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On Law Invariant Coherent Risk Measures

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1 Introduction

The idea of coherent risk measures has been introduced by Artzner, Delbaen, Eber and Heath [1]. We think of a special class of coherent risk measures and give a characterization of it. Let \((\Omega, \mathcal{F}, P)\) be a probability space. We denote \(L^\infty(\Omega, \mathcal{F}, P)\) by \(L^\infty\). Following [1], we give the following definition.

Definition 1 We say that a map \(\rho : L^\infty \to \mathbb{R}\) is a coherent risk measure if the following are satisfied.

1. If \(X \geq 0\), then \(\rho(X) \leq 0\).
2. Subadditivity : \(\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)\).
3. Positive homogeneity : for \(\lambda > 0\) we have \(\rho(\lambda X) = \lambda \rho(X)\).
4. For every constant \(c\) we have \(\rho(X + c) = \rho(X) - c\).

Then Delbaen [2] proved the following.

Theorem 2 Let \(\rho\) be a coherent risk measure. Then the following conditions are equivalent.

1. There is a (closed convex) set of probability measures \(Q\) such that any \(Q \in Q\) is absolutely continuous with respect to \(P\) and for \(X \in L^\infty\)
   \(\rho(X) = \sup\{E^Q[-X]; Q \in Q\}\).

2. \(\rho\) satisfies the Fatou property, i.e., if \(\{X_n\}_{n=1}^\infty \subset L^\infty\) are uniformly bounded and converging to \(X\) in probability, then
   \(\rho(X) \leq \liminf_{n \to \infty} \rho(X_n)\).

3. If \(X_n\) is a uniformly bounded sequence that decreases to \(X\), then \(\rho(X_n)\) tends to \(\rho(X)\).

Now we introduce the following notion.

Definition 3 We say that a map \(\rho : L^\infty \to \mathbb{R}\) is law invariant, if \(\rho(X) = \rho(Y)\) whenever \(X, Y \in L^\infty\) have the same probability law.

Our purpose is to characterize law invariant coherent risk measures with the Fatou property.

Let \(D\) be the set of probability distribution functions of bounded random variables, i.e., \(D\) is the set of non-decreasing right-continuous functions \(F\) on \(\mathbb{R}\) such that there are \(z_0, z_1 \in \mathbb{R}\) for which \(F(z) = 0, z < z_0\) and \(F(z) = 1, z \geq z_1\). Let us define \(Z : [0, 1) \times D \to \mathbb{R}\) by

\[Z(x, F) = \inf\{z; F(z) > x\}, \quad x \in [0, 1), \ F \in D.\]
Then \( Z(\cdot, F) : [0, 1) \to \mathbb{R} \) is non-decreasing and right continuous. We denote by \( F_X \) the probability distribution function of a random variable \( X \).

For each \( \alpha \in (0, 1) \), let \( \rho_\alpha : L^\infty \to \mathbb{R} \) be given by

\[
\rho_\alpha(X) = \alpha^{-1} \int_{1-\alpha}^1 Z(x, F_{-X}) \, dx, \quad X \in L^\infty.
\]

Also, we define \( \rho_0 : L^\infty \to \mathbb{R} \) by

\[
\rho_0(X) = \text{ess.sup}(-X) \quad X \in L^\infty.
\]

Then it is easy to see that \( \rho_\alpha(X) : [0, 1) \to \mathbb{R} \) is a non-increasing continuous function for any \( X \in L^\infty \).

We will show later that \( \rho_\alpha, \alpha \in [0, 1) \), is a law invariant coherent risk measure with the Fatou property. Actually \( \rho_\alpha \) is the same as \( WCM_\alpha \) in [1].

From now on, we assume the following.

(Assumption) \((\Omega, \mathcal{F}, P)\) is a standard probability space and \( P \) is non-atomic.

Our main results are the following.

**Theorem 4** Let \( \rho : L^\infty \to \mathbb{R} \). Then the following conditions are equivalent.

1. There is a (compact convex) set \( \mathcal{M}_0 \) of probability measures on \([0, 1]\) such that

\[
\rho(X) = \sup \{ \int_0^1 \rho_\alpha(X)m(d\alpha); m \in \mathcal{M}_0 \}, \quad X \in L^\infty.
\]

2. \( \rho \) is a law invariant coherent risk measure with the Fatou property.

