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A topological approach to a group structure through monomials

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1 Introduction

In order to investigate the structure of a group G, we tried to study the set $\mathrm{Mon}_1(G)$ of all monomials over G. We can naturally define two distinct operations of such monomials. One is based on an ordinary multiplication of G and another is based on substitution. A monomial over G can also be regarded as a function over G. Such a recognition shows that the set of all monomials over a group and certain subsets of it have some algebraic structures. Actually $\mathrm{Mon}_1(G)$ forms a group under ordinary multiplication and it also forms a monoid under substitution. In this paper we gave a special attention to a subset $\mathrm{SMon}_1(G)$ of $\mathrm{Mon}_1(G)$ each of whose element sends the identity of G to itself. $\mathrm{SMon}_1(G)$ and its certain quotient set form semi-distributive ring (SDR) and the group algebra $Z[G]$ respectively. In the former part of this paper, some algebraic properties of $\mathrm{SMon}_1(G)$ as an SDR are discussed. Using $\mathrm{Mon}_1(G)$ we defined a topology over G. In order to visualize inclusion relation of these closed sets, we defined a semi-lattices and a lattice whose point is a closed set of the topology.The main purpose of this paper is to show the relationship between the shape of the lattice and the structure of the group.

2 Preliminary

Let G be a group and F_n be a free group which is generated by n invariants $X_1, \ldots, X_n$. We regard a monomial over a group with invariants $X_1, \ldots, X_n$ as an element of the free product $G * F_n$ of a group G and a free group F_n. We denote the set of all monomials over G with n invariants by $\mathrm{Mon}_n(G)$ which is the same set to $G * F_n$. For elements $f(X) = (123)X^2(23)X^{-1}, g(X) = (132)X^{-1}(23)$ of $\mathrm{Mon}_1(S_3)$, two different operations "$\cdot$" and " $\circ$ " can be defined naturally as follows;

$$\begin{aligned} f(X) \cdot g(X) &= (123)X^2(23)X^{-1}(132)X^{-1}(23), \\ g(X) \cdot f(X) &= (132)X^{-1}(23)(123)X^2(23)X^{-1} = (132)X^{-1}(12)X^2(23)X^{-1}. \end{aligned}$$

Hence it is clear to see $1 = 1_G$ is the identity element of $\mathrm{Mon}_1(S_3)$ and

$$f(X)^{-1} = X(23)X^{-2}(132), \text{ and } \quad g(X)^{-1} = (23)X(123).$$

$$\begin{aligned} f(X) \circ g(X) &= f(g(X)) = (12)X^{-3}(23)X \quad \text{and} \\ g(X) \circ f(X) &= g(f(X)) = X^{-1}(23)X^2(12). \end{aligned}$$

It is clear to see that X is the identity element for the operation "$\circ$".Therefore we can see that $\mathrm{Mon}_1(S_3)$ forms a group under the operation " $\cdot$" with the identity element 1_G and forms a monoid under the operation "$\circ$ " with the identity element X. But it is not easy to analize the algebraic structure of $\mathrm{Mon}_n(G)$, so we are going to investigate certain subsets and quotient set of $\mathrm{Mon}_n(G)$ with which we are able to deal more easily. Take it for granted to call $f(X) \cdot g(X)$ and $f(X) \circ g(X)$ the ordinary product and the composition product of $f(X)$ and $g(X)$ respectively.

Definition 1 Let G be a group, $X_1, \ldots, X_n$ be invariants. F_n is a free group generated by $X_1, \ldots, X_{n-1}$ and X_n.
(1) $\mathrm{Mon}_n(G)$ denotes the set of all monomials over G, which is the same thing to the free product $G * F_n$ of G and F_n.

A general form of an element $f(X)$ of $\mathrm{Mon}_1(G)$ is as follows;

$$f(X) = \begin{cases} a_1X^{e_1}a_2X^{e_2} \cdots a_rX^{e_r}a_0 & \text{or} \\ a & \end{cases},$$

where $\quad a_1, a_2, \cdots a_r \in G - \{1\}, \quad a_0, a_1, a \in G, \quad e_1, \cdots, e_r \in \mathbb{Z} - \{0\}.$

(2) For an element $f(X_1, \ldots, X_n)$ of $\mathrm{Mon}_n(G)$, $\deg_i f(X)$ is defined to be the sum of powers of invariant X_i which appear in $f(X)$.

(3) For an element $f(X)$ of $\mathrm{Mon}_n(G)$, $\deg f(X)$ is defined to be the sum $\sum_{i=1}^{n} \deg_i f(X)$ of $\deg_i f(X)$'s for all i.

