda^{|I|}\rangle} \), there is a map \( w : \check{I} \rightarrow (\text{supp } v)^\check{I} \) in \( \hat{V}(X)^{\langle\Lambda\rangle} \) such that \( v = j_*(w)(\delta) \) holds in \( \hat{V}(X)^{\langle\Lambda\rangle} \).

**Proof.** Since \( \text{rng } v \subseteq \Lambda^{|I|} \), \( \bigcup_{g \in \text{rng } v} \text{rng } g \subseteq \Lambda \). Therefore we can define \( w : (\text{supp } v)^I \rightarrow \mathcal{B}(\Lambda) \) by

\[
w(s) = \bigwedge_{i \in I} v(s(i))(i).
\]

Then we get \( w \) as required. First, we show \( w \in \hat{V}(X)^{\langle\Lambda\rangle} \). If \( s \neq s' \), then there is \( i_0 \in I \) such that \( s(i_0) \neq s'(i_0) \). Since \( \text{rng } v \) is pairwise disjoint, we have

\[
w(s) \land w(s') \leq v(s(i_0))(i_0) \land v(s'(i_0))(i_0) = 0.
\]

There is \( C \subseteq \Lambda^0 \) such that \( \bigcup_{g \in \text{rng } v} \text{rng } g \subseteq C \). Then we have \( \text{rng } w \subseteq C \) and then

\[
\bigvee_{s \in (\text{supp } v)^I} w(s) = \bigvee_{s \in (\text{supp } v)^I} \bigwedge_{i \in I} v(s(i))(i) = \bigwedge_{i \in I} \bigvee_{y \in \text{supp } v} v(y)(i) = 1.
\]
We have thus shown $w \in \hat{V}(X)^{\langle\Lambda\rangle}$. For each $i \in I$, since $w(s) \leq v(s(i))(i)$ holds for every $s \in (\text{supp } v)^I$, we have

$$\left[ v = j_*(w)(\delta)\right]_{\Lambda^{|I|}}(i) = \left[ v = j_*(w)(\check{i})\right]_{\Lambda^{|I|}}(i) = \left( \bigvee \{ v(y) \land j(w(s)) \land \check{i}(x) \mid y = s(x) \} \right)(i) = \bigvee_{s \in (\text{supp } v)^I} (v(s(i))(i) \land w(s)) = \bigvee_{s \in (\text{supp } v)^I} w(s) = 1.$$

We have thus proved the theorem. \hfill \Box

Let $\mathcal{U}$ be an ultrafilter of an LACA $\Lambda$. A local ultralimit $\rho: \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow *V(X)$ is a bounded elementary embedding satisfying $\rho(\check{x}/\mathcal{U}) = *x$ for every $x \in V(X)$.

**Theorem 4.7** (Local Ultraproduct Theorem). Let $\rho: \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow *V(X)$ be a local ultralimit and let $p$ be an internal element of $*V(X)$. Then there is a local ultralimit $\tau: \hat{V}(X)^{\langle\Lambda^{|I|}\rangle}/\mathcal{V} \rightarrow *V(X)$ such that the following conditions hold.

(i) The index set $I$ is a set relative to $V(X)$ and $|I| = \text{nos}(p)$.

(ii) Let $j: \Lambda \rightarrow \Lambda^{|I|}$ be the canonical embedding. Then $\mathcal{U} = j^{-1}\mathcal{V}$ and $\rho = \tau \circ j_*$.

(iii) The submodel $\text{rng } \tau$ of $*V(X)$ is the minimal bounded elementary submodel of $*V(X)$ that contains $\{p\} \cup \text{rng } \rho$.

$$\xymatrix{ \hat{V}(X)^{\langle\Lambda^{|I|}\rangle}/\mathcal{V} \ar[r]_{\tau} & *V(X) \\
V(X) \ar[u]_{j_*} \ar[r]_{\rho} & \hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \\
p \in \text{rng } \tau. & }$$

**Proof.** Let $I$ be a set relative to $V(X)$ such that $p \in *I$ and $|I| = \text{nos}(p)$. We have identified $\mathcal{B}(\Lambda^{|I|})$ with $\mathcal{P}(I)^{\langle\Lambda\rangle}$. Define $\mathcal{V} \subseteq \mathcal{B}(\Lambda^{|I|})$ by