**Theorem 5** If \( m_1 \) and \( m_2 \) are probability measures on \([0, 1]\), and if

\[
\int_0^1 \rho_\alpha(X)m_1(d\alpha) = \int_0^1 \rho_\alpha(X)m_2(d\alpha), \quad \text{for all } X \in L^\infty,
\]

then \( m_1 = m_2 \).

**Definition 6** (1) We say that a pair \( X \) and \( Y \) of random variables is comonotone, if

\[
(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \quad P(d\omega) \otimes P(d\omega') \text{ - a.s.}
\]

(2) We say that a map \( \rho : L^\infty \to \mathbb{R} \) is comonotone, if

\[
\rho(X + Y) = \rho(X) + \rho(Y)
\]

for any comonotone pair \( X, Y \in L^\infty \).

**Theorem 7** Let \( \rho : L^\infty \to \mathbb{R} \). Then the following conditions are equivalent.

1. There is a probability measure \( m \) on \([0, 1]\) such that for \( X \in L^\infty \)

\[
\rho(X) = \int_0^1 \rho_\alpha(X)m(d\alpha), \quad X \in L^\infty.
\]

2. \( \rho \) is a law invariant and comonotone coherent risk measure with the Fatou property.
We define $VaR_{\alpha} : L^\infty \to \mathbb{R}$, $\alpha \in (0, 1)$, by

$VaR_{\alpha}(X) = \sup\{z \in \mathbb{R}; F_{-X}(z) < 1 - \alpha\}$.

Theorem 9 Let $\alpha \in (0, 1)$. If $\rho$ is law invariant coherent risk measure such that

$\rho(X) \geq VaR_{\alpha}(X), \quad X \in L^\infty,$

then we have

$\rho(X) \geq \rho_{\alpha}(X), \quad X \in L^\infty.$

The author thanks Prof. Delbaen for useful discussions. In particular, Theorems 7 and 9 are suggested by him.

2 Key Lemma

Since we assume that $(\Omega, \mathcal{F})$ is a standard probability space and $P$ be non-atomic, we may assume that our basic probability space $(\Omega, \mathcal{F}, P)$ is a Lebesgue space, i.e., $\Omega = [0, 1)$, $\mathcal{F}$ is the Borel algebra over $[0, 1)$, and $P$ is the Lebesgue measure $\mu$ on $[0, 1)$. Therefore we assume so throughout this paper.

Let $G$ be the set of non-decreasing right-continuous probability density functions on $[0, 1)$. In this section, we will prove the following.

Lemma 10 Let $\rho : L^\infty \to \mathbb{R}$. Then the following conditions are equivalent.

(1) There is a subset $G_0$ of $G$ such that

$\rho(X) = \sup\{\int_{0}^{1} Z(x, F_{-X})g(x)dx; g \in G_0\}, \quad X \in L^\infty.$

(2) $\rho$ is a law invariant coherent risk measure with the Fatou property.

Let $P$ denote the set of probability measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to $P$. For any $Q \in P$, $Y_Q$ denotes the Radon-Nykodim density $dQ/dP$. Let $\mathcal{F}_n$, $n \geq 1$, be a sub-$\sigma$-algebra of $\mathcal{F}$ generated by $1_{[2^{-n}(k-1), 2^{-n}k)}, k = 1, \ldots, 2^n$. Let $\mathcal{X}$ be the set of all bounded random variables $X$ such that $X$ is $\mathcal{F}_n$-measurable for some $n$.

Then we have the following.

Lemma 11 Let $Q \in P$ and $X \in \mathcal{X}$. Then we have

$\int_{0}^{1} Z(x, F_X)Z(x, F_{Y_Q})dx = \sup\{E^Q[\tilde{X}]; \tilde{X} \in \mathcal{X}, F_{\tilde{X}} = F_X\}

= \sup\{E^Q[X]; \tilde{Q} \in P, F_{\tilde{Q}} = F_{Y_Q}\}.$

We make some preparations before proving Lemma 11. We easily see the following.
Proposition 12. Let \( x_{k}, k = 1, 2, \ldots, n \), be a sequence of numbers, and let \( y_{k}, k = 1, 2, \ldots, n \), be a sequence of non-negative numbers. If \( x_{i_{1}} \leq x_{i_{2}} \leq \cdots \leq x_{i_{m}}, y_{j_{1}} \leq y_{j_{2}} \leq \cdots \leq y_{j_{n}} \), and \( \{i_{1}, i_{2}, \ldots, i_{n}\} = \{j_{1}, j_{2}, \ldots, j_{n}\} = \{1, 2, \ldots, n\} \), then

\[
\sum_{k=1}^{n} x_{k} y_{k} \leq \sum_{k=1}^{n} x_{ik} y_{jk}.
\]

Also we have the following (see Williams [3] Chapters 3 and 17).

