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NONSTANDARD UNIVERSE

MASAHIKO MURAKAMI
DEPARTMENT OF MATHEMATICS
HOSEI UNIVERSITY

ABSTRACT. The nonstandard universes are frameworks of nonstandard analysis. We find sheaf representation for a nonstandard universe in Theorem 3.7.

1. NONSTANDARD Universe

Definitions 1.1 (superstructure, base set). Given a set $X$, we define the iterated power set $V_n(X)$ over $X$ recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The superstructure $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set $X$ is said to be a base set if $\emptyset \not\in X$ and each element of $X$ is disjoint from $V(X)$.

Definition 1.2 (nonstandard universe). A nonstandard universe is a triple $(V(X), V(Y), \star)$ such that:

1. $X$ and $Y$ are infinite base sets.
2. (Transfer Principle) The symbol $\star$ is a map from $V(X)$ into $V(Y)$ such that

$$V(X) \models \varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad V(Y) \models \varphi(\star a_1, \ldots, \star a_n)$$

holds for any bounded formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in V(X)$.
3. $\star X = Y$.
4. For every infinite subset of $A$ of $X$, $\{\star a \mid a \in A\}$ is a proper subset of $\star A$.

Definitions 1.3 (standard, internal). For $a \in V(\star X)$, we call $a$ a standard if there is an $x \in V(X)$ such that $a = \star x$.

For $a \in V(\star X)$, we call $a$ internal if there is an $x \in V(X)$ such that $a \in \star x$. We denote by $V(\star X)$ the set of all internal elements in $V(\star X)$.

From now on, we denote a nonstandard universe by single $\star V(X)$.

Definitions 1.4 (norm, radius). The norm (of standardness) of an internal element $a$ is a cardinal defined by

$$\text{nos}(a) = \min \{|x| \mid a \in \star x\}.$$ 

The radius of $\star V(X)$ is a cardinal defined by

$$\text{rad}(\star V(X)) = \min \{\kappa \mid \forall y \in \star V(X) \text{ nos}(y) < \kappa\}.$$
Definition 1.5 (covering number). Let $a$ be an internal element. The local ultra-power at $a$ is defined by

$$V(X)[a] = \{(w)(a) \mid w \in V(X) \text{ and } a \in ^*(\text{dom}(w))\}.$$ 

For a subset $E \subseteq \mathcal{V}(X)$, we denote

$$V(X)[E] = \bigcup\{V(X)[s] \mid s \text{ is a finite subset of } E\}.$$ 

The covering number of $\mathcal{V}(X)$ is defined by

$$\text{cov}(\mathcal{V}(X)) = \min \{|E| \mid E \subseteq \mathcal{V}(X) \text{ and } V(X)[E] = \mathcal{V}(X)\}.$$ 

2. Locally atomic complete algebra

Definition 2.1 (regular complete subalgebra). Let $\langle \mathcal{B}, \wedge, \vee, \neg, 0_{\mathcal{B}}, 1_{\mathcal{B}} \rangle$ be a Boolean algebra. A subset $C \subseteq \mathcal{B}$ is said to be a regular complete subalgebra of $\mathcal{B}$ if $C$ is a complete subalgebra of $\mathcal{B}$ and the inclusion map is also complete.

Notation. Let $\mathcal{B}$ be a Boolean algebra. For a subset $S \subseteq \mathcal{P}(\mathcal{B})$, we denote

$$S^0 = \{C \in S \mid C \text{ is a regular complete subalgebra of } \mathcal{B}\}.$$ 

Definition 2.2 (LCA). A locally complete algebra (LCA) is a set $\Lambda$ of subsets of a Boolean algebra $\mathcal{B}$ satisfying the conditions below.

1. $\bigcup\Lambda = \mathcal{B}$.
2. If $S_1, S_2 \in \Lambda$ then $S_1 \cup S_2 \in \Lambda$.
3. If $S \in \Lambda$ and $T \subseteq S$ then $T \in \Lambda$.
4. For every $S \in \Lambda$, there is a $C \in \Lambda^0$ containing $S$.

For an LCA $\Lambda$, we denote by $\mathcal{B}(\Lambda)$ the Boolean algebra $\bigcup\Lambda$. We call the Boolean algebra $\mathcal{B}(\Lambda)$ the base Boolean algebra of $\Lambda$.

Definition 2.3 (LACA). An LCA $\Lambda$ is a locally atomic complete algebra (LACA) if every $C \in \Lambda^0$ is atomic. We denote the set of atoms of $C \in \Lambda^0$ by $\text{Atom}(C)$.

Definition 2.4 (homomorphism). We introduce notation $R^\mathcal{S} = \{R^S \mid S \in \mathcal{S}\}$. Let $\Lambda$ and $\Xi$ be LCAs. A Boolean homomorphism $f : \mathcal{B}(\Lambda) \to \mathcal{B}(\Xi)$ is a pseudo-homomorphism of LCAs if $f^\mathcal{S} \subseteq \Xi$. We denotes a pseudo-homomorphism by $f : \Lambda \to \Xi$. A pseudo-homomorphism $h : \Lambda \to \Xi$ of LCAs is a (complete) homomorphism if $\bigvee h^S = h(\bigvee S)$ for all $S \in \Lambda$. An embedding or monomorphism $j : \Lambda \to \Xi$ is an injective homomorphism.

Definition 2.5 (subLCA). A subLCA of an LCA $\Lambda$ is a nonempty subset of $\Lambda$ which is itself an LCA and the inclusion map is an embedding.