$$g \in \mathcal{V} \iff p \in \rho(g/\mathcal{U}).$$

Then $\mathcal{V}$ is an ultrafilter of $\Lambda^{|I|}$. Let $b$ be an element of $\mathcal{U}$. From (1) of Lemma 4.5, $\rho(j(b)/\mathcal{U})$ coincides $\check{b}$. Then we have $j(b) \in \mathcal{V}$ from the definition of $\mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ are maximal filters, we obtain $j^{-1}\mathcal{V} = \mathcal{U}$. Let $\varphi(x_1, \ldots, x_r)$ be a $\Delta_0$-formula of $L_\mathcal{E}$ with only $x_1, \ldots, x_r$ free. Let $v_1, \ldots, v_r$ be elements of $\hat{V}(X)^{\langle\Lambda^{|I|}\rangle}$. By Theorem 4.6, there are maps $w_1, \ldots, w_r$ from $\check{I}$ in $\hat{V}(X)^{\langle\Lambda\rangle}$ such that $v_k = j_*(w_k)(\check{\delta})$ hold, where $\delta$ is the diagonal element of $I$ on $\Lambda$. Putting $g_0 = \{ i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i)) \}$ in $\hat{V}(X)^{\langle\Lambda\rangle}$, we have from (2) of Lemma 4.5

$$g_0 = \left[ \delta \in j_*(g_0) \right]_{\Lambda^{|I|}} = \left[ \delta \in j_*(\{ i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i)) \}) \right]_{\Lambda^{|I|}} = \left[ \delta \in \{ i \in \check{I} \mid \varphi(j_*(w_1)(i), \ldots, j_*(w_r)(i)) \} \right]_{\Lambda^{|I|}} = \left[ \varphi(j_*(w_1)(\delta), \ldots, j_*(w_r)(\delta)) \right]_{\Lambda^{|I|}} = \left[ \varphi(v_1, \ldots, v_r) \right]_{\Lambda^{|I|}}$$
and we have

$$
\tilde{V}(X)^{(\Lambda)} / \mathcal{U} \models g_0 / \mathcal{U} = \{ i \in \tilde{I} / \mathcal{U} \mid \varphi((w_1 / \mathcal{U})(i), \ldots, (w_r / \mathcal{U})(i)) \}
$$

$$
\rho(g_0 / \mathcal{U}) \models \{ i \in \mathcal{T} \mid \varphi(\rho(w_1 / \mathcal{U})(i), \ldots, \rho(w_r / \mathcal{U})(i)) \}.
$$

By the definition of \( \mathcal{V} \), we obtain

$$
g_0 \in \mathcal{V} \text{ iff } \rho(g_0 / \mathcal{U}) \models [\varphi(v_1, \ldots, v_r)]_{\Lambda^{|I|}} \in \mathcal{V} \text{ iff } \varphi(\rho(w_1 / \mathcal{U})(p), \ldots, \rho(w_r / \mathcal{U})(p)).
$$

The case \( \varphi(x_1, x_2) \equiv \"x_1 = x_2\" \) enables us to define the operation \( u / \mathcal{V} \mapsto \rho(w / \mathcal{U})(p) \) where \( v = j_*(u)(\delta) \) holds in \( \tilde{V}(X)^{(\Lambda^{[J]})} \). Thus, defining \( \tau : \tilde{V}(X)^{(\Lambda^{[J]})} / \mathcal{V} \rightarrow \mathcal{V}(X) \)
by \( \tau(v / \mathcal{V}) = \rho(w / \mathcal{U})(p) \) where \( v = j_*(u)(\delta) \) holds in \( \tilde{V}(X)^{(\Lambda^{[J]})} \), we get \( \tau \) as required. In fact, it is clear in the preceding context that \( \tau \) is a bounded elementary embedding of \( \tilde{V}(X)^{(\Lambda^{[J]})} / \mathcal{V} \) into \( \mathcal{V}(X) \). Let \( \iota \) be the identity map on \( I \), then we see \( \tau(j_*(u / \mathcal{U})(\delta / \mathcal{V})) = \rho(u / \mathcal{U})(p) = \iota(p) = p \). For \( u / \mathcal{U} \in \tilde{V}(X)^{(\Lambda)} / \mathcal{U} \), let \( \tilde{u} \) be the constant map from \( \tilde{I} \) onto \( \{ u \} \) in \( \tilde{V}(X)^{(\Lambda)} \), then we have \( \tau(j_*(u / \mathcal{U})) = \rho(\tilde{u} / \mathcal{U})(p) = \rho(u / \mathcal{U}) \).

Suppose a bounded elementary submodel \( W \) of \( \mathcal{V}(X) \) contains \( \{ p \} \cup \text{rng} \rho \). From the definition of \( \tau \), \( \tau(v / \mathcal{V}) = \rho(w / \mathcal{U})(p) \in W \) for some \( w / \mathcal{U} \in \tilde{V}(X)^{(\Lambda)} / \mathcal{U} \). Therefore \( \text{rng} \tau \) is the minimum. We have completed the proof of Theorem 4.7. \( \square \)

Let \( \{ j^d_d : \Lambda_d \rightarrow \Lambda_{d'} \}_{d \leq d', d' \in D} \) be an embedding system of LACAs with direct limit \( \{ j_d : \Lambda_d \rightarrow \Lambda \}_{d \in D} \). Let \( \mathcal{U} \) be an ultrafilter of \( \Lambda \), then each \( \mathcal{U}_d = j_d^{-1} u \mathcal{U} \) is an ultrafilter of \( \Lambda_d \).