Proposition 13. (1) For any \( F \in \mathcal{D} \), the probability distribution function of the law of \( Z(x, F) \) under \( \mu(dx) \) is \( F \).
(2) If \( F_{n} \in \mathcal{D} \) converges to \( F \) weakly, then \( Z(x, F_{n}) \) converges to \( Z(x, F) \) for \( \mu - a.s. \).

Now let us prove Lemma 11. Let \( X \in \mathcal{X} \). Then \( X \) is \( \mathcal{F}_{n} \)-measurable for some \( n \geq 1 \).

Let \( Y_{m} = E[Y_{Q}|\mathcal{F}_{m}] \), \( m \geq n \). Then for any \( m \geq n \), we have

\[
X(\omega) = \sum_{k=1}^{2^{m}} x_{m,k}1_{[(k-1)2^{-m}, k2^{-m})}(\omega), \quad Y_{m}(\omega) = \sum_{k=1}^{2^{m}} y_{m,k}1_{[(k-1)2^{-m}, k2^{-m})}(\omega), \quad P - a.s.,
\]

where \( x_{m,k} = 2^{m}E[P[X, [(k-1)2^{-m}, k2^{-m})] \text{ and } y_{m,k} = 2^{m}E[P[Y_{Q}, [(k-1)2^{-m}, k2^{-m})]] \text{,}
\)

\( k = 1, 2, \ldots, 2^{m} \). Let \( \sigma_{m} \) and \( \tau_{m} \) be a permutation on \( \{1, 2, \ldots, 2^{m}\} \) such that

\[
x_{m,\sigma_{m}(1)} \leq x_{m,\sigma_{m}(2)} \leq \cdots \leq x_{m,\sigma_{m}(2^{m})} \text{ and } y_{m,\tau_{m}(1)} \leq y_{m,\tau_{m}(2)} \leq \cdots \leq y_{m,\tau_{m}(2^{m})}.
\]

Then one can easily obtain that

\[
Z(x, F_{X}) = \sum_{k=1}^{2^{m}} x_{\sigma_{m}(k)}1_{[(k-1)2^{-m}, k2^{-m})}(x), \quad Z(x, F_{Y_{m}}) = \sum_{k=1}^{2^{m}} y_{\tau_{m}(k)}1_{[(k-1)2^{-m}, k2^{-m})}(x),
\]

and so

\[
E^{Q}[X] = E[XY_{Q}] = E[XY_{m}] = 2^{-m} \sum_{k=1}^{2^{m}} x_{m,k}y_{m,k} \leq 2^{-m} \sum_{k=1}^{2^{m}} x_{m,\sigma_{m}(k)}y_{m,\tau_{m}(k)}
\]

\[
= \int_{0}^{1} Z(x, F_{X})Z(x, F_{Y_{m}})dx.
\]

Since \( Y_{m} = E[Y_{Q}|\mathcal{F}_{m}] \) converges to \( Y \) \( P \)-a.s., we see by Proposition 13 that \( Z(x, F_{Y_{m}}) \) converges to \( Z(x, F_{Y}) \) for \( \mu - a.e.x \). Since \( \{Y_{m}\}_{m=n}^{\infty} \) are uniformly integrable, \( \{Z(x, F_{Y_{m}})\}_{m=n}^{\infty} \) are also uniformly integrable by Proposition 13 (1). Therefore letting \( m \to \infty \), we have

\[
E^{Q}[X] \leq \int_{0}^{1} Z(x, F_{X})Z(x, F_{Y_{Q}})dx \tag{1}
\]

for any \( X \in \mathcal{X} \). Let

\[
\tilde{X}_{m}(\omega) = \sum_{k=1}^{2^{m}} x_{m,\sigma_{m}(\tau_{m}^{-1}(k))}1_{[(k-1)2^{-m}, k2^{-m})}(\omega).
\]
Then one can easily see that the probability distributions of $X$ and $X_m$ under $P$ are the same and $X_m \in \mathcal{X}$. Also, we have

$$E^Q[X_m] = 2^{-m} \sum_{k=1}^{2^m} x_{m,\sigma_m(\tau_m^{-1}(k))} y_{m,k}$$

$$= \int_0^1 Z(x, F_X) Z(x, F_{Y_m}) dx.$$ 

So letting $m \to \infty$, we have

$$\sup\{E^Q[X]; X \in \mathcal{X}, F_X = F_X\} \geq \int_0^1 Z(x, F_X) Z(x, F_{Y_m}) dx. \quad (2)$$