Definition 2.6 (generator). Let $\Lambda$ be an LCA. A subset $\mathcal{G} \subseteq \Lambda^0$ is a generator of $\Lambda$ or $\mathcal{G}$ generates $\Lambda$ if $\Lambda$ is the only subLCA of $\Lambda$ containing $\mathcal{G}$.
Definitions 2.7 (radius, covering number, diameter). The radius of an LCA $\Lambda$ is a cardinal defined by

$$\text{rad}(\Lambda) = \min \{ \kappa \mid \forall C \in \Lambda \diamond \ |\text{Atom}(C)| < \kappa \}$$

The covering number of an LCA $\Lambda$ is a cardinal defined by

$$\text{cov}(\Lambda) = \min \{|\mathcal{G}| \mid \mathcal{G} \text{ is a generator of } \Lambda \}.$$ 

The diameter of an LCA $\Lambda$ is a cardinal defined by

$$\text{diam}(\Lambda) = \min \left\{ \sum_{C \in \mathcal{G}} |\text{Atom}(C)| \mid \mathcal{G} \text{ is a generator of } \Lambda \right\}.$$ 

Definition 2.8 (direct product). Let $I$ be an index set. The direct product $\Lambda^[[I]]$ of the LCA $\Lambda$ is defined by:

$$\Lambda^[[I]] = \{ S \subseteq B(\Lambda)^I \mid \bigcup_{g \in S} \text{rng } g \in \Lambda \}$$

with the pointwise Boolean operations on $B(\Lambda^[[I]]) = \bigcup \Lambda^[[I]] \subseteq B(\Lambda)^I$. Then $\Lambda^[[I]]$ is an LCA. The LCA $\Lambda$ is embedded into $\Lambda^[[I]]$ by the canonical embedding $b \mapsto I \times \{b\}$.

Definitions 2.9 (embedding system, direct limit). The embedding system of LCAs is a family of embeddings

$$\mathcal{E} = \{ j^d_{d'} : \Lambda_d \to \Lambda_{d'} \}_{d \leq d', d, d' \in D}$$

satisfying $j^d_{d'} \circ j^{d''}_{d} = j^{d''}_{d}$ for all $d \leq d' \leq d''$, where $D$ is an upper direct set. The direct limit of $\mathcal{E}$ is $\bigcup \{ j^d_d : B(\Lambda_d) \to B(\Lambda_{d'}) \}_{d \leq d', d, d' \in D}$ as Boolean algebras.

Definition 2.10 (ultrafilter). Let $\Lambda$ be an LCA. A subset $\mathcal{U}$ of $B(\Lambda)$ is an ultrafilter of an LCA $\Lambda$ if it is an ultrafilter of the base Boolean algebra $B(\Lambda)$.

3. ULTRALIMIT

Definition 3.1 (LACA-valued model). Let $\Lambda$ be an LACA and let $M$ be a model for a language $\mathcal{L}$. The $B(\Lambda)$-valued universe of $M$ is defined by

$$M^{\langle \Lambda \rangle} = \left\{ u : M \to B(\Lambda) \mid u(x) \land u(y) = 0 \text{ for } x \neq y, \text{ rng } u \in \Lambda, \text{ } \lor \text{rng } u = 1 \right\}.$$ 

For $u \in M^{\langle \Lambda \rangle}$, the support of $u$ is a subset of $M$ defined by

$$\text{supp } u = \{ x \in M \mid u(x) \neq 0 \}.$$ 

To each function $F$ of $\mathcal{L}(M)$ and each $u_1, \ldots, u_n \in M^{\langle \Lambda \rangle}$, we assign a $\check{F}(u_1, \ldots, u_n) \in M^{\langle \Lambda \rangle}$ by:

$$\check{F}(u_1, \ldots, u_n)(y) = \lor \left\{ \bigwedge_{i=1}^{n} u_i(x_i) \mid M \models y = F(x_1, \ldots, x_n) \right\} \text{ for } y \in M.$$
We regard a constant of \( L(M) \) as a function without any variables. Note that \( \bigwedge_{i=1}^{n} u_{i}(x_{i}) = 1 \) if \( n = 0 \). To each sentence \( \varphi \) of \( L(M^{\langle A \rangle}) \) we assign a truth value \( [\varphi] \in \mathcal{B}(A) \) by following recursive rules:

\[
[u = v] = \bigvee \{ u(x) \land v(x) \mid x \in M \},
\]

\[
[R(u_{1}, \ldots, u_{m})] = \bigvee \left\{ \bigwedge_{i=1}^{m} u_{i}(x_{i}) \mid \mathfrak{M} \models R(x_{1}, \ldots, x_{m}) \right\},
\]

\[
[-\varphi] = -[\varphi],
\]

\[
[\varphi_{1} \lor \varphi_{2}] = [\varphi_{1}] \lor [\varphi_{2}],
\]

\[
[\exists x \varphi(x)] = \bigvee \{ [\varphi(u)] \mid u \in M^{\langle A \rangle} \},
\]

where \( R \) is any predicate in \( L \).

**Definition 3.2 (LACA-valued superstructure).** Let \( A \) be an LACA. The \( A \)-valued superstructure of \( V(X) \) is defined by

\[
\hat{V}(X)^{\langle A \rangle} = \{ u \in V(X)^{\langle A \rangle} \mid \text{supp } u \in V(X) \}.
\]

While the truth values range over \( \overline{\mathcal{B}(A)} \) on this definition, we shall see \([\varphi]_{A} \in \mathcal{B}(A)\).

