An equilibrium model of
the short-term stock price behavior*

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Abstract

The author proposes a new equilibrium model for stock price processes. We first consider our one-period formulation, and then its continuous-time analogue. The dynamics of the resulting price process is determined by the distribution of risk tolerance among the agents, and for some special case we recover the Black-Scholes stock price model.

1   Introduction

The theory of equilibrium under uncertainty was introduced by Arrow [1]. There is now an extensive literature on the stochastic (or dynamic) equilibria as well, see e.g. Duffie [4] and Karatzas [7]. Their equilibrium approach is formulated in an abstract framework, and is capable of dealing with the general case. However, since only exogenous events are used for contingencies, the theory does not always seem to be suitable for vividly describing the energy in the stock market. See also Radner [12].

In order to give more realistic models, there is an increasing interest in the market microstructure theory. Two of the earliest works of the field are Grossman & Stiglitz [5] and Kyle [8]. Here the market mechanism for determining the stock price is in spotlight, and asymmetric information among the agents often plays a major role. For references see e.g. O'Hara's book [10] and an expository paper of Biais & Rochet [3].

In the present article, we try to give a new type of equilibrium formulation for stock price processes. For the following three reasons our approach may be suitable for describing short-term behavior.

- The current stock price is assumed to be given, and the probability distribution of the price at the next moment will be determined only by the agents' strategies. The fundamental value of the stock is not considered.

- The agents maximize their utility functions moment by moment, *i.e.* they are myopic.

- No consumption, no production is considered.

Our approach also has the following features.

- It is possible to model both a market with the presence of large investors and a market with small investors only.

- Our formulation is in a sense a game-theoretic one. Even small investors are not purely price takers: each of them is able to give infinitesimal (but direct) impact for the determination of the price at the next moment.

- The difference in risk preferences among the agents, rather than informational asymmetries, is a major factor.

- The resulting stock price process has a time-varying volatility.

This paper is organized as follows. In Section 2 we give our one-period formulation and result. The assertions there are proved in Section 3. A continuous-time analogue is considered in Section 4, and a possible extension is mentioned in Section 5. The appendix is a technical note on the applicability of the stochastic Fubini theorem.

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2 One-period model and result

Throughout this paper, R_+ is defined as the set of all nonnegative real numbers and R_{++} the set of all strictly positive numbers. For $x \in R$, define x^+ as $\max\{x, 0\}$.

Definition 2.1 Let $(A, \mathcal{A}, \nu)$ be a measure space, where $\{a\} \in \mathcal{A}$ for all $a \in A$. Suppose the three measurable functions are given:

$$
\begin{aligned}
V &: & A &\to R_{++}, \\
\xi &: & A &\to R, \\
\gamma &: & A &\to R_{++}.
\end{aligned}
$$

For notational simplicity we denote $V(a)$ by V_a; the same rule is applied for all measurable functions on A.

Interpretation. The measure space $(A, \mathcal{A}, \nu)$ is interpreted as the set of all agents in the market; this setting is used for the theory of large economies, c.f. Hildenbrand [6]. Each $a \in A$ with $\nu(\{a\}) > 0$ is interpreted as a large investor, and every $a \in A$ with $\nu(\{a\}) = 0$ is a small investor. We consider a securities market where the agents exchange two assets, the stock and the riskless bond. No consumption is considered, every agent is solely interested in the rate of return of her portfolio. Assume that the interest rate is zero and that the stock pays no dividends. We also assume that the initial stock price is already given. Each agent $a \in A$ is initially endowed with the stock worth $V_a \xi_a$ yen and the bond worth $V_a(1 - \xi_a)$ yen, in total V_a yen. As we will see in Definition 2.2 below, every agent is risk-averse, and γ_a represents agent a's willingness to take risk.

Our price change mechanism is as follows. First each agent $a \in A$ decides how much to trade assets and how much to exert influence. Then the stock price changes, where the mean μ and the variance σ^2 of the rate of change are determined by the agents' strategies. Furthermore, for every investor, the return of her portfolio turns out to maximize her utility U_a given the other agents' strategies, so things are somewhat like the Nash-equilibrium formulation.