Let

$$Y_m(\omega) = \sum_{k=1}^{2^m} 1_{[((k-1)2^{-m}, k2^{-m})]}(\omega) Y_Q(\omega - k2^{-m} + \tau_m(\sigma_m^{-1}(k))2^{-m}).$$

Then one can easily see that the probability distributions of $Y_Q$ and $\tilde{Y}_m$ under $P$ are the same. Let $\tilde{Q} = \tilde{Y}_m$. Then we have

$$E^\tilde{Q}[X] = 2^{-m} \sum_{k=1}^{2^m} x_{m,k} y_{m,\tau_m(\sigma_m^{-1}(k))} = \int_0^1 Z(x, F_X) Z(x, F_{Y_m}) dx.$$ 

So letting $m \to \infty$, we have

$$\sup\{E^\tilde{Q}[X]; \tilde{Q} \in \mathcal{P}, F_{Y_Q} = F_{Y_Q}\} \geq \int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx. \quad (3)$$

We have Lemma 11 from Equations (1), (2) and (3).

This completes the proof of Lemma 11.

**Proposition 14** Let $Q \in \mathcal{P}$. Then for any $X \in L^\infty$, we have

$$\int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx = \sup\{E^\tilde{Q}[X]; \tilde{Q} \in \mathcal{P}, F_{Y_{\Phi}} = F_{Y_Q}\}.$$ 

**Proof.** Let $\tilde{Y}$ be a random variable whose distribution is the same as that of $Y_Q$. Let $X_n = E[X|\mathcal{F}_n]$, $n \geq 1$. Then for any $m \geq 1$, we have

$$E[|X - X_n|\tilde{Y}] \leq E[|X - X_n||\tilde{Y} \wedge m]| + \|X - X_n\|_{\infty} E[|\tilde{Y}|, \tilde{Y} > m]$$

$$\leq (m + 2 \|X\|_{\infty} E[|\tilde{Y}|, \tilde{Y} > m].$$

So we have

$$\sup\{E^\tilde{Q}[|X - X_n|]; \tilde{Q} \in \mathcal{P}, F_{Y_Q} = F_{Y_Q}\} \to 0, \quad n \to \infty.$$ 

By Proposition 13, we have

$$\int_0^1 Z(x, F_{X_n}) Z(x, F_{Y_{Q}}) dx \to \int_0^1 Z(x, F_X) Z(x, F_{Y_Q}) dx, \quad n \to \infty.$$
Therefore we have our assertion from Lemma 11. This completes the proof.

Now let us prove Lemma 10.

Proof of Lemma 10. (1) $\Rightarrow$ (2) Let $\mathcal{G}_0$ be a subset of $\mathcal{G}$, and $\rho : L^\infty \to \mathbb{R}$ be given by

$$
\rho(X) = \sup \{ \int_0^1 Z(x, F_{-X})g(x)dx; \ g \in \mathcal{G}_0 \}, \quad X \in L^\infty.
$$

Then it is obvious that $\rho$ is law invariant. So it is sufficient to prove that $\rho$ is a coherent risk measure with the Fatou property. Let $Q_0$ be the set of $Q \in \mathcal{P}$ such that $Z(\cdot, Y_Q) \in \mathcal{G}_0$. Then by Proposition 14, we have

$$
\rho(X) = \sup \{ E^Q[-X]; \ Q \in Q_0 \}, \quad X \in L^\infty.
$$

So by Theorem 2, we see that $\rho$ is a coherent risk measure with the Fatou property. This implies our assertion.