**Theorem 3.1.** Let \( \varphi(x_{1}, \ldots, x_{r}) \) be a formula of \( L \) with only \( x_{1}, \ldots, x_{r} \) free. For \( u_{1}, \ldots, u_{r} \in M^{\langle A \rangle} \),

\[
([\varphi(u_{1}, \ldots, u_{r})]_{A} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathfrak{M} \models \varphi(x_{1}, \ldots, x_{r}) \right\}.
\]

**Proof.** For \( \varphi \) either "\( x_{1} = x_{2} \)" or \( R \), (*) holds by definition. If (*) holds for an atomic formula \( \varphi(x) \) then, by simple calculus of Boolean algebra, (*) holds for \( \varphi(F(x_{1}, \ldots, x_{n})) \). Thus, by induction, (*) holds for \( \varphi \) atomic. Suppose (*) holds for \( \varphi, \varphi_{1} \) and \( \varphi_{2} \). Since there is an atomic \( C \in \Lambda^{0} \) containing all the ranges of \( u_{1}, \ldots, u_{r} \) and every range of \( u_{i} \) is a partition of unity except for \( 0 \),

\[
[-\varphi]_{A} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathfrak{M} \models -\varphi(x_{1}, \ldots, x_{r}) \right\}.
\]

It is easy to see:

\[
[\varphi_{1} \lor \varphi_{2}]_{A} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid \mathfrak{M} \models \varphi_{1}(x_{1}, \ldots, x_{r}) \lor \varphi_{2}(x_{1}, \ldots, x_{r}) \right\}.
\]

Since \([\varphi(u)]_{A} = \bigvee_{x \in M} (u(x) \land [\varphi(x)])_{A} \), we have \([\exists x \varphi(x)]_{A} = \bigvee_{x \in M} [\varphi(x)]_{A} \).

Therefore (*) holds for \( \exists x \varphi(x) \). \( \square \)

Similarly, we shall obtain the superstructure version.

**Corollary 3.2.** Let \( \varphi(x_{1}, \ldots, x_{r}) \) be a formula of \( L_{\in} \) with only \( x_{1}, \ldots, x_{r} \) free. For \( u_{1}, \ldots, u_{r} \in \hat{V}(X)^{\langle A \rangle} \),

\[
[\varphi(u_{1}, \ldots, u_{r})]_{A} = \bigvee \left\{ \bigwedge_{i=1}^{r} u_{i}(x_{i}) \mid V(X) \models \varphi(x_{1}, \ldots, x_{r}) \right\}.
\]
By the theorem and the corollary above, we have a fundamental property 
\([u = v] \land [\varphi(u)] \leq [\varphi(v)]\). We have just introduced \(\mathcal{B}(\Lambda)\)-valued model
\(\mathcal{M}^{\langle\Lambda\rangle} = (M^{\langle\Lambda\rangle}, \hat{R}, \hat{F}, \check{c})\) and \(\mathcal{B}(\Lambda)\)-valued superstructure \(\hat{V}(X)^{\langle\Lambda\rangle}\). We say that
a sentence \(\varphi\) of \(\mathcal{L}(M^{\langle\Lambda\rangle})\) holds in \(\mathcal{M}^{\langle\Lambda\rangle}\) if \([\varphi]_{\Lambda} = 1\) and that a sentence \(\psi\) of
\(\mathcal{L} \in \hat{V}(X)^{\langle\Lambda\rangle}\) holds in \(\hat{V}(X)^{\langle\Lambda\rangle}\) if \([\psi]_{\Lambda} = 1\). Theorem 3.1 and Corollary 3.2
follow that we consider the values \(u(x)\) only for \(x \in \text{supp } u\). For \(E \subseteq M\), we may
regard \(E^{\langle\Lambda\rangle}\) as a subset of \(M^{\langle\Lambda\rangle}\) by extending the domain of \(u \in E^{\langle\Lambda\rangle}\) to \(M\).
This means that we define for \(u \in E^{\langle\Lambda\rangle}\)

\[u(x) = 0 \text{ if } x \notin E.\]

In the superstructure version, if \(E\) is a set relative to \(V(X)\) then we may assume

\[E^{\langle\Lambda\rangle} = \{u \in \hat{V}(X)^{\langle\Lambda\rangle} | u \in \hat{E} \text{ holds in } \hat{V}(X)^{\langle\Lambda\rangle}\}.\]

Theorem 3.3 (Maximum principle). Let \(\varphi(x)\) be a formula of \(\mathcal{L}(M^{\langle\Lambda\rangle})\) with only
\(x\) free. Then there is \(u \in M^{\langle\Lambda\rangle}\) such that \([\varphi(u)]_{\Lambda} = [\exists x \varphi(x)]_{\Lambda}\).

Proof. Let \(\{a_\zeta\}_{\zeta < \alpha}\) be a well-ordering for \(M\). By theorem 3.1, there is \(C \in \Lambda^0\)
containing \(\{[\varphi(x)] | x \in M\}\). Putting \(b_\zeta = [\varphi(a_\zeta)] \land \neg \bigvee_{\zeta < \alpha} [\varphi(a_\zeta)]\), we have
\(\{b_\zeta\}_{\zeta < \alpha} \subseteq C\). Since \(\{b_\zeta\}_{\zeta < \alpha}\) is a pairwise disjoint family, we can pick \(u \in M^{(C)}\)
with \(u(a_\zeta) \geq b_\zeta\). Then \([\varphi(u)] \geq u(a_\zeta) \land [\varphi(a_\zeta)] \geq b_\zeta\) for any \(\zeta < \alpha\). Since
\(\exists x \varphi(x) = \bigvee_{\zeta < \alpha} \varphi(a_\zeta) = \bigvee_{\zeta < \alpha} b_\zeta\), we have \([\varphi(u)] \geq [\exists x \varphi(x)].\)

Corollary 4.4. Let \(\varphi(x)\) be a formula of \(\mathcal{L} \in \hat{V}(X)^{\langle\Lambda\rangle}\) with only \(x\) free and let \(v\)
be an element of \(\hat{V}(X)^{\langle\Lambda\rangle}\). Then there is \(u \in \hat{V}(X)^{\langle\Lambda\rangle}\) such that \(\varphi(u)]_{\Lambda} = [\exists x \in v \varphi(x)]_{\Lambda}\).