Definition 2.2 Fix $c_1, c_2 \in R_{++}$ throughout this paper. For each pair of measurable functions

$$X \;:\; A \to R,$$
$$\rho \;:\; A \to R_+$$

satisfying the following three conditions

$$\left. \begin{array}{rcl} \int_A |X_a|\, \rho_a \, d\nu & < & \infty, \\ \int_A |X_a|\, \rho_a^2 \, d\nu & < & \infty, \\ 0 < \quad \int_A X_a^+ \, d\nu & < & \infty, \end{array} \right\} \quad (*)$$

we define $\mu \in R$, $\sigma \in R_+$, and two R-valued measurable functions θ and U on A as

$$\mu \;=\; \mu(X, \rho) \;\overset{\text{def}}{=}\; c_1 \, \frac{\int_A X_a \rho_a \, d\nu}{\int_A X_a^+ \, d\nu},$$

$$\sigma^2 \;=\; \sigma^2(X, \rho) \;\overset{\text{def}}{=}\; c_2 \, \frac{\int_A |X_a|\, \rho_a^2 \, d\nu}{\int_A X_a^+ \, d\nu},$$

$$\theta_a \;=\; \theta_a(X_a) \;\overset{\text{def}}{=}\; \xi_a + \frac{X_a}{V_a},$$

$$U_a \;=\; U_a(X, \rho) \;\overset{\text{def}}{=}\; \theta_a \mu - \frac{1}{2\gamma_a} \theta_a^2 \sigma^2.$$

Definition 2.3 A pair (X^*, ρ^*) of measurable functions

$$X^* \ : \ A \to R,$$

$$\rho^* \ : \ A \to R_+,$$

is said to be an *equilibrium* (resp. a *weak equilibrium*) if it satisfies the conditions (∗) in Definition 2.2 as well as the following (i-1), (i-2), and (ii). Here we use the notations:

$$\mu^* \ \overset{\text{def}}{=} \ \mu(X^*, \rho^*),$$

$$\sigma^* \ \overset{\text{def}}{=} \ \sigma(X^*, \rho^*),$$

$$\theta_a^* \ \overset{\text{def}}{=} \ \theta_a(X_a^*).$$

(i-1) For every $a \in A$ with $\nu(\{a\}) > 0$, the pair $(X_a^*, \rho_a^*) \in R \times R_+$ maximizes (resp. locally maximizes)

$$U_a\Big(\ \{ \ \{X_b^*\}_{b \neq a}, \ X_a \ \}, \ \{ \ \{\rho_b^*\}_{b \neq a}, \ \rho_a \ \} \ \Big)$$

subject to $X_a \in R$ and $\rho_a \in R_+$.

(i-2) For almost every $a \in A$ with $\nu(\{a\}) = 0$, θ_a^* solves the problem

$$\max_{\theta_a \in R} \ \Big\{ \ \theta_a \cdot \mu^* \ - \ \frac{1}{2\gamma_a} \theta_a^2 \cdot (\sigma^*)^2 \ \Big\}$$

with μ^* and σ^* considered to be given constants, and ρ_a^* solves

$$\max_{\rho_a \in R_+} \ \Big\{ \ \theta_a^* \cdot c_1 X_a^* \rho_a \ - \ \frac{1}{2\gamma_a} (\theta_a^*)^2 \cdot c_2 |X_a^*| \rho_a^2 \ \Big\}.$$

(ii) $\displaystyle\int_A X_a^* \, d\nu \ = \ 0.$ (Market Clearing Condition)

Interpretation. Each agent $a \in A$ buys the stock worth X_a^* yen and exerts influence ρ_a^*. After the exchange and before the price change, her portfolio consists of the stock worth $V_a \cdot \theta_a^*$ yen and the bond worth $V_a \cdot \{1 - \theta_a^*\}$ yen. Moreover $\int_A (X_a^*)^+ \, d\nu = \frac{1}{2} \int_A |X_a^*| \, d\nu$ represents the trading volume.

The condition (i-2) is heuristically derived as follows. Every small investor wants to maximize the (non-rigorous) expression

$$\theta_a(X_a) \cdot c_1 \frac{\int_A X_b^* \rho_b^* \, d\nu \ + \ X_a \, \rho_a \, \Delta\nu(a)}{\int_A (X_b^*)^+ \, d\nu \ + \ X_a^+ \, \Delta\nu(a)}$$

$$- \ \frac{1}{2\gamma_a} \theta_a^2(X_a) \cdot c_2 \frac{\int_A |X_b^*| (\rho_b^*)^2 \, d\nu \ + \ |X_a| \, \rho_a^2 \, \Delta\nu(a)}{\int_A (X_b^*)^+ \, d\nu \ + \ X_a^+ \, \Delta\nu(a)}$$

$$\approx \ \theta_a(X_a) \cdot \Big\{ \mu^* \ + \ \frac{c_1 X_a \rho_a \ - \ \mu^* X_a^+}{\int_A (X_b^*)^+ \, d\nu} \, \Delta\nu(a) \Big\}$$