(2) $\Rightarrow$ (1) Let $\rho$ be a law invariant coherent risk measure with the Fatou property. Let $\mathcal{P}_0$ be the set of $Q \in \mathcal{P}$ such that $E^Q[-X] \leq \rho(X)$ for all $X \in L^\infty$. Then by Theorem 2 we have

$$
\rho(X) = \sup \{ E^Q[-X]; \ Q \in \mathcal{P}_0 \}, \quad X \in L^\infty.
$$

Take a $Q \in \mathcal{P}_0$ and $X \in L^\infty$, and fix them for a while. Let $\tilde{X}(\omega) = Z(\omega; F_X)$, $\omega \in \Omega = [0, 1)$. Then we have $\rho(\tilde{X}) = \rho(X)$. Let $U_n, n \geq 1$, be random variables defined by

$$
U_n = \begin{cases} 
\tilde{X}(\omega + 2^{-n}), & \omega \in [0, 1 - 2^{-n}), \\
\|X\|_\infty, & \omega \in [1 - 2^{-n}, 1)
\end{cases}
$$

Then we see that $U_n \downarrow \tilde{X}, P - a.s$. Let $V_n = E^P[\tilde{X}|\mathcal{F}_n]$. Then we see that $V_n \leq U_n, P - a.s.$ and that $V_n \to \tilde{X}, P - a.s.$ So by Theorem 2 we have

$$
\lim_{n \to \infty} \inf_{\lim} \rho(V_n) \leq \lim_{n \to \infty} \rho(U_n) = \rho(\tilde{X}).
$$

On the other hand, by Lemma 11 and Proposition 14 we have

$$
E^Q[-X] \leq \int_0^1 Z(x, F_{-X})Z(x, F_{Y_Q})dx
= \lim_{n \to \infty} \int_0^1 Z(x, F_{-V_n})Z(x, F_{Y_Q})dx
= \lim \sup_{n \to \infty} \{ E^Q[-\tilde{V}]; \tilde{V} \in \mathcal{X}, F_{\tilde{V}} = F_{V_n} \}
\leq \lim \inf_{n \to \infty} \rho(V_n) \leq \rho(X).
$$

Thus letting $\mathcal{G}_0 = \{ Z(\cdot, F_{Y_Q}); \ Q \in \mathcal{P}_0 \}$, we see that

$$
\rho(X) = \sup \{ \int_0^1 Z(x, F_{-X})g(x)dx; \ g \in \mathcal{G}_0 \}.
$$

This implies our assertion. This completes the proof of Lemma 10.
3 Proof of Theorem 4

In this section, we prove Theorem 4. Let \( g \in \mathcal{G} \), and let \( \tilde{g} : \mathbb{R} \to \mathbb{R} \) be given by 
\[
\tilde{g}(t) = 0, t < 0, \quad \tilde{g}(t) = g(t), t \in [0, 1), \quad \text{and} \quad \tilde{g}(t) = g(1 - t), t \geq 1.
\]
Then we have for any \( X \in L^\infty \)
\[
\int_0^1 Z(x, F_-X)g(x)dx = \int_{[0,1)} \left( \int_x^1 Z(y; F_-X)dy \right) d\tilde{g}(x) = \int_{[0,1)} \rho_{1-x}(X)(1-x) d\tilde{g}(x).
\]
Letting \( X = -1 \), we have
\[
1 = \int_0^1 g(x)dx = \int_{[0,1)} (1-x) d\tilde{g}(x).
\]
From this observation and Lemma 10, we have the following.

Proposition 15 Let \( \rho : L^\infty \to \mathbb{R} \). Then the following conditions are equivalent.
(1) There is a set \( \mathcal{M}_0 \) of probability measures on \((0, 1]\) such that for \( X \in L^\infty \)
\[
\rho(X) = \sup \left\{ \int_{[0,1]} \rho_\alpha(X)m(d\alpha); m \in \mathcal{M}_0 \right\}.
\]
(2) \( \rho \) is a law invariant coherent risk measure with the Fatou property.

Now we prove Theorem 4. For each probability measure \( m \) on \([0, 1]\), let \( \nu_n(m), n \geq 1 \), be a probability measure on \((0, 1]\) given by
\[
\nu_n(m)(A) = m(A \cap (0, 1]) + m(\{0\}) \delta_{1/n}(A), \quad \text{for a Borel set in} \ [0, 1].
\]
Then we see that for any \( X \in L^\infty \)
\[
\int_{[0,1]} \rho_\alpha(X)m(d\alpha) = \sup_n \int_{[0,1]} \rho_\alpha(X)\nu_n(m)(d\alpha).
\]
This and Proposition 15 imply Theorem 4. This completes the proof of Theorem 4.