Note that an element $f(X) = f(X_1, \dots, X_n)$ of $\mathrm{Mon}_n(G)$ can be regarded as a mapping of G^n to G when an element $((g_1, \dots, g_n)$ of G^n is substituted into $f(X)$.

Example 1 *Let G be the dihedral group D_{10} of order 10. An element $f(X_1, X_2)$ $(13524)X_1^{-1}(12)(35)X_2$ of $\mathrm{Mon}_2(D_{10})$ sends an element $((12345), (14253))$ of $D_{10} \times D_{10}$ to $(14)(23)$.*

(4) $\mathrm{SMon}_n(G)$ stands for a normal subgroup of $\mathrm{Mon}_n(G)$ defined by

$$\mathrm{SMon}_n(G) = \{f(X) = f(X_1, \dots, X_n) \in \mathrm{Mon}_n(G) | f(1, \dots, 1) = 1\}.$$

It is obvious that $\mathrm{SMon}_n(G)$ is a normal subgroup and a submonoid of $\mathrm{Mon}_n(G)$ under operations "$\cdot$ " and " $\circ$ " respectively. We write this situation that $(\mathrm{SMon}_n(G), \cdot)$ forms a subgroup of $(\mathrm{Mon}_n(G), \cdot)$ and $(\mathrm{SMon}_n(G), \circ)$ forms a submonoid of $(\mathrm{Mon}_n(G), \circ)$. Seeing an element of $\mathrm{Mon}_n(G)$ as a function or a mapping of G^n to G, it is observed that two distinct elements $f(X)$ and $h(X)$ are possible to work as the same function, namely $f(g_1, \dots, g_n)$ coinsides with $h(g_1, \dots, g_n)$ for any element $(g_1, \dots, g_n)$ of G^n. This fact urges us to divide $\mathrm{Mon}_n(G)$ into some classes each of which is a collection of elements which stand for the same function of G^n to G. In order to formulate the situation above, we define $\mathrm{I}_n(G)$ and $\mathrm{PMon}_n(G)$ as follows. $\mathrm{I}_n(G)$ is a subset of $\mathrm{Mon}_n(G)$ any of whose element $f(X) = f(X_1, \dots, X_n)$ satisfies that $f(g_1, \dots, g_n) = 1$ for any element $(g_1, \dots, g_n)$ of G^n. $\mathrm{PMon}_n(G)$ is defined to be a quotient group $\mathrm{Mon}_n(G)/\mathrm{I}_n(G)$ of $\mathrm{Mon}_n(G)$ by $\mathrm{I}_n(G)$ and it is claer that each equivalent class is a collection of the same functions on G. $\mathrm{PSMon}_n(G)$ can be defined in the same way:

$$\mathrm{PSMon}_n(G) = \mathrm{SMon}_n(G)/\mathrm{I}_n(G).$$

Following propositions show the structures of $\mathrm{PMon}_1(G)$for some known

group G.

Theorem 1 *Let G be an abelian group. Then $\mathrm{PMon}_n(G)$ is isomorphic to $\mathbb{Z}_{\mathrm{exp}G}^n \times G$.*

Remark 1 *For a dihedal group D_{2p} of order $2p$ for a prime p, $\mathrm{PSMon}_1(D_{2p})$ is isomorphic to $\mathbb{Z}_p \times (\mathbb{Z}_p^2 : \mathbb{Z}_2)$.*

Theorem 2 *Let G be a group.* $\mathrm{PMon}_1(G) \simeq G \times \cdots \times G$ ($|G|$times) *if and only if $G \simeq \mathbb{Z}_2$ or G is a non abelian simple group.*

3 Semi Distributive Rings

Definition 2 Let $R = (R, \cdot, \circ)$ be a set in which two distinct operations are defined. $R = (R, \cdot, \circ)$ is said to be a left SDR if R satisfies following four conditions
(1)$(R, \cdot)$ forms a group with the identity 1_R,
(2)$(R, \circ)$ forms a monoid with the identity X,
(3)$1_R \circ a = a \circ 1_R = 1_R$ holds for any $a \in R$,
(4)Left distributity holds, namely

$$(x \cdot y) \circ z = (x \circ z)(y \circ z) \text{ for any } x, y \text{ and } z \in R.$$

In order to see the some analogues of considerations which appear in ordiary ring theory, we give definitions of an ideal, homomorphisms of SDR, and the homomorphism theorem for them as follows.

Definition 3 Let R be an SDR. A subset I of R is said to be an ideal of R if I satisfies following three conditions:
(1)I is a normal subgroup of $(R, \cdot)$
(2)$R \circ I \circ R \subseteq I$,
(3)$(a \cdot I) \circ (b \cdot I) \subseteq (a \circ b) \cdot I$ for any $a, b \in R$, equivqlently $a \circ (b \cdot i) \cdot I \subseteq (a \circ b) \cdot I$ for any $i \in I$ and $a, b \in R$.

