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On quasi-minimal structures

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1 Introduction

Unlike the model theory of \((\mathbb{C}, +, \cdot, 0, 1)\), we do not know hardly anything about the model theory of \((\mathbb{C}, +, \cdot, \exp, 0, 1)\). This situation is very different from the one concerning the model theory of \((\mathbb{R}, +, \cdot, <, \exp, 0, 1)\) or of \((\mathbb{R}, +, \cdot, <, 0, 1, f)_{f \in \text{An}([0, 1])}\), where \(\text{An}([0, 1]) = \{ f \mid f : U \to \mathbb{R} \text{ is analytic for } U \text{ some open } \supset [0, 1]^n \}\).

First attempts to investigate the model theory of \((\mathbb{C}, +, \cdot, \exp, 0, 1)\) are made by B. Zil'ber who has conjectured that the structure is a quasi-minimal structure which is a generalization of minimal structures.

Definition 1. An uncountable structure is called quasi-minimal if its definable sets are at most countable or co-countable.

The conjecture has not yet been answered neither affirmatively nor negatively. As a minor contribution to this line of research we study basic properties of quasi-minimal structures. It is well known that we can define a combinatorial geometry on minimal structures using a closure operation. It is then very natural to define a similar geometry on quasi-minimal structures.

We thank TSUBOI Akito of the University of Tsukuba for his valuable comments and remarks.
2 Pre-Geometry

In this note we only work with countable languages $L$. We also assume that the reader is familiar with basic model theory.

**Definition 2.** Let $M$ be an uncountable structure and $A \subseteq M$. Then

$$
ccl_M(A) = \{ b \in M : b \models \varphi, \varphi^M \text{ is countable for some } \varphi \in L(A) \}
$$

We omit the subscript $M$ if it is clear from context.

**Definition 3.** Let $X$ be a set and $cl$ be a function from $P(X)$ to $P(X)$, where $P(X)$ denotes the set of all subsets of $X$. If $X$ and the function $cl$ satisfy the following properties, we say that $(X, cl)$ is a pre-geometry. Let $A \subseteq X$ and $b, c \in X$.

(I) $A \subseteq cl(A)$.

(II) (Finite Character) $b \in cl(A) \Rightarrow b \in cl(A_0)$ for some finite $A_0 \subseteq A$.

(III) (Transfer Property) $cl(cl(A)) = cl(A)$.

(IV) (Exchange Property) $b \in cl(Ac) - cl(A) \Rightarrow c \in cl(Ab)$.

Let $M$ be an uncountable structure. We first show that $(M, ccl)$ satisfies these properties under some conditions.

**Proposition 4.** For any infinite structure $M$, $(M, ccl)$ satisfies (I) and (II).

**Proof:** Clear by the definition of $ccl$ since the language is countable.

**Lemma 5.** Suppose $M$ is a quasi-minimal structure. Let $A \subseteq M$, $|A| < |M|$ and $b, c \in M - ccl(A)$. Then $tp(b/A) = tp(c/A)$.

**Proof:** If $tp(b/A) \neq tp(c/A)$, then there is a formula $\varphi(x) \in L(A)$ such that both $\varphi(b)$ and $\neg \varphi(c)$ hold. By the quasi-minimality of $M$, either $\varphi$ or $\neg \varphi$ is countable. Hence either $b$ or $c$ is in $ccl(A)$. This contradicts to the assumption on $b, c$. 
Proposition 6. Assume that $M$ is quasi-minimal and homogeneous. Then $(M, \text{ccl})$ satisfies the transfer property (III).

Proof: Let $A \subset M$. We show that $\text{ccl}(\text{ccl}(A)) = \text{ccl}(A)$. Clearly $\text{ccl}(\text{ccl}(A)) \supset \text{ccl}(A)$ holds by (I). For the other direction, it is enough to show that $\text{ccl}(\text{ccl}(A)) \subset \text{ccl}(A)$ for finite $A \subset M$, since $\text{ccl}(A) = \bigcup \{\text{ccl}(B) : B \subset M, |B| < \aleph_0\}$ by (II). Assume that there is an element $b \in \text{ccl}(\text{ccl}(A)) - \text{ccl}(A)$. Since $|\text{ccl}(A)| \leq |A| + |L|$, there is an element $c \in M - \text{ccl}(\text{ccl}(A))$. By Lemma 5, we have $\text{tp}(b/A) = \text{tp}(c/A)$. So by the homogeneity assumption on $M$, there is an $A$-automorphism $f$ of $M$ such that $f(b) = c$. Since $b$ is in $\text{ccl}(\text{ccl}(A))$, $c$ is also in $\text{ccl}(\text{ccl}(A))$. This contradicts to the assumption on $c$.

Proposition 7. Assume that $M$ is quasi-minimal, homogeneous and $|M| \geq \aleph_2$. Then $(M, \text{ccl})$ satisfies the exchange property (IV).