Proof. Since there is \(n\) such that \(\text{supp } v \subseteq V_{n+1}(X)\),

\[[x \in v] = \bigvee \{v(y) | x \in y \in \text{supp } v\} = 0 \text{ for } x \notin V_n(X).\]

Therefore we can choose \(u\) whose support is a subset of \(V_n(X).\)

Definition 3.3 (ultralimit). We denote by \(u/\mathcal{U}\) the equivalence class of \(u \in M^{\langle\Lambda\rangle}\)
by the equivalence relation

\[x \sim u y \equiv [x = y]_{\Lambda} \in \mathcal{U}.\]

The ultralimit \(\mathcal{M}^{\langle\Lambda\rangle}/\mathcal{U}\) of \(\mathcal{M}\) modulo \(\mathcal{U}\) of \(\Lambda\) is defined by:

\[M^{\langle\Lambda\rangle}/\mathcal{U} = \{u/\mathcal{U} | u \in M^{\langle\Lambda\rangle}\}.\]

\[\hat{F}/\mathcal{U}(u_1/\mathcal{U}, \ldots, u_n/\mathcal{U}) = (\hat{F}(u_1, \ldots, u_n))/\mathcal{U}.\]

\[\mathcal{M}^{\langle\Lambda\rangle}/\mathcal{U} \models \mathcal{R}(u_1, \ldots, u_m) \iff [\mathcal{R}(u_1, \ldots, u_m)] \in \mathcal{U}.\]

Definition 3.4 (bounded ultralimit). We denote by \(u/\mathcal{U}\) the equivalence class of \(u \in \hat{V}(X)^{\langle\Lambda\rangle}\) by the equivalence relation

\[x \sim u y \equiv [x = y]_{\Lambda} \in \mathcal{U}.\]
The bounded ultralimit $\hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U}$ of $V(X)$ modulo $\mathfrak{U}$ of $\Lambda$ is defined by:

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} = \{ u/\mathfrak{U} \mid u \in \hat{V}(X)^{\langle\Lambda\rangle} \},$$

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \models u/\mathfrak{U} \in v/\mathfrak{U} \iff [u \in v] \in \mathfrak{U}.$$  

**Theorem 3.5** (Łoś Principle of Ultralimits). Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in M^{\langle\Lambda\rangle}$,

$$M^{\langle\Lambda\rangle}/\mathfrak{U} \models \varphi(u_1/\mathfrak{U}, \ldots, u_r/\mathfrak{U}) \iff [\varphi(u_1, \ldots, u_r)] \in \mathfrak{U}.$$  

**Proof.** The proof proceeds by induction on the complexity of formulae. The only nontrivial step is the case where $\varphi$ is of the form $\exists x \psi(x)$. Suppose $[\exists x \psi(x)] \in \mathfrak{U}$. By the maximal principle (Theorem 3.3), there is $u$ satisfying $[\psi(u)] = [\exists x \psi(x)]$. Then $M^{\langle\Lambda\rangle} \models \psi(u/\mathfrak{U})$ by the induction assumption. We have thus $M^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Conversely, suppose $M^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Then there is some $u$ such that $M^{\langle\Lambda\rangle} \models \psi(u/\mathfrak{U})$. By the induction assumption, $[\exists x \psi(x)] \geq [\psi(u/\mathfrak{U})] \in \mathfrak{U}$.  

**Corollary 3.6** (Łoś-Mostowski Principle of Bounded Ultralimits). Let $\varphi(x_1, \ldots, x_r)$ be a $\Delta_0$-formula of $\mathcal{L}_\in$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \hat{V}(X)^{\langle\Lambda\rangle}$,

$$\hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U} \models \varphi(u_1/\mathfrak{U}, \ldots, u_r/\mathfrak{U}) \iff [\varphi(u_1, \ldots, u_r)] \in \mathfrak{U}.$$  

**Proof.** The proof is similar to that of Theorem 3.5. The only different part is the if-part of the case where $\varphi$ is of the form $\exists x \in \underline{y} \psi(x)$. Suppose $[\exists x \in u_k \psi(x)] \in \mathfrak{U}$. It follows from Corollary 3.4 that there is $u \in \hat{V}(X)^{\langle\Lambda\rangle}$ satisfying $[u \in u_k \land \psi(u)] = [\exists x \in u_k \psi(x)].$  

A bounded ultralimit is a pre-nonstandard universe: that satisfies (1),(2) and (3) of Definition 1.2 with Mostowski collapsing.

**Definition 3.5** (atlas). An atlas is a pair $\langle\Lambda, \mathfrak{U}\rangle$ of an LACA $\Lambda$ and an ultrafilter of $\Lambda$ such that $\text{rad}(\hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U}) = \text{rad}(\Lambda)$ and $\text{cov}(\hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U}) = \text{cov}(\Lambda)$.

**Theorem 3.7** (Sheaf representation Theorem for Nonstandard Universes). For any nonstandard universe $^*V(X)$, there is an atlas $\langle\Lambda, \mathfrak{U}\rangle$ such that $\hat{V}(X)^{\langle\Lambda\rangle}/\mathfrak{U}$ isomorphic to $^*V(X)$.

We prove the theorem in the next section.