$$- \ \frac{1}{2\gamma_a} \theta_a^2(X_a) \cdot \Big\{ (\sigma^*)^2 \ + \ \frac{c_2 |X_a| \rho_a^2 \ - \ (\sigma^*)^2 X_a^+}{\int_A (X_b^*)^+ \, d\nu} \, \Delta\nu(a) \Big\}$$

subject to $X_a \in R$ and $\rho_a \in R_+$, which leads to (i-2). It should be remarked that, for the maximization of utility, the agent here does not have to know all of the other agents' strategies; she uses only the values of μ^* and σ^*.

The greater influence the agent exerts, the more favorable μ will be for her, but at the same time the more volatile the change will be. The author believes that the influence ρ_a^* can be formulated as the extent of "infinitesimal randomization" of each agent's excess demand (hence the source of randomness is purely endogenous), but to be frank he himself still does not have a 100% clear vision of ρ^*.

Remark If $\nu(\{a\}) = 0$ for all $a \in A$ then, by definition, there is no distinction between an equilibrium and a weak equilibrium. Note also that in case of $\sharp A < \infty$, the number of unknown variables $\{X_a^*\}_{a \in A}$ and $\{\rho_a^*\}_{a \in A}$ is $2\sharp A$, while the number of equations they have to satisfy to be a (weak) equilibrium is essentially $2\sharp A + 1$.

We proceed with the following three assumptions:

Assumption 1 $\quad \displaystyle\int_A V_a |\xi_a| \, d\nu < \infty \qquad$ and $\qquad S \overset{\text{def}}{=} \displaystyle\int_A V_a \xi_a \, d\nu > 0;$

Assumption 2 $\quad \Gamma \overset{\text{def}}{=} \displaystyle\int_A V_a \gamma_a \, d\nu < \infty;$

Assumption 3 $\quad \dfrac{\xi_a}{\gamma_a}$ is not constant in a.

The integral S represents the initial total market value of the stock.

Theorem 2.4

(1) *Under the above assumptions, the following* (X^*, ρ^*) *is a weak equilibrium:*

$$X_a^* = V_a \left\{ \frac{S}{\Gamma} \gamma_a - \xi_a \right\};$$

$$\rho_a^* = \begin{cases} \frac{c_1}{c_2} \frac{\Gamma}{S} & \text{if } X_a^* > 0, \\ 0 & \text{if } X_a^* < 0, \\ \text{any nonnegative number} & \text{if } X_a^* = 0. \end{cases}$$

For this weak equilibrium, we have that

$$\mu^* = C^2 \frac{\Gamma}{S},$$

$$\sigma^* = C\,\frac{\Gamma}{S},$$

$$U_a(X^*, \rho^*) = \frac{C^2}{2}\,\gamma_a,$$

where $C \overset{\text{def}}{=} \frac{c_1}{\sqrt{c_2}}$.

(2)   Assume further that every $a \in A$ satisfies $\nu(\{a\}) = 0$. Then the above pair (X^*, ρ^*) is the unique equilibrium satisfying $\sigma^* > 0$, except for the arbitrariness of ρ_a^* for $a \in A$ with $X_a^* = 0$. Moreover, there exists an equilibrium with $\sigma^* = 0$ if and only if $\nu(\{a \in A : \xi_a < 0\}) > 0$.

The proof will be given in Section 3.

Remarks 1. Suppose a group of agents $B \in \mathcal{A}$ get together to form a single large (institutional) investor i with

$$V_i\,\nu(\{i\}) \overset{\text{def}}{=} \int_B V_a\,d\nu,$$

$$\xi_i \overset{\text{def}}{=} \frac{\int_B V_a\xi_a\,d\nu}{\int_B V_a\,d\nu},$$

$$\gamma_i \overset{\text{def}}{=} \frac{\int_B V_a\gamma_a\,d\nu}{\int_B V_a\,d\nu}.$$

For this new market, as far as Assumption 3 is satisfied, we get the same values of S and Γ, and consequently the same μ^* and σ^*. On the other hand, the trading volume $\int_A (X_a^*)^+\,d\nu$ is no longer the same.

2.   It is possible to generalize our theorem for a wider class of utility functions, e.g.

$$U_a(X, \rho) \overset{\text{def}}{=} \theta_a\,\mu - \frac{1}{\eta_a\gamma_a}\,|\theta_a\,\sigma|^{\eta_a},$$

where $\eta_a > 1$.