Proof: By the finite character (II), it is enough to show the exchange property (IV) assuming that $A \subset M$ is finite. Suppose that there are elements $b, c \in M$ such that $b \in \text{ccl}(Ac) - \text{ccl}(A)$ and $c \notin \text{ccl}(Ab)$. By Lemma 5, we have $\text{tp}(b/A) = \text{tp}(c/A)$. Let $p(x, y) = \text{tp}(bc/A)$. We construct a sequence $(b_i)_{i \leq \omega_1}$ such that $b_0 = b$, $b_1 = c$ and $i < j \Rightarrow \text{tp}(b_ib_j/A) = \text{tp}(bc/A)$. Suppose that we have chosen $b_j (j < i)$.

Claim. $\cap_{j<i} p(b_j, M) \neq \emptyset$.

Proof of claim: Since $\cap_{j<i} p(b_j, M) = M - \cup_{j<i} (M - p(b_j, M))$, it is enough to show that $M - p(b_j, M)$ is countable for each $j$. Let $d \in M - p(b_j, M)$. Then there is a formula $\varphi(b_j, y) \in p(b_j, y)$ such that $\neg \varphi(b_j, d)$ holds. Since $\varphi(b, c)$ and $\text{tp}(b/A) = \text{tp}(b_j/A)$, $\varphi(b_j, y)$ is not countable in $M$ by the homogeneity of $M$. Hence $\neg \varphi(b_j, y)$ is countable by the quasi-minimality of $M$ and $d \in \text{ccl}(Ab_j)$. So $M - p(b_j, M)$ is countable. This completes the proof of Claim.

Now we finish the proof of proposition. Let $b_i \in \cap_{j<i} p(b_j, M)$. By the definition of $b_i$, $B = \{b_i : i < \omega_1\} \subset \text{ccl}(Ab_{\omega_1})$. But $B$ is uncountable. This is a contradiction.
3 Some Examples

1. Any strongly minimal structure is quasi-minimal.

2. Let $M$ be uncountable, $P$ a unary predicate and $|M^P|$ countable, then $(M, P)$ is quasi-minimal.

3. Let $T$ be a theory of an equivalence relation $E$ with infinitely many infinite equivalent classes. Then $T$ has quasi-minimal models such as;

   (a) $E$ is an equivalence relation with uncountably many countable equivalence classes.

   (b) $E$ is an equivalence relation with one uncountable class and countable countable classes.

4. There are no quasi-minimal random graphs.

   **Proof:** Assume that $(M, R)$ is a quasi-minimal random graph. Let $a \in M$ and $b, c \in M - \text{ccl}(a)$. By Lemma 5, we may assume without loss of generality that any element in $M - \text{ccl}(a)$ is connected to $a$. Since $M$ is a random graph, there is an element $d \in M$ such that $R(d, a) \land R(d, b) \land \neg R(d, c)$ holds. Then $d \in \text{ccl}(a)$, because $R(d, a)$ holds. Hence $\text{tp}(b/d) = \text{tp}(c/d)$, but $R(d, b) \land \neg R(d, c)$ holds as well. This is a contradiction.

5. (a) $(\omega_1, <)$ is not quasi-minimal, since the successor points are definable.

   (b) $(\omega_1 \times \mathbb{Z}, <)$ (< is the lexicographic order) is quasi-minimal but not homogeneous.

   (c) $(\omega_1 \times \mathbb{Q}, <)$ is quasi-minimal and homogeneous. But the exchange property (IV) does not holds (since the cardinality is less than $\aleph_2$).

4 Remarks

Unlike strongly minimal sets, the first-order property of quasi-minimal sets are not easily understood. The reason for this is that for two elementarily
equivalent structures $M$ and $N$, the quasi-minimality of $M$ may or may not imply the quasi-minimality of $N$. This forbids us to employ the usual model theoretic tools such as compactness arguments.

Another difficulty is that the luck of natural interesting examples. Zil'ber's conjecture on the structure $(\mathbb{C}, +, \cdot, \exp, 0, 1)$ seems plausible but at this moment we do not know how to study the structure. As a very small first step we notice the following:

**Remark 8.** It seems very natural to claim that for a quasi-minimal structure $(M, \cdots)$, the expanded structure $(M, \cdots, P_i(i \in \omega))$ is also quasi-minimal where each $P_i$ is a unary predicate whose interpretation is a countable subset of $M$. As a corollary to this we have that $(\mathbb{C}, +, \cdot, 0, 1, P_i(i \in \omega))$ is quasi-minimal where each $P_i$ is a unary predicate whose interpretation is a countable subset of $\mathbb{C}$.

**Remark 9.** In Section 2 we studied the basic pre-geometric properties of quasi-minimal sets. Although our proof used the additional homogeneity assumption on the structure, it is not clear whether this assumption is necessary.

**Remark 10.** In model theory we often work in a saturated model. It seems that our usual arguments for constructing saturated models are not enough to define a saturated quasi-minimal structures.

5 References