4 Proof of Theorem 5

We give some computation on \( \rho_\alpha \) in this section.

Proposition 16 Let \( c \in (0, 1) \) and \( X_c(\omega) = 1_{[1-c, 1)}(\omega), \ \omega \in \Omega = [0, 1). \)
(1) We have
\[
\rho_\alpha(-X_c) = 1 \wedge \frac{c}{\alpha} \quad \alpha \in (0, 1]
\]
(2) Let \( m \) be a probability measure on \([0, 1]\) and let \( f(s) = \int_{[0,1]} \rho_\alpha(-X_s)m(d\alpha), s \in (0, 1] \). Then \( f(c) \) is differentiable at \( s = c \in (0, 1) \) such that \( m(\{c\}) = 0 \), and
\[
\frac{df}{ds}(c) = \int_{(c,1]} \frac{1}{\alpha} m(d\alpha).
\]
Proof. Noting that \( F_{X_\epsilon}(x) = 1_{[1-\alpha,1]}(x), \ x \in [0,1], \) we easily have the assertion (1). Then we have for \( 0 < s < t < 1 \)

\[
\frac{f(t) - f(s)}{t - s} = \int_{[t,1]} \frac{1}{s} m(d\alpha) + \frac{1}{t - s} \int_{(s,t]} \frac{\alpha - s}{\alpha} m(d\alpha).
\]

This proves the assertion (2).

Theorem 5 is an easy consequence of Proposition 16 (2).

5 Supporting measures and Proof of Theorem 9

Let \( \mathcal{M} \) denote the set of all probability measures on \([0,1]\). Then \( \mathcal{M} \) is a compact metric space with the Prohorov metric. Let \( \rho \) be a law invariant coherent risk measure with the Fatou property. Let

\[
\mathcal{M}(\rho) = \{ m \in \mathcal{M}; \int_{[0,1]} \rho_\alpha(X)m(d\alpha) \leq \rho(X) \ \text{for all} \ X \in L^\infty \}.
\]

Since \( \rho_\alpha(X) \) is continuous in \( \alpha \in [0,1] \), \( \mathcal{M}(\rho) \) is a closed convex subset of \( \mathcal{M} \). Then from Theorem 4 we have

\[
\rho(X) = \sup \{ \int_{[0,1]} \rho_\alpha(X)m(d\alpha); m \in \mathcal{M}(\rho) \}, \quad X \in L^\infty.
\]

For each \( X \in L^\infty \) let

\[
\tilde{\mathcal{M}}(X;\rho) = \{ m \in \mathcal{M}(\rho); \int_{[0,1]} \rho_\alpha(X)m(d\alpha) = \rho(X) \}.
\]

From the compactness of \( \mathcal{M}(\rho) \) we see that \( \tilde{\mathcal{M}}(X;\rho) \neq \emptyset \). It is obvious that \( \tilde{\mathcal{M}}(X;\rho) \) depends only on the distribution \( F_X \) of \( X \), and so we denote it by \( \tilde{\mathcal{M}}(F_X;\rho) \).

Now we prove Theorem 9. Let \( \rho \) be a law invariant coherent risk measure such that

\[
\rho(X) \geq \text{VaR}_\alpha(X), \quad X \in L^\infty.
\]

Let \( X_\epsilon(\omega) = 1_{[1-\alpha-\epsilon,1]}(\omega), \ \omega \in \Omega = [0,1], \ \epsilon \in (0,1-\alpha) \), and let \( m_\epsilon \in \tilde{\mathcal{M}}(X_\epsilon;\rho) \). Then by Proposition 16 we see that

\[
\rho(-X_\epsilon) = \int_{[0,1]} 1_{\frac{\alpha + \epsilon}{s}} m_\epsilon(ds).
\]

On the other hand, we have \( \text{VaR}_\alpha(-X_\epsilon) = 1 \). So we see that \( m_\epsilon([0,\alpha + \epsilon]) = 1 \). Since \( \mathcal{M}(\rho) \) is compact, we see that there is an \( m \in \mathcal{M}(\rho) \) such that \( m([0,\alpha]) = 1 \). Therefore we see that \( \rho_\alpha(X) \leq \rho(X), \ X \in L^\infty \).