Following three are examples of an ideal of an SDR.
(i)An ideal of an arbitrary ring.
(ii)The commutator subgroup $[\mathrm{SMon}_n(G)^n, \mathrm{SMon}_n(G)^n]$ is an ideal of an SDR $\mathrm{SMon}_n(G)^n$.
(iii) For a rational integer m, the inverse image $\deg(m\mathbb{Z})$ of an ideal $m\mathbb{Z}$ of $\mathbb{Z}$ is an ideal of an SDR $\mathrm{SMon}_1(G)$.

Definition 4 Let A and B be SDR's. A mapping φ of A to B is said to be an SDR homomorphism if φ preserves two operations "$\cdot$ " and "$\circ$" and sends the identity element of A to that of B.

Note that a kernel of φ is an ideal of A. A proposition which is similar to the homomorphism theorem for an ordinary ring holds as follows.

Theorem 3 *Let A and B be* SDR*'s and φ be an* SDR *homomorphism of A to B, then the image of φ is isomorphic to the quotient* SDR $A/\ker\varphi$ *as an* SDR, *i.e. Im $\varphi \simeq A/\ker\varphi$. as an* SDR.

Following are some fundamental properties of $\mathrm{SMon}_1(G)$.

Theorem 4 *Let* $\mathrm{U}(\mathrm{SMon}_1(G))$ *be the set of invertible elements of* $(\mathrm{SMon}_1(G), \circ)$, *then* $\mathrm{U}(\mathrm{SMon}_1(G))$ *is isomorphic to* $Z_2 \times G$.

Corollary 1 *Let* G *and* H *be groups as an* SDR $\mathrm{PSMon}_1(G)$ *is isomorphic to* $\mathrm{PSMon}_1(H)$ *as an* SDR *if and only if* G *is isomorphic to* H *as a group.*

Theorem 5 *Let* G *and* H *be groups.* $\mathrm{PSMon}_1(G \times H)$ *is isomorphic to* $\mathrm{PSMon}_1(G) \times \mathrm{PSMon}_1(H)$ *as an* SDR *if and only if* $\exp(H^{ab})$ *is prime to* $\exp(G^{ab})$.

Theorem 6 $\mathrm{SMon}_1(G)^{ab}$ *is isomorphic to* $Z[G]$ *as an* SDR.

4 Topologies on G defined by some set of monomials

In this section we will try to consider some topologies ,semi-lattices and lattices which are defined by a set Λ of monomials. Take it for granted to choose $\mathrm{SMon}_1(G)$ as such Λ which woud reflect the strcture of G. Roughly speaking, a closed set of the topology considered here is a set of solutions of an equation which is defined by an element of $\mathrm{SMon}_1(G)$.

Definition 5 *Let* Λ *be a subset of* $\mathrm{Mon}_1(G)$. *We define the set of solutions* $\mathcal{S}ol_G(\Lambda)$ *of equations each of which is defined by an monomial of* Λ *as follows*

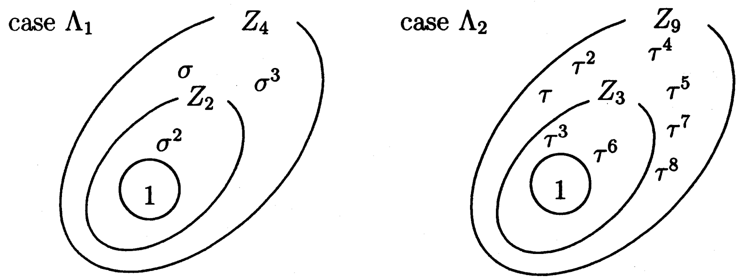
$$\mathcal{S}ol_G(\Lambda) := \{g \in G | f(g) = 1 \text{ for any } f(x) \in \Lambda\})$$

Let Λ *be a subset of* $\mathrm{SMon}_1(\sigma)$. $\mathcal{F}_{\sigma\Lambda}$ *is defined to be the collection of solution set of any subset* Δ *of* Λ *defined as follows*

$$\mathcal{F}_{o\Lambda} := \{\mathcal{S}ol_G(\Delta) | \Delta \subseteq \Lambda\}$$

Definition 6 $\mathcal{F}_\Lambda$ *is defined to be the weakest topology which contains* $\mathcal{F}_{o\Lambda}$ *as a collection of closed sets.*