**4. LOCAL ULTRALIMITS**

We shall see that a homomorphism of LACAs induces an elementary embedding of ultralimits and a bounded elementary embedding of bounded ultralimits. Let $h: \Lambda \rightarrow \Xi$ be a homomorphism. The induced map $h_*: M^{\langle\Lambda\rangle} \rightarrow M^{\langle\Xi\rangle}$ is defined by $h_*(u) = h \circ u$. Then we have the lemma below.

**Lemma 4.1.** Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in M^{\langle\Lambda\rangle}$

$$[\varphi(h_*(u_1), \ldots, h_*(u_r))]_\Xi = h([\varphi(u_1, \ldots, u_r)]_\Lambda).$$
Proof. There is $C \in \Lambda^\circ$ containing all the ranges of $u_k$. Since $h \upharpoonright C$ is complete, we have from Theorem 3.1
\[
\bigvee \left\{ \bigwedge_{i=1}^r h(u_i(x_i)) \bigg| \mathfrak{M} \models \varphi(x_1, \ldots, x_r) \right\} = h \left( \bigvee \left\{ \bigwedge_{i=1}^r u_i(x_i) \bigg| \mathfrak{M} \models \varphi(x_1, \ldots, x_r) \right\} \right)
\]
\[
\llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_\mathfrak{M} = h(\llbracket \varphi(u_1, \ldots, u_r) \rrbracket_\Lambda).
\]
We have thus proved the lemma. □

For $u \in \hat{V}(X)^\langle\Lambda\rangle$, since $\text{supp}(h \circ u) \subseteq \text{supp} u$, we can define the induced map $h_* : \hat{V}(X)^\langle\Lambda\rangle \to \hat{V}(X)^\langle\Lambda\rangle$ similarly.

Corollary 4.2. Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}_\in$ with only $x_1, \ldots, x_r$ free. For $u_1, \ldots, u_r \in \hat{V}(X)^\langle\Lambda\rangle$
\[
\llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_\mathfrak{M} = h(\llbracket \varphi(u_1, \ldots, u_r) \rrbracket_\Lambda).
\]
Proof. Using Corollary 3.2, we see the proof is similar to that of Lemma 4.1. □

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters of $\Lambda$ and $\Xi$, respectively. Suppose $h^{-1}\mathcal{U} \subseteq \mathcal{V}$. Then we have from Lemma 4.1 or from Corollary 4.2
\[
[u = u']_\Lambda \in \mathcal{U} \iff [h_*(u) = h_*(u')]_\Lambda \in \mathcal{V}.
\]
Therefore we can define the injection $h_* : M^\langle\Lambda\rangle / \mathcal{U} \to M^\langle\Xi\rangle / \mathcal{V}$, denoted by same $h_*$, by $h_*(u/\mathcal{U}) = h_*(u)/\mathcal{V}$. Since $\text{supp} h_*(u) \subseteq \text{supp} u$, we can define the injection $h_* : \hat{V}(X)^\langle\Lambda\rangle / \mathcal{U} \to \hat{V}(X)^\langle\Xi\rangle / \mathcal{V}$ similarly.

Lemma 4.3. The injection $h_*$ is an elementary embedding of $M^\langle\Lambda\rangle / \mathcal{U}$ into $M^\langle\Xi\rangle / \mathcal{V}$.

Proof. Let $\varphi(x_1, \ldots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \ldots, x_r$ free. From Theorem 3.5, we have for $u_1, \ldots, u_r \in M^\langle\Lambda\rangle$
\[
h(\llbracket \varphi(u_1, \ldots, u_r) \rrbracket_\Lambda) \in \mathcal{V} \iff \llbracket \varphi(u_1, \ldots, u_r) \rrbracket_\Lambda \in h^{-1}\mathcal{U} \mathcal{V}
\]
\[
\llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_\mathfrak{M} \in \mathcal{V} \iff \llbracket \varphi(h_*(u_1), \ldots, h_*(u_r)) \rrbracket_\mathfrak{M} \in \mathcal{V}
\]
\[
\mathfrak{M}^\langle\Xi\rangle / \mathcal{V} \models \varphi(h_*(u_1/\mathcal{U}), \ldots, h_*(u_r/\mathcal{U})) \iff \mathfrak{M}^\langle\Lambda\rangle / \mathcal{U} \models \varphi(u_1, \ldots, u_r).
\]
□

Corollary 4.4. The injection $h_*$ is a bounded elementary embedding of $\hat{V}(X)^\langle\Lambda\rangle / \mathcal{U}$ into $\hat{V}(X)^\langle\Xi\rangle / \mathcal{V}$.

Proof. Using Corollary 3.6, we see the proof is similar to that of Lemma 4.3. □

Let $I$ be a set relative to $V(X)$. We shall find a one-to-one correspondence between $\mathcal{P}(I)^\langle\Lambda\rangle$ and $\mathcal{B}(\Lambda^\circ I)$. Note that $\mathcal{P}(I)^\langle\Lambda\rangle$ is the set of "the subsets of $I$ in $\hat{V}(X)^\langle\Lambda\rangle"$. For $A \in \mathcal{P}(I)^\langle\Lambda\rangle$, there is $C \in \Lambda^\circ$ such that $\text{rng} A \subseteq C$. Define $g : I \to \mathcal{B}(\Lambda)$ by $g(i) = [i \in A]_\Lambda$. Then we have $\text{rng} g \subseteq C$ and $g \in \mathcal{B}(\Lambda^\circ I)$. Conversely, for $g \in \mathcal{B}(\Lambda^\circ I)$, there is $C \in \Lambda^\circ$ such that $\text{rng} g \subseteq C$. Define $A : \mathcal{P}(I) \to C$ by
\[
A(x) = \bigwedge_{i \in I} \text{sg}_x(i, g(i)), \quad \text{where} \quad \text{sg}_x(i, b) = \begin{cases} b & \text{if } i \in x, \\ -b & \text{if } i \in I \setminus x. \end{cases}
\]
Since $C$ is completely distributive, we have $A \in \mathcal{P}(I)^{\langle\Lambda\rangle}$. Suppose $g(i) = [i \in A]_\Lambda$ and $g'(i) = [i \in A']_\Lambda$. Then we see $(g \wedge g')(i) = [i \in A \cap A']_\Lambda$ and $(\neg g)(i) = [i \not\in I \setminus A]_\Lambda$. In the context above, the relation $g(i) = [i \in A]_\Lambda$ sets up a one-to-one correspondence between $\mathcal{P}(I)^{\langle\Lambda\rangle}$ and $\mathfrak{B}(\Lambda^{[I]})$ as Boolean algebras. From now on, we identify $\mathcal{P}(I)^{\langle\Lambda\rangle}$ with $\mathfrak{B}(\Lambda^{[I]})$.