Example 2.5   Suppose there exists an $a \in A$ with $\nu(\{a\}) > 0$ such that $X_b^* > 0$ for all $b \neq a$; we will here show that the weak equilibrium (X^*, ρ^*) of Theorem 2.4 is not an equilibrium. By the market clearing condition (ii) we have

$$\int_{A\setminus\{a\}} (X_b^*)^+\,d\nu = -X_a^*\,\nu(\{a\})$$

$$= V_a\left\{\xi_a - \frac{S}{\Gamma}\,\gamma_a\right\}\nu(\{a\})$$

$$< V_a\xi_a\,\nu(\{a\}).$$

Also if $X_a \uparrow -V_a\xi_a$ then $\theta_a(X_a) \uparrow 0$, which implies that

$$\lim_{X_a \uparrow -V_a\xi_a} U_a\Big(\big\{\ \{X_b^*\}_{b\neq a},\ X_a\ \big\},\ \big\{\ \{\rho_b^*\}_{b\neq a},\ \tfrac{c_1}{c_2}\tfrac{\gamma_a}{|\theta_a(X_a)|}\ \big\}\Big)$$

$$= \frac{C^2}{2}\gamma_a \cdot \frac{V_a\xi_a\,\nu(\{a\})}{\int_{A\setminus\{a\}}(X_b^*)^+\,d\nu}$$

$$> \frac{C^2}{2}\gamma_a$$

$$= U_a(X^*, \rho^*).$$

Thus our X_a^* and ρ_a^* do not globally maximize agent a's utility.

We can generalize the above example to give

Proposition 2.6   *For each* $a \in A$ *with* $\nu(\{a\}) > 0$:

(1)   *There exists no* $(\tilde{X}_a, \tilde{\rho}_a) \in R \times R_+$ *satisfying the following two properties.*

- $\tilde{X}_a \geq 0$ *or* $\theta_a(\tilde{X}_a) \geq 0$,

- $U_a\Big(\big\{\ \{X_b^*\}_{b\neq a},\ \tilde{X}_a\ \big\},\ \big\{\ \{\rho_b^*\}_{b\neq a},\ \tilde{\rho}_a\ \big\}\Big) > U_a(X^*, \rho^*),$

where (X^*, ρ^*) *is the weak equilibrium of Theorem 2.4.*

(2)   *Suppose* $\xi_a \geq 0$ *and* $\int_{A\setminus\{a\}}(X_b^*)^+\,d\nu > 0$. *Then, in order that there exists some* $(\tilde{X}_a, \tilde{\rho}_a) \in R \times R_+$ *satisfying the following two properties*

- $\tilde{X}_a < 0$ *and* $\theta_a(\tilde{X}_a) < 0$,

- $U_a\Big(\big\{\ \{X_b^*\}_{b\neq a},\ \tilde{X}_a\ \big\},\ \big\{\ \{\rho_b^*\}_{b\neq a},\ \tilde{\rho}_a\ \big\}\Big) > U_a(X^*, \rho^*),$

it is necessary and sufficient that

$$\frac{V_a\xi_a\,\nu(\{a\})}{\int_{A\setminus\{a\}}(X_b^*)^+\,d\nu} + \left\{\left(\frac{V_a\,\frac{S}{\Gamma}\,\gamma_a\,\nu(\{a\})}{2\int_{A\setminus\{a\}}(X_b^*)^+\,d\nu} - 1\right)^+\right\}^2 > 1.$$

The proof uses only standard methods (such as considering some separate cases and differentiating the utility), so it is omitted here.

3   Proof of Theorem 2.4

Proof of Theorem 2.4   (1)   It is easily seen that the pair (X^*, ρ^*) satisfies the conditions (i-2) and (ii). The expression for μ^* is shown as follows:

$$
\begin{aligned}
\mu^* &= c_1 \frac{\int_A X_a^* \rho_a^* \, d\nu}{\int_A (X_a^*)^+ \, d\nu} \\
&= c_1 \cdot \frac{c_1}{c_2} \frac{\Gamma}{S} \cdot \frac{\int_A (X_a^*)^+ \, d\nu}{\int_A (X_a^*)^+ \, d\nu} \\
&= C^2 \frac{\Gamma}{S}.
\end{aligned}
$$

The expression for σ^* can be proved the same way. Here Assumptions 1 and 2 guarantee the integrability of X^* and other quantities. X_a^* is not identically zero by Assumption 3, hence $\int_A (X_a^*)^+ \, d\nu > 0$.

We will divide the proof of the validity of condition (i-1) into the following four cases of $a \in A$ with $\nu(\{a\}) > 0$.