Example 2 *Let* σ, τ *be generations of* Z_4, Z_9 *respectively and* Λ_1, Λ_2 *be* $\mathrm{SMon}_1(Z_4)$, $\mathrm{SMon}_1(Z_9)$ *respectively.Then* $\mathcal{F}_{o\Lambda_1}$ *and* $\mathcal{F}_{o\Lambda_2}$ *can be drawn as*



Theorem 7 *Let G^{ab} stand for $G/[G,G]$ for a goup G. If $|G^{ab}|$ is prime to $|H^{ab}|$, then*

$$\mathcal{F}_{\mathrm{SMon}_1(G\times H)} \quad \text{is homeomorphic to} \quad \mathcal{F}_{\mathrm{SMon}_1(G)} \times \mathcal{F}_{\mathrm{SMon}_1(H)}$$

Definition 7 *Let $(L, \subseteq)$ be a poset with binary relation $\subseteq$.*

i) *$(L, \subseteq)$ is said to be a semi-lattice if L has the greatest lower bound $x \wedge y$ for any pair of elements x and y of L.*

ii) *A semi-lattice $(L, \subseteq)$ is said to be a lattice if L has the least lower bound $x \vee y$ for any pair of elements x and y of L.*

Remark 2 *Let $\subseteq$ be an inclusion relation.*

i) *$(\mathcal{F}_{0\mathrm{SMon}_1(G)}, \subseteq)$ forms a semi-lattice and is denoted by $\mathcal{L}_{0(G)}$.*

ii) *$(\mathcal{F}_{\mathrm{SMon}_1(G)}, \subseteq)$ forms a lattice and is denoted by $\mathcal{L}_{(G)}$.*

Definition 8 *Let L and K be lattices. A mapping $\sigma : L \to K$ is said to be a (lattice)homomorphism if*

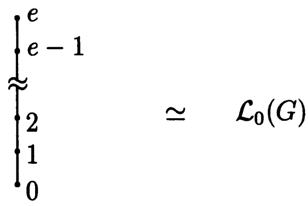
$$(x \wedge y)^\sigma = x^\sigma \wedge y^\sigma \text{ and } (x \vee y)^\sigma = x^\sigma \vee y^\sigma$$

for any pair of elements x, y of L.

A lattice homorphism σ is said to be a lattice isomorphism if it is bijective.A semi-lattice homomorphism and a semi-lattice isomorphism can be defined similarly. $L \simeq K$ stands for that there exists a (semi)lattice isomorphism betwen two (semi)lattices L and K.

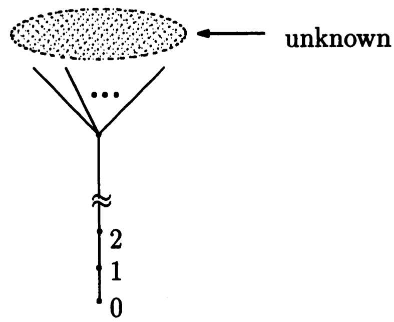
Following propositions are the main theorem of this paper which stand for a relation between the shape of a (semi)lattice and the structure of the

Theorem 8 i) *G is an abelian p-group such that* $\exp(G) = p^e$ *for some positive integer e if and only if $\mathcal{L}_0(G)$ is isomorphic to the following semi-lattice.*



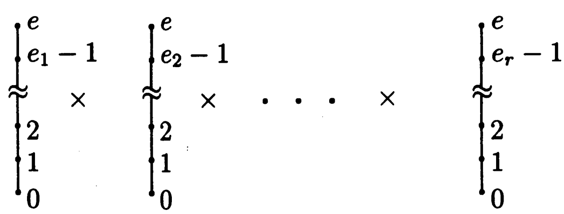
In this case, $\mathcal{L}_0(G)$ is isomophic to $\mathcal{L}(G)$ as a lattice.

ii) *G is a finite p-group if and only if $\mathcal{L}_0(G)$ is ispmorhpic to the following semi-lattice.*



$\mathcal{L}(G)$ is also isomorphic to a lattice which is drawn as above. Whereas it does not always imply that $\mathcal{L}_0(G)$ is isomorphic to $\mathcal{L}(G)$ as a semi-lattice.

iii) *G is an abelian group such that* $\exp(G) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ *for some prime p_i ($p_i \neq p_j$* if *$i \neq j$) and some integer e_i if and only if $\mathcal{L}_0(G)$ is isomorp to the following semi-lattice.*



In this case $\mathcal{L}_0(G)$ is not isomorphic to $\mathcal{L}(G)$ as a semi-lattice for r ≥ 2.

References

[1] Abe, S.- Iiyori, N., *A theory of monomials over groups*, in preparation.