We shall define the special element $\delta \in I^{\langle\Lambda^{[I]\rangle}} \subseteq \check{V}(X)^{\langle\Lambda^{[I]\rangle}}$ by

$$\delta(x)(i) = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{if } x \neq i. \end{cases}$$

We call the $\delta$ diagonal element of $I$ on $\Lambda$. Let $j : \Lambda \to \Lambda^{[I]}$ be the canonical embedding. Then $j$ is also a Boolean monomorphism of $\mathfrak{B}(\Lambda)$ into $\mathcal{P}(I)^{\langle\Lambda\rangle}$. The diagonal element $\delta$ has following properties.

**Lemma 4.5.** The following statements hold.

1. $[j(b) = \check{I}]_\Lambda = b$ for every $b \in \mathfrak{B}(\Lambda)$.
2. $[\delta \in j_*(g)]_{\Lambda^{[I]}} = g$ for every $g \in \mathcal{P}(I)^{\langle\Lambda\rangle}$.

**Proof.** Since $[i \in j(b)]_\Lambda = j(b)(i) = b$ for all $i \in I$, $[j(b) = \check{I}]_\Lambda = [j(b) \supseteq \check{I}]_\Lambda = \bigwedge_{i \in I} [i \in j(b)]_\Lambda = b$. From the definition of $\delta$, it is clear that $[\delta = \check{I}]_{\Lambda^{[I]}}(i) = \delta(i)(i) = 1$. Then we have $[\delta \in j_*(g)]_{\Lambda^{[I]}}(i) = [i \in j_*(g)]_{\Lambda^{[I]}}(i) = [i \in g]_\Lambda = g(i)$.

**Theorem 4.6.** For any $v \in \check{V}(X)^{\langle\Lambda^{[I]\rangle}}$, there is a map $w : \check{I} \to (\text{supp } v)^\vee$ in $\check{V}(X)^{\langle\Lambda\rangle}$ such that $v = j_*(w)(\delta)$ holds in $\check{V}(X)^{\langle\Lambda\rangle}$.

**Proof.** Since $\text{rng } v \in \Lambda^{[I]}$, $\bigcup_{g \in \text{rng } v} \text{rng } g \in \Lambda$. Therefore we can define $w : (\text{supp } v)^\check{I} \to \mathfrak{B}(\Lambda)$ by

$$w(s) = \bigvee_{i \in I} v(s(i))(i).$$

Then we get $w$ as required. First, we show $w \in \check{V}(X)^{\langle\Lambda\rangle}$. If $s \neq s'$, then there is $i_0 \in I$ such that $s(i_0) \neq s'(i_0)$. Since $\text{rng } v$ is pairwise disjoint, we have

$$w(s) \wedge w(s') \leq v(s(i_0))(i_0) \wedge v(s'(i_0))(i_0) = 0.$$
We have thus shown $w \in \hat{V}(X)^{\langle \Lambda \rangle}$. For each $i \in I$, since $w(s) \leq v(s(i))(i)$ holds for every $s \in (\text{supp } v)^I$, we have
\[
\begin{align*}
\langle v = j_*(w)(\delta) \rangle_{\Lambda[I]}(i) &= \langle v = j_*(w)(\check{i}) \rangle_{\Lambda[I]}(i) \\
&= \left( \bigvee \{ v(y) \wedge j(w(s)) \wedge \check{i}(x) \mid y = s(x) \} \right)(i) \\
&= \bigvee_{s \in (\text{supp } v)^I} (v(s(i))(i) \wedge w(s)) \\
&= \bigvee_{s \in (\text{supp } v)^I} w(s) = 1.
\end{align*}
\]
We have thus proved the theorem. \hfill \Box

Let $\mathcal{U}$ be an ultrafilter of an LACA $\Lambda$. A **local ultralimit** $\rho : \hat{V}(X)^{\langle \Lambda \rangle} / \mathcal{U} \to \star V(X)$ is a bounded elementary embedding satisfying $\rho(\check{x}/\mathcal{U}) = \check{x}$ for every $x \in V(X)$.

**Theorem 4.7** (Local Ultralimit Theorem). Let $\rho : \hat{V}(X)^{\langle \Lambda \rangle} / \mathcal{U} \to \star V(X)$ be a local ultralimit and let $p$ be an internal element of $\star V(X)$. Then there is a local ultralimit $\tau : \hat{V}(X)^{\langle \Lambda[I] \rangle} / \mathcal{V} \to \star V(X)$ such that the following conditions hold.