Case 1: $X_a^* > 0$ *and* $\int_{A \setminus \{a\}} (X_b^*)^+ \, d\nu > 0$.   We see that

$$
\left(\frac{\partial}{\partial X_a}\right)^k \mu(X^*, \rho^*) = \left(\frac{\partial}{\partial X_a}\right)^k \sigma^2(X^*, \rho^*) = 0 \qquad (\Diamond)
$$

for every $k \in N$, thus

$$
\frac{\partial}{\partial X_a} U_a(X^*, \rho^*) = \frac{\partial}{\partial \rho_a} U_a(X^*, \rho^*) = 0
$$

and a little more calculation shows

$$
\left(\frac{\partial}{\partial X_a}\right)^2 U_a(X^*, \rho^*) = -\frac{1}{V_a^2 \gamma_a} \cdot C^2 \left(\frac{\Gamma}{S}\right)^2,
$$

$$
\left(\frac{\partial}{\partial \rho_a}\right)^2 U_a(X^*, \rho^*) = -\left(\frac{S}{\Gamma}\right)^2 \gamma_a \cdot c_2 \frac{X_a^* \nu(\{a\})}{\int_A (X_b^*)^+ \, d\nu},
$$

$$
\frac{\partial^2}{\partial X_a \, \partial \rho_a} U_a(X^*, \rho^*) = -\frac{c_1}{V_a} \cdot \frac{X_a^* \nu(\{a\})}{\int_A (X_b^*)^+ \, d\nu}.
$$

The determinant of the Hessian is therefore

$$
\left(\frac{c_1}{V_a}\right)^2 \left\{ \frac{X_a^* \nu(\{a\})}{\int_A (X_b^*)^+ \, d\nu} - \left(\frac{X_a^* \nu(\{a\})}{\int_A (X_b^*)^+ \, d\nu}\right)^2 \right\} > 0.
$$

Case 2: $X_a^* > 0$ *and* $\int_{A\backslash\{a\}} (X_b^*)^+ \, d\nu = 0$. In this case we see that $X_b^* \leq 0$ for a.e. $b \neq a$, so if $X_a > 0$ then

$$U_a\Big(\big\{ \{X_b^*\}_{b\neq a}, X_a \big\}, \big\{ \{\rho_b^*\}_{b\neq a}, \rho_a \big\} \Big)$$

$$= \theta_a(X_a) \cdot c_1 \rho_a - \frac{1}{2\gamma_a} \theta_a^2(X_a) \cdot c_2 \rho_a^2,$$

which is clearly maximized by our X_a^* and ρ_a^*.

Case 3: $X_a^* < 0$. If $X_a < 0$ and $\theta_a(X_a) > 0$, then it is easy to see that

$$\max_{\rho_a \in \mathbf{R}_+} U_a\Big(\big\{ \{X_b^*\}_{b\neq a}, X_a \big\}, \big\{ \{\rho_b^*\}_{b\neq a}, \rho_a \big\} \Big)$$

is solved by $\rho_a^* = 0$. Also, given $\rho_a^* = 0$ our X_a^* locally maximizes the utility, since the equality $(\Diamond)$ in Case 1 holds for this case too.

Case 4: $X_a^* = 0$. In this case it is trivial that

$$U_a\Big(X^*, \big\{ \{\rho_b^*\}_{b\neq a}, \rho_a \big\} \Big)$$

does not depend on ρ_a. A calculation reveals that the right partial derivative of U_a with respect to X_a at $\Big(X^*, \big\{ \{\rho_b^*\}_{b\neq a}, \rho_a \big\} \Big)$ is

$$-\frac{c_2 \gamma_a}{2 \int_A (X_b^*)^+ \, d\nu} \Big(\frac{S}{\Gamma}\Big)^2 \Big\{ \rho_a - \frac{c_1}{c_2} \frac{\Gamma}{S} \Big\}^2 \nu(\{a\}),$$

which is strictly negative if $\rho_a \neq \frac{c_1}{c_2} \frac{\Gamma}{S}$. Likewise the left derivative can be shown to be strictly positive if $\rho_a \neq 0$. We consider the cases $\rho_a = 0$, $\frac{c_1}{c_2} \frac{\Gamma}{S}$ as well and can show that our our X_a^* and ρ_a^* locally maximize the utility. $\square$

Proof of Theorem 2.4 (2) The pair (X^*, ρ^*) in the statement of the Theorem is the unique equilibrium satisfying $\sigma^* > 0$. Indeed, if $\sigma^* > 0$ then it follows from the condition (i-2) that

$$\theta_a^* = \frac{\mu^*}{(\sigma^*)^2} \gamma_a,$$

$$X_a^* = V_a \Big\{ \frac{\mu^*}{(\sigma^*)^2} \gamma_a - \xi_a \Big\}.$$

This together with (ii) implies

$$\frac{\mu^*}{(\sigma^*)^2} = \frac{\int_A V_a \xi_a \, d\nu}{\int_A V_a \gamma_a \, d\nu} = \frac{S}{\Gamma},$$

which gives X_a^*, ρ_a^*, and consequently μ^* and σ^*.