This completes the proof of Theorem 9.
6 Proof of Theorem 7

Proposition 17 Let $X, Y$ be comonotone random variables and $a, b \in \mathbb{R}$. Then $\{X \geq a\} \subset \{Y \geq b\}$ $P$-a.s. or $\{Y \geq b\} \subset \{X \geq a\}$ $P$-a.s.

Proof. Let $C = \{X \geq a\} \cap \{Y < b\}$ and $D = \{X < a\} \cap \{Y \geq b\}$. Then for $(\omega, \omega') \in C \times D$,

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) < 0.$$ 

This implies that $P(C) = 0$ or $P(D) = 0$. So we have our assertion.

As an immediate consequence, we have the following.

Corollary 18 Let $X, Y$ be comonotone random variables. Then we have

$$P(X + Y \geq a + b) \geq P(X \geq a) \land P(Y \geq b), \quad a, b \in \mathbb{R}.$$ 

Proposition 19 Let $X, Y \in L^{\infty}$ be comonotone and $a, b \in \mathbb{R}$. Then

$$Z(x, F_{X+Y}) = Z(x, F_{X}) + Z(x, F_{Y}), \quad x \in [0, 1)$$

Proof. By the definition of $Z(x, F_{X})$ we have $F_{X}(Z(x, F_{X})-) \leq x$. So we have $P(X \geq Z(x, F_{X})) \geq 1 - x$. Similarly we have $P(Y \geq Z(x, F_{Y})) \geq 1 - x$. Therefore by Corollary 18 we have

$$P(X + Y \geq Z(x, F_{X}) + Z(x, F_{Y})) \geq 1 - x, \quad x \in [0, 1).$$

Note that Let $Z(x, F_{X+Y}) = \sup\{z \in \mathbb{R}; F_{X+Y}(z) \leq x\}$, $x \in (0, 1)$. So we see that $Z(x, F_{X}) + Z(x, F_{Y}) \leq Z(x, F_{X+Y})$, $x \in (0, 1)$. On the other hand, we have

$$\int_{[0,1]} (Z(x, F_{X}) + Z(x, F_{Y})) \mu(dx) = E[X] + E[Y] = \int_{[0,1]} Z(x, F_{X+Y}) \mu(dx).$$

So we see that

$$Z(x, F_{X}) + Z(x, F_{Y}) = Z(x, F_{X+Y}), \quad \mu$-a.e.$x.$

Since both sides are right continuous, we have our assertion.

Proposition 20 $\rho_{\alpha}, \alpha \in [0, 1]$, are comonotone.

Proof. For each $\alpha \in (0, 1)$, we see that $\rho_{\alpha}$ is comonotone from the definition of $\rho_{\alpha}$ and Proposition 19. Letting $\alpha \downarrow 0$, we see that $\rho_{0}$ is also comonotone.

Proposition 21 Let $\rho$ be a comonotone law invariant coherent risk measure with the Fatou property. Then $\bigcap_{i=1}^{n} \mathcal{M}(F_{i}; \rho) \neq \emptyset$ for any $n \geq 1$ and $F_{1}, F_{2}, \ldots, F_{n} \in \mathcal{D}$.

Proof. Let $X_{i}(\omega) = Z(\omega, F_{i})$, $\omega \in \Omega = [0, 1)$, $i = 1, \ldots, n$. Then $\sum_{i=1}^{k} X_{i}$ and $X_{k+1}$ are comonotone for each $k = 1, \ldots, n-1$. Let $X = \sum_{i=1}^{n} X_{i}$. Then we have

$$\rho(X) = \sum_{i=1}^{n} \rho(X_{i}).$$
Let $m \in \tilde{\mathcal{M}}(X; \rho)$. Then we have
\[
\sum_{i=1}^{n} \int_{[0,1]} \rho_{\alpha}(X_i) m(d\alpha) = \rho(X) = \sum_{i=1}^{n} \rho(X_i).
\]

Also, we have
\[
\int_{[0,1]} \rho_{\alpha}(X_i) m(d\alpha) \leq \rho(X), \quad i = 1, \ldots, n.
\]
So we have
\[
\int_{[0,1]} \rho_{\alpha}(X_i) m(d\alpha) = \rho(X), \quad i = 1, \ldots, n.
\]
This implies $m \in \tilde{\mathcal{M}}(X_i; \rho), \ i = 1, \ldots, n$. So we have our assertion.