(i) The index set $I$ is a set relative to $V(X)$ and $|I| = \text{nos}(p)$.
(ii) Let $j : \Lambda \to \Lambda[I]$ be the canonical embedding. Then $\mathcal{U} = j^{-1}\mathcal{V}$ and $\rho = \tau \circ j_*$. 
(iii) The submodel rng $\tau$ of $\star V(X)$ is the minimal bounded elementary submodel of $\star V(X)$ that contains $\{p\} \cup \text{rng } \rho$.

\[
\begin{array}{c}
\hat{V}(X)^{\langle \Lambda[I] \rangle} / \mathcal{V} \xrightarrow{\tau} \star V(X) \\
V(X) \xrightarrow{j_*} \hat{V}(X)^{\langle \Lambda \rangle} / \mathcal{U} \xrightarrow{\rho} \star V(X)
\end{array}
\]

Proof. Let $I$ be a set relative to $V(X)$ such that $p \in I$ and $|I| = \text{nos}(p)$. We have identified $\mathcal{B}(\Lambda[I])$ with $\mathcal{P}(I)^{\langle \Lambda \rangle}$. Define $\mathcal{V} \subseteq \mathcal{B}(\Lambda[I])$ by
\[
g \in \mathcal{V} \iff p \in \rho(g/\mathcal{U}).
\]
Then $\mathcal{V}$ is an ultrafilter of $\Lambda[I]$. Let $b$ be an element of $\mathcal{U}$. From (1) of Lemma 4.5, $\rho(j(b)/\mathcal{U})$ coincides $\check{b}$. Then we have $j(b) \in \mathcal{V}$ from the definition of $\mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ are maximal filters, we obtain $j^{-1}\mathcal{V} = \mathcal{U}$. Let $\varphi(x_1, \ldots, x_r)$ be a $\Delta_0$-formula of $\mathcal{L}_\mathcal{E}$ with only $x_1, \ldots, x_r$ free. Let $w_1, \ldots, w_r$ be elements of $\hat{V}(X)^{\langle \Lambda \rangle}$. By Theorem 4.6, there are maps $w_1, \ldots, w_r$ from $\check{I}$ in $\hat{V}(X)^{\langle \Lambda \rangle}$ such that $v_k = j_*(w_k)(\delta)$ hold, where $\delta$ is the diagonal element of $I$ on $\Lambda$. Putting $g_0 = \{ i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i)) \}$ in $\hat{V}(X)^{\langle \Lambda \rangle}$, we have from (2) of Lemma 4.5
\[
g_0 = \langle \delta \in j_*(g_0) \rangle_{\Lambda[I]} \\
= \langle \delta \in j_*(\{ i \in \check{I} \mid \varphi(w_1(i), \ldots, w_r(i)) \}) \rangle_{\Lambda[I]} \\
= \langle \delta \in \{ i \in \check{I} \mid \varphi(j_*(w_1)(i), \ldots, j_*(w_r)(i)) \} \rangle_{\Lambda[I]} \\
= \langle \varphi(j_*(w_1)(\delta), \ldots, j_*(w_r)(\delta)) \rangle_{\Lambda[I]} \\
= \langle \varphi(v_1, \ldots, v_r) \rangle_{\Lambda[I]}
\]
and we have
\[ \hat{V}(X)^{\langle \Lambda \rangle}/U \models g_0/U = \{ i \in I/U \mid \varphi((w_1/U)(i), \ldots, (w_r/U)(i)) \} \]
\[ \rho(g_0/U) = \{ i \in I \mid \varphi(\rho(w_1/U)(i), \ldots, \rho(w_r/U)(i)) \}. \]

By the definition of \( \mathcal{V} \), we obtain
\[ g_0 \in \mathcal{V} \iff p \in \rho(g_0/U) \]
\[ [\varphi(v_1, \ldots, v_r)]_{\Lambda[I]} \in \mathcal{V} \iff \varphi(\rho(w_1/U)(p), \ldots, \rho(w_r/U)(p)). \]

The case \( \varphi(x_1, x_2) \equiv "x_1 = x_2" \) enables us to define the operation \( v/\mathcal{V} \mapsto \rho(w/U)(p) \) where \( v = j_*(w)(\delta) \) holds in \( \hat{V}(X)^{\langle \Lambda[I] \rangle} \). Thus, defining \( \tau : \hat{V}(X)^{\langle \Lambda[I] \rangle}/\mathcal{V} \to \mathcal{V}(X) \) by \( \tau(v/\mathcal{V}) = \rho(w/U)(p) \) where \( v = j_*(w)(\delta) \) holds in \( \hat{V}(X)^{\langle \Lambda[I] \rangle} \), we get \( \tau \) as required. In fact, it is clear in the preceding context that \( \tau \) is a bounded elementary embedding of \( \hat{V}(X)^{\langle \Lambda[I] \rangle}/\mathcal{V} \) into \( \mathcal{V}(X) \). Let \( \iota \) be the identity map on \( I \), then we see \( \tau(j_*(\iota/U)(\delta/\mathcal{V})) = \rho(\iota/U)(p) = \iota(p) = p \). For \( u/U \in \hat{V}(X)^{\langle \Lambda \rangle}/U \), let \( \tilde{u} \) be the constant map from \( I \) onto \( \{ u \} \) in \( \hat{V}(X)^{\langle \Lambda \rangle} \), then we have \( \tau(j_*(u/U)) = \rho(\tilde{u}/U)(p) = \rho(u/U) \). Suppose a bounded elementary submodel \( W \) of \( \mathcal{V}(X) \) contains \( \{ p \} \cup \text{rng} \rho \). From the definition of \( \tau \), \( \tau(v/\mathcal{V}) = \rho(w/U)(p) \in W \) for some \( w/U \in \hat{V}(X)^{\langle \Lambda \rangle}/U \).