Next we consider the possibility that there exists an equilibrium satisfying $\sigma^* = 0$. For such an equilibrium we have $X^* \rho^* \equiv 0$, thus by (i-2)

$$\theta_a^* \cdot X_a^* = \left(\xi_a + \frac{X_a^*}{V_a} \right) \cdot X_a^* \leq 0$$

for a.e. $a \in A$. Such an X^* exists if and only if $\nu(\{ a : \xi_a < 0 \}) > 0$. $\quad\square$

4 Continuous-time analogue

Our one-period model of Section 2 is extended to the multi-period one as follows. Initially, each agent a is endowed with $V_a(0)$ and $\xi_a(0)$. In this section her risk preference γ_a is assumed not to change over time. (For the case of time-varying $\gamma_a(t)$ see Section 5.) The total market value of the stock changes from $S(0)$ to $S(1)$, where the rate of change $\frac{S(1)-S(0)}{S(0)}$ has mean $\mu^*(0)$ and standard deviation $\sigma^*(0)$, as we saw in Section 2. The change in price determines $V_a(1)$ and $\xi_a(1)$ for each agent (no additional endowment is considered), and our equilibrium argument restarts from there, giving $S(2)$, $S(3)$, $\cdots$ successively. Note that each agent maximizes her utility moment by moment.

Now we consider the continuous-time analogues of the price process $S(t)$ and the portfolio value $V_a(t)$, without rigorously building the model. Assumptions 1 through 3 are modified here as:

Assumption 1' $\displaystyle\int_A V_a(0) \, |\xi_a(0)| \, d\nu < \infty$

and $S(0) \overset{\text{def}}{=} \displaystyle\int_A V_a(0) \, \xi_a(0) \, d\nu > 0$;

Assumption 2' $\displaystyle\int_A V_a(0) \, \gamma_a \, d\nu < \infty$;

Assumption 3' $\dfrac{\xi_a(0)}{\gamma_a}$ is not constant in a.

In addition we need

Assumption 4 γ_a is not constant in a,

otherwise $\frac{\xi_a(t)}{\gamma_a}$ would be constant in a for $t > 0$, which would violate the "$t > 0$" version of Assumption 3'. It is also convenient to assume that

Assumption 5 $S(0) = \displaystyle\int_A V_a(0) \, d\nu$,

i.e. the net amount of bond in the market is zero. If $S(0) < \int_A V_a(0) \, d\nu$ then, as we will see later, $S(t)$ could go negative for some $t > 0$.

Under the above five assumptions, the stock price process $S(t)$ satisfies, by analogy with Section 2,

$$\frac{dS(t)}{S(t)} = \sigma^*(t)\,dW(t) + \mu^*(t)\,dt$$

$$= \frac{\int_A V_a(t)\,\gamma_a\,d\nu}{S(t)}\left(C\,dW(t) + C^2\,dt\right),$$

where W is a standard one-dimensional Brownian motion starting from the origin. Furthermore, agent a's portfolio value $V_a(t)$ satisfies

$$\frac{dV_a(t)}{V_a(t)} = \theta_a^*(t)\,\frac{dS(t)}{S(t)}$$

$$= \gamma_a\left(C\,dW(t) + C^2\,dt\right),$$

thus

$$V_a(t) = V_a(0)\exp\left\{C\gamma_a\,W(t) + C^2\left(\gamma_a - \frac{\gamma_a^2}{2}\right)t\right\}.$$

It follows that if Assumption 2' is satisfied, then the same property $\int_A V_a(t)\gamma_a\,d\nu < \infty$ holds for all $t > 0$, a.s.