Now let us prove Theorem 7. Suppose that $m \in \mathcal{M}$ and $\rho: L^\infty \to \mathbb{R}$ is given by
\[
\rho(X) = \int_{[0,1]} \rho_{\alpha}(X) m(d\alpha), \quad X \in L^\infty.
\]
Then by Theorem 4 and Proposition 20, we see that $\rho$ is comonotone and law invariant.

On the other hand, suppose that $\rho$ is a comonotone law invariant coherent risk measure with the Fatou property. Then by Proposition 21 and the fact that $\mathcal{M}$ is compact, we see that $\bigcap \{\mathcal{M}(F; \rho); F \in \mathcal{D}\} \neq \emptyset$. Let $m$ be an element of this set. Then we see that
\[
\rho(X) = \int_{[0,1]} \rho_{\alpha}(X) m(d\alpha), \quad X \in L^\infty.
\]
This completes the proof of Theorem 7.

7 A Remark

For each $\alpha \in (0, 1]$ let $\varphi_{\alpha}: [0, 1] \to [0, 1]$ be given by
\[
\varphi_{\alpha}(t) = \frac{t}{\alpha} \wedge 1, \quad t \in [0, 1].
\]

Then we have the following.

**Proposition 22** For any $\alpha \in (0, 1]$ and $X \in L^\infty$ satisfying $X \leq 0, P - a.s.$, we have the following.
\[
\rho_{\alpha}(X) = \int_{0}^{\infty} \varphi_{\alpha}(P(-X > y)) dy.
\]

**Proof.** Let $\alpha \in (0, 1)$ and $X \in L^\infty$ such that $X \leq 0$ and $X$ has a continuous strictly increasing distribution on $(\text{ess.inf} \ X, \text{ess.sup} \ X)$. Then we see that $Z(x, F_{-X}) = F_{-X}^{-1}(x)$, $x \in (0, 1)$. Let $q_\alpha \in (0, \infty)$ be such that $F_{-X}(q_\alpha) = 1 - \alpha$. Then we have
\[
\rho_{\alpha}(X) = \frac{1}{\alpha} \int_{q_\alpha}^{\infty} y d(1 - F_{-X}(y))
\]
\[
= -\frac{1}{\alpha} \int_{q_\alpha}^{\infty} y d(1 - F_{-X}(y)) + \frac{1}{\alpha} \int_{q_\alpha}^{\infty} (1 - F_{-X}(y)) dy
\]
\[
\int_0^\infty \varphi_{\alpha}(P(-X > y))dy.
\]

Since any nonpositive random variables is approximated by such random variables in probability, we have our assertion for \(\alpha \in (0, 1)\). Letting \(\alpha \uparrow 1\), we also have our assertion for \(\alpha = 1\). This completes the proof.

Let \(m \in \mathcal{M}\), and let \(\varphi(t; m) = \int_{(0,1]} \varphi_{\alpha}(t)m(d\alpha)\), \(t \in [0,1]\). Then we see that \(\varphi(\cdot, m) : [0,1] \to [0,1]\) is a continuous increasing concave function with \(\varphi(0) = 0\), and \(\varphi(1) = 1 - m(\{0\})\). We also see that

\[
\frac{d}{dt} \varphi(t; m) = \int_t^1 \frac{1}{\alpha}m(d\alpha),
\]

for any continuous point \(t \in (0,1)\) of the measure \(m\). So \(\varphi(\cdot, m)\) determines \(m\).

For any nonpositive \(X \in L^\infty\) we have

\[
\int_0^\infty \rho_{\alpha}(X)m(d\alpha) = m(\{0\})ess.sup(-X) + \int_0^\infty \varphi(P(-X > y); m)dy.
\]

These observations imply the following.

**Theorem 23** Let \(\rho : L^\infty \to \mathbb{R}\). Then the following are equivalent.

1. \(\rho\) is a law invariant and comonotone coherent risk measure with the Fatou property.
2. There is a continuous nondecreasing concave function \(\varphi : [0,1] \to [0,1]\) such that

\[
\rho(X) = (1 - \varphi(1))ess.sup(-X) + \int_0^\infty \varphi(P(-X > y))dy
\]

for any nonpositive \(X \in L^\infty\).

**References**