**Theorem 4.8** (Elementary Net Theorem of Ultralimits). Let \( \mathcal{M} \) and \( \Re \) be models for \( \mathcal{L} \). Suppose there are elementary embeddings \( \tau_d : \mathcal{M}^{\langle\langle \Lambda_d \rangle\rangle} / \mathcal{U}_d \rightarrow \Re \) satisfying the condition \( \tau_d = \tau_{d'} \circ j^d_{d'} \ast \) for \( d \leq d' \). Then there is an elementary embedding \( \tau : \mathcal{M}^{\langle\langle \Lambda \rangle\rangle} / \mathcal{U} \rightarrow \Re \) such that \( \tau_d = \tau \circ j_d \ast \) for \( d \in D \).

- **Proof.** Let \( v \) be an element of \( \mathcal{M}^{\langle\langle \Lambda \rangle\rangle} \). Since \( \text{rng} v \in \Lambda \), \( \{ j_d \upharpoonright S \mid d \in D \text{ and } S \in \Lambda_d \} \) from the definition of direct limits, there is \( u \in \mathcal{M}^{\langle\langle \Lambda_d \rangle\rangle} \) such that \( v = j_d \ast(u) \).

Therefore defining \( \tau(v / \mathcal{U}) = \tau_d(u / \mathcal{U}_d) \) where \( v = j_d \ast(u) \), we get \( \tau \) as required. Let \( \varphi(x_1, \ldots, x_r) \) be a formula of \( \mathcal{L} \) with only \( x_1, \ldots, x_r \) free and let \( u_1, \ldots, u_r \) be elements \( M^{\langle\langle \Lambda \rangle\rangle} \). Then there are \( d \in D \) and \( u_1, \ldots, u_r \in M^{\langle\langle \Lambda_d \rangle\rangle} \) such that \( u_k = j_d \ast(u_k) \). We conclude as below.

$$
\mathcal{M}^{\langle\langle \Lambda_d \rangle\rangle} / \mathcal{U}_d \models \varphi(u_1 / \mathcal{U}_d, \ldots, u_r / \mathcal{U}_d) \text{ iff } \Re \models \varphi(\tau_d(u_1 / \mathcal{U}), \ldots, \tau_d(u_r / \mathcal{U}))
$$

$$
\mathcal{M}^{\langle\langle \Lambda \rangle\rangle} / \mathcal{U} \models \varphi(v_1 / \mathcal{U}, \ldots, v_r / \mathcal{U}) \text{ iff } \Re \models \varphi(\tau(v_1 / \mathcal{U}), \ldots, \tau(v_r / \mathcal{U}))
$$


Theorem 4.9 (Bounded Elementary Net Theorem of Bounded Ultralimits).
Suppose there are local ultralimits $\tau_d : \hat{V}(X)^{\langle \Lambda_d \rangle} / \mathfrak{U}_d \to \star V(X)$ satisfying the condition $\tau_d = \tau_d' \circ j_{d*}$ for $d \leq d'$. Then there is a local ultralimit $\tau : \hat{V}(X)^{\langle \Lambda \rangle} / \mathfrak{U} \to \star V(X)$ such that $\tau_d = \tau \circ j_{d*}$ for $d \in D$.

Proof. Similar to the proof of Theorem 4.8.

We call the pair $\langle \Lambda, \mathfrak{U} \rangle$ in Theorem 4.8 or Theorem 4.7 the direct limit of $\{\langle \Lambda_d, \mathfrak{U}_d \rangle\}_{d \leq d', d, d' \in D}$.

Proof of Theorem 3.7 Let $\{p_\zeta\}_{\zeta < \kappa}$ be a sequence in $\star V(X)$ with $\kappa = \text{cov}(\star V(X))$. We define local ultralimits $\{\rho_\zeta : \hat{V}^{\langle \Lambda_\zeta \rangle} / \mathfrak{U}_\zeta \to \star V(X)\}_{\zeta < \kappa}$ of $\star V(X)$ by:

- $\Lambda_0 = \mathcal{P}(\{0, 1\})$, $\mathfrak{U}_0 = \{1\}$.
- $\Lambda_{\zeta+1} = \Lambda^{[I_\zeta]}$, where $|I_\zeta| = \text{nos}(p_\zeta)$, $p_\zeta \in \text{rng} \rho_{\zeta+1}$ in Theorem 4.7.
- $\langle \Lambda_\lambda, \mathfrak{U}_\lambda \rangle$ is the direct limit of $\{\langle \Lambda_\zeta, \mathfrak{U}_\zeta \rangle_{\zeta < \lambda}\}$ in Theorem 4.9.

Then the direct limit $\langle \Lambda, \mathfrak{U} \rangle$ of $\{\langle \Lambda_\zeta, \mathfrak{U}_\zeta \rangle\}_{\zeta < \kappa}$ is an atlas of $\star V(X)$.

References


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