We see that

$$S(t) = S(0) + \left\{\int_A V_a(t)\,d\nu - \int_A V_a(0)\,d\nu\right\} \qquad (\clubsuit)$$

$$= \int_A V_a(t)\,d\nu \qquad \text{(by Assumption 5)}$$

$$= S(0)\,\frac{\int_A V_a(0)\exp\left\{C\gamma_a\,W(t) + C^2\left(\gamma_a - \frac{\gamma_a^2}{2}\right)t\right\}d\nu}{\int_A V_a(0)\,d\nu}$$

(for a rigorous treatment of $(\clubsuit)$ see the appendix). The dynamics of S is thus determined by the distribution of γ with respect to the measure $\frac{V_a(0)\,d\nu(a)}{\int_A V_b(0)\,d\nu(b)}$ on A. The value of $S(t)$ is increasing with respect to the value of $W(t)$, so the process S is Markovian. Also,

$$\frac{dS(t)}{S(t)} = \frac{\int_A V_a(t)\,\gamma_a\,d\nu}{\int_A V_a(t)\,d\nu}\left(C\,dW(t) + C^2\,dt\right)$$

and hence the volatility of S is C times the market mean of γ_a weighted by $V_a(t)$. We give three examples.

Example 4.1 Assumption 4 prohibits γ from being constant in a, but if the distribution of γ is concentrated in a very small neighborhood of one single value, then the resulting process is close to a constant-volatility geometric Brownian motion, i.e. the Black-Scholes stock price model.

Example 4.2 Suppose the distribution of γ has the density

$$\frac{1}{Z} \exp(-\eta x^2 + \lambda x) \qquad \text{for} \quad x \in R_{++}.$$

Here $\eta \in R_{++}$, $\lambda \in R$, and Z is the normalizing constant. A calculation then yields that

$$S(t) = S(0) \cdot \sqrt{\frac{2\eta}{C^2 t + 2\eta}} \frac{1}{f(\frac{\lambda}{\sqrt{2\eta}})} \cdot f\left(\frac{CW(t) + C^2 t + \lambda}{\sqrt{C^2 t + 2\eta}}\right),$$

where

$$f(x) \overset{\text{def}}{=} \sqrt{2\pi} \exp(\frac{x^2}{2}) \Phi(x)$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Moreover,

$$f(x) \sim \sqrt{2\pi} \exp(\frac{x^2}{2}) \qquad (x \to \infty),$$

$$f(x) \sim \frac{1}{|x|} \qquad (x \to -\infty).$$

This fact implies that the distribution of $\log S(t)$ has a fatter tail on the right than the normal distributions, and a thinner tail on the left.

Example 4.3 Suppose that the density function is

$$\lambda \exp\left\{-\lambda(x - \ell)\right\} \qquad \text{for} \quad x > \ell$$

and 0 for $0 < x \leq \ell$, where $\lambda \in R_{++}$ and $\ell \in R_+$. Then

$$S(t) = S(0) \cdot \frac{\lambda}{\sqrt{C^2 t}} \exp\left\{C\ell W(t) + C^2(\ell - \frac{\ell^2}{2})t\right\}$$
$$\cdot f\left(\frac{CW(t) + (1 - \ell)C^2 t - \lambda}{\sqrt{C^2 t}}\right)$$

with the function $f(\cdot)$ defined in the previous example. If $\ell > 0$ then the distribution of $\log S(t)$ has roughly the same tail on the left as a normal distribution.

Remarks 1. We can generalize the above observations to show the following. In general, if $\operatorname{ess\,inf} \gamma_a > 0$ then the left tail of $\log S(t)$ is roughly the same as that of a normal distribution, whereas if $\operatorname{ess\,inf} \gamma_a = 0$ then the left tail is thinner than the normal distributions. Likewise, if $\operatorname{ess\,sup} \gamma_a < \infty$ then $\log S(t)$ has roughly the same tail on the right as a normal distribution, and otherwise the right tail is fatter. The right tail of $\log S(t)$ cannot be thinner than the left.

2. The price process S has a unique equivalent martingale measure Q, under which

$$W^Q(t) \overset{\text{def}}{=} W(t) + Ct$$

is a Brownian motion. The market is thus complete. Since S is the sum (or integral) of some constant-volatility geometric Brownian motions driven by the same W^Q, we have that $\forall T \in R_{++}$, $\forall K \in R_{++}$, $\exists k : A \to R_{++}$,

$$(S_T - K)^+ = \int_A \{V_a(T) - k_a\}^+ \, d\nu.$$

Thus the price of a European call option on S is the sum of some Black-Scholes formulae, just as Jamshidian's trick works for options on coupon-bearing bonds (see e.g. p.170 of Baxter & Rennie [2] or p.298 of Musiela & Rutkowski [9]).