Therefore \( \text{rng} \tau \) is the minimum. We have completed the proof of Theorem 4.7. \( \square \)

Let \( \{ j_{d'} : \Lambda_d \to \Lambda_{d'} \}_{d \leq d', d \in D} \) be an embedding system of LACAs with direct limit \( \{ j_d : \Lambda_d \to \Lambda \}_{d \in D} \). Let \( U \) be an ultrafilter of \( \Lambda \), then each \( U_d = j_d^{-1}U \) is an ultrafilter of \( \Lambda_d \).

**Theorem 4.8 (Elementary Net Theorem of Ultraproducts).** Let \( \mathcal{M} \) and \( \mathcal{N} \) be models for \( \mathcal{L} \). Suppose there are elementary embeddings \( \tau_d : \mathcal{M}^{\langle \Lambda_d \rangle}/U_d \to \mathcal{N} \) satisfying the condition \( \tau_d = \tau_d \circ j_{d*}^{d'} \) for \( d \leq d' \). Then there is an elementary embedding \( \tau : \mathcal{M}^{\langle \Lambda \rangle}/U \to \mathcal{N} \) such that \( \tau_d = \tau \circ j_{d*} \) for \( d \in D \).

**Proof.** Let \( v \) be an element of \( \mathcal{M}^{\langle \Lambda \rangle} \). Since \( \text{rng} v \in \Lambda = \{ j_d'' S \mid d \in D \text{ and } S \in \Lambda_d \} \) from the definition of direct limits, there is \( u \in \mathcal{M}^{\langle \Lambda_d \rangle} \) such that \( v = j_{d*}(u) \). Therefore defining \( \tau(v/U) = \tau_d(u/U_d) \) where \( v = j_{d*}(u) \), we get \( \tau \) as required.

Let \( \varphi(x_1, \ldots, x_r) \) be a formula of \( \mathcal{L} \) with only \( x_1, \ldots, x_r \) free and let \( v_1, \ldots, v_r \) be elements \( \mathcal{M}^{\langle \Lambda \rangle} \). Then there are \( d \in D \) and \( u_1, \ldots, u_r \in \mathcal{M}^{\langle \Lambda_d \rangle} \) such that \( v_k = j_{d*}(u_k) \). We conclude as below.

\[ \mathcal{M}^{\langle \Lambda_d \rangle}/U_d \models \varphi(u_1/U_d, \ldots, u_r/U_d) \iff \mathcal{N} \models \varphi(\tau_d(u_1/U_d), \ldots, \tau_d(u_r/U_d)) \]
\[ \mathcal{M}^{\langle \Lambda \rangle}/U \models \varphi(v_1/U, \ldots, v_r/U) \iff \mathcal{N} \models \varphi(\tau(v_1/U), \ldots, \tau(v_r/U)). \]
Theorem 4.9 (Bounded Elementary Net Theorem of Bounded Ultralimits).
Suppose there are local ultralimits \( \tau_d: \hat{V}(X)^{\langle \Lambda_d \rangle}/\mathfrak{U}_d \to \star V(X) \) satisfying
the condition \( \tau_d = \tau_{d'} \circ j_{d*}^{d'} \) for \( d \leq d' \). Then there is a local ultralimit
\( \tau: \hat{V}(X)^{\langle \Lambda \rangle}/\mathfrak{U} \to \star V(X) \) such that \( \tau_d = \tau \circ j_{d*} \) for \( d \in D \).

\[
\begin{array}{c}
\hat{V}(X)^{\langle \Lambda_d \rangle}/\mathfrak{U}_d \\
\downarrow j_{d*}^{d'} \\
\hat{V}(X)^{\langle \Lambda_d \rangle}/\mathfrak{U}_d \\
\end{array}
\begin{array}{c}
\star V(X) \\
\downarrow \tau \\
\star V(X) \\
\end{array}
\]

Proof. Similar to the proof of Theorem 4.8. \( \square \)

We call the pair \( \langle \Lambda, \mathfrak{U} \rangle \) in Theorem 4.8 or Theorem 4.7 the direct limit of
\( \{ \langle \Lambda_d, \mathfrak{U}_d \rangle \}_{d \leq d', d, d' \in D} \).

Proof of Theorem 3.7 Let \( \{ p_\zeta \}_{\zeta < \kappa} \) be a sequence in \( \star V(X) \) with \( \kappa = \text{cov}(\star V(X)) \).
We define local ultralimits \( \{ \rho_\zeta: \hat{V}^{\langle \Lambda_\zeta \rangle}/\mathfrak{U}_\zeta \to \star V(X) \}_{\zeta < \kappa} \) of \( \star V(X) \) by:
\[
\begin{align*}
\Lambda_0 &= \mathcal{P}(\{0,1\}), \quad \mathfrak{U}_0 = \{1\}, \\
\Lambda_{\zeta+1} &= \Lambda^{[I_\zeta]}, \quad |I_\zeta| = \text{nos}(p_\zeta), \quad p_\zeta \in \text{rng} \rho_{\zeta+1} \quad \text{in Theorem 4.7}. \\
\end{align*}
\]

\( \langle \Lambda_\lambda, \mathfrak{U}_\lambda \rangle \) is the direct limit of \( \{ \langle \Lambda_\zeta, \mathfrak{U}_\zeta \rangle_{\zeta < \lambda} \} \) in Theorem 4.9.

Then the direct limit \( \langle \Lambda, \mathfrak{U} \rangle \) of \( \{ \langle \Lambda_\zeta, \mathfrak{U}_\zeta \rangle \}_{\zeta < \kappa} \) is an atlas of \( \star V(X) \). \( \square \)

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E-mail address: muramasa@ms.u-tokyo.ac.jp