5 Conclusion

In this paper we considered an equilibrium model for describing the short-term stock price behavior. Under mild assumptions we proved that there exists a weak equilibrium, which is the unique equilibrium if every agent is a small investor and her initial endowment of the stock is non-negative. A market with the presence of large investors is shown to be less stable than a market with small investors only. The dynamics of the price process depends on the distribution of risk tolerance among the agents; in particular, if the distribution is concentrated in a small neighborhood of one single value, then the resulting process is close to the well-known Black-Scholes model. If a group of agents get together to form a single institutional investor, then we have the same price dynamics but the trading volume is no longer the same.

A better understanding of the influence ρ^*, one of the key concepts for our equilibrium mechanism, is left for future studies.

In Section 4 we considered only the case that each agent's risk tolerance, γ_a, is constant through time, but it is also possible to consider the case of time-varying $\gamma_a(t)$. For instance, $\gamma_a(t)$ can be formulated so that it increases with respect to the value of $V_a(t)$. Our equilibrium price process still exists if Assumptions 1 through 5, appropriately modified, are all satisfied. If one of those models fails to satisfy Assumption 2 for some finite time t, $i.e.$ if

$$P\left[\int_A V_a(t)\gamma_a(t) \, d\nu = \infty \text{ for some } t > 0 \right] > 0,$$

then we can interpret the explosion as the burst of a price bubble.

6 Appendix: Applicability of the stochastic Fubini theorem

The equality (♣) in Section 4 is intuitively clear, but to prove it rigorously we need to apply the stochastic Fubini theorem (c.f. Protter [11], p.160). Mathematically we formulate our problem as follows.

Definition A.1 Let $\big(w(t)\big)_t$ be a standard one-dimensional Brownian motion, starting from the origin, on a certain filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We also consider another measure space $(B, \mathcal{B}, m)$ and an R_+-valued measurable function κ on it. For each $b \in B$ let $\big(V_b(t)\big)_t$ be an adapted process satisfying

$$dV_b(t) = V_b(t)\,\kappa_b\,dw(t),$$

i.e.,

$$V_b(t) = V_b(0)\,\exp\left\{ \kappa_b\,w(t) - \frac{\kappa_b^2}{2}t \right\}.$$

Moreover $V.(0)$ is assumed to be an R_+-valued measurable function on B.

Proposition A.2 *Suppose the integrals*

$$\int_B V_b(0)\,dm \qquad and \qquad \int_B V_b(0)\,\kappa_b\,dm$$

are both finite. Then

$$\Sigma(t) \overset{\text{def}}{=} \int_B V_b(t)\,dm \qquad and \qquad \int_B V_b(t)\,\kappa_b\,dm$$

are both finite for all $t > 0$, a.s. Furthermore we have that

$$d\Sigma(t) = \left(\int_B V_b(t)\,\kappa_b\,dm \right) dw(t).$$

Remark Here κ_b corresponds to $C\gamma_a$ in Section 4. The Brownian motion $w(t)$ can be viewed as $W(t) + Ct$ after some appropriate measure change.

Proof It is easy to prove the first half of our assertion, since

$$\sup_{b \in B}\,\exp\left\{ \kappa_b\,w(t) - \frac{\kappa_b^2}{2}t \right\} < \infty$$

for all $t > 0$, a.s. To prove the second half, we use a version of the stochastic Fubini theorem (Theorem IV.46 of [11]) and reduce the problem to showing that

$$\int_0^T dt \int_B dm\,V_b(0)\,\kappa_b^2\,\exp\left\{ 2\kappa_b\,w(t) - \kappa_b^2\,t \right\} < \infty$$

for all $T > 0$, a.s. Since

$$\sup_{b \in B}\,\kappa_b\,\exp\left\{ 2\kappa_b\,w(t) - \kappa_b^2\,t \right\}$$

$$\leq \max_{x \in R_+}\,x\,\exp\left\{ 2x\,w(t) - x^2\,t \right\}$$

$$= \frac{w(t) + \sqrt{w(t)^2 + 2t}}{2t}\,\exp\left\{ \frac{w(t)^2 + w(t)\,\sqrt{w(t)^2 + 2t}}{2t} - \frac{1}{2} \right\},$$

it suffices to show that

$$\int_{0+} \frac{w(t) + \sqrt{w(t)^2 + 2t}}{2t} \exp\left\{ \frac{w(t)^2 + w(t)\sqrt{w(t)^2 + 2t}}{2t} - \frac{1}{2} \right\} dt < \infty \quad \text{a.s.}$$

The last inequality holds by the law of the iterated logarithm. □

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