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Reasoning about Dominant Actions

— Logic of Decision Theory II —

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ABSTRACT. The logic of 'utility maximizers' L^{um} is proposed which is an extension of a system of modal logic for two players. The sound models according to L^{um} are given in terms of game theory. It is shown for the models that two utility maximizing players must take the same action if they mutually believe that each takes a dominant action, even when they have different informations. We remark that the logic L^{um} has the finite model property.

1. INTRODUCTION

The purpose of the paper is to develop a formal theory of decision making processes among two players under uncertainty based on modal logics rather than on probability measures (as in the standard theory).

Recently researchers in such diverse fields as Game Theory, Logics, Artificial Intelligence, and Computer Science have become interested in

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reasoning about knowledge. There are pragmatic concerns about the relationship between knowledge and actions, and there are also concerns about the complexity of computing knowledge. Of most interest to us is the emphasis on considering situation involving the knowledge of a group of players rather than that of a single player although logicians tend to focus on the process of reasoning about knowledge in the latter case.

We propose the formal theory of 'utility maximizers' that consists of a formal system and a class of models, in which each player takes the actions being the best response to the other players' actions. In the system we use a 'logic of belief' in stead of using the logics of knowledge, and we show the two results in the models: First, each player chooses an action when he simply believes that it is dominant, and second the utility maximizing players must take the same action if they mutually believe that each takes a dominant action, even when they have different informations.

There are other kinds of theories of decision making. The theory of 'agreeing to disagree' is most interesting, in which all players must make the same prediction about an event (c.f. Aumann [1], Bacharach [2] and Matsuhisa [6].) It is noted that the latter one of the two results mentioned above is a variation of the 'agreeing to disagree' theorem.

This paper organizes as follows: Section 2 presents a game with belief. In Section 3 we propose a system for 'utility maximizers' that is an extension of a system of modal logic. The sound models according to the system are given in terms of the game theory with belief. Section 4 presents the logic of utility maximizers and shows the two theorems: the completeness and the finite model property for the logic. The most important idea is that of a canonical model such as the sentences true in the model are precisely the theorems of the logic. The idea of Γ-filtration is also important in the argument for the finite model property. The main problems in proving the two theorems become those of defining a canonical model and its Γ-filtration. In Section 5 we show that two utility maximizers in each sound model must take the same actions if they mutually believe that each takes a dominant action, even when they have different informations. Example (Prisoner Dilemma) demonstrates that they does not always take the same actions in case that each player simply believes that he takes a dominant action. Section 6 presents the logic for 'agreement of dominant actions' L^{ada} and remarks that the logic has the finite model property.

2. The Model

Let Ω be a non-empty set called a *state-space*, N a set of two *players* $\{1, 2\}$, and let 2^Ω be the family of all subsets of Ω. Each member of 2^Ω is called an *event* and each element of Ω called a *state*.

2.1. **Information and Belief** (Binmore [3]). An *information structure* $(P_i)_{i \in N}$ is a class of mappings P_i of Ω into 2^Ω. Given our interpretation, player i for whom $P_i(\omega) \subseteq E$ believes, in the state ω, that some state in the event E has occurred. In this case we say that in the state ω the player i believes E.

Player i's *belief operator* is an operator B_i on 2^Ω such that $B_i E$ is the set of states in which i believes that E has occurred; that is,

$$B_i E = \{\omega \in \Omega | P_i(\omega) \subseteqq E\}. \tag{1}$$

We note that i's belief operator satisfies the following properties: For every E, F of 2^Ω,

N: $B_i \Omega = \Omega$ and $B_i \emptyset = \emptyset$;

K: $B_i(E \cap F) = B_i E \cap B_i F$;

The set $P_i(\omega)$ will be interpreted as the set of all the states of nature that i believes to be possible at ω, and $B_i E$ will be interpreted as the set of states of nature for which i believes E to be possible. We will therefore call P_i i's *possibility operator* on Ω and also will call $P_i(\omega)$ i's *possibility set* at ω. An event E is said to be an i's *truism* if $E \subseteqq B_i E$

We should note that the information structure P_i is uniquely determined by the belief operator B_i with $P_i(\omega) = \bigcap_{\omega \in B_i E} E$.

2.2. **Game and Belief.** By a *game* for two players we mean a triple $\langle N, (A_i)_{i=1,2}, (V_i)_{i=1,2} \rangle$ with the following structure and interpretations: N is a of players $\{1, 2\}$, A_i is a finite set of i's *available actions* (or i's pure strategies) and V_i is an i's *utility-function* of $A_1 \times A_2$ into $\mathbb{R}$. We denote by A_{-i} the set A_j for $j \neq i$.

An action a_i in A_i is called *dominant* for i if $V_i(a_i, a_{-i}) \geq V_i(b, a_{-i})$ for all $b \in A_i$ and for all $a_{-i} \in A_{-i}$.

Example 1. (Prisoners' dilemma:) Let A be a set of two available actions $\{a_1, a_2\}$ which is common for players $1, 2$. The utility functions (V_1, V_2) are given by Table 1.

In this example we can plainly observe that the action a_1 is dominant for each player 1 and 2. □

TABLE 1

	Player 2		
	(V_1, V_2)	a_1	a_2
Player 1	a_1	1, 1	3, 0
	a_2	0, 3	2, 2

Definition. By a *game with belief* we mean a tuple

$$\mathcal{V} = \langle \Omega, (P_*)_{*=1,2,E}, (A_i)_{i=1,2}, (V_i)_{i=1,2} \rangle$$

with the following structures:

- Ω is a state-space;
- $P_i : \Omega \longrightarrow 2^\Omega$ is i's information function for $i = 1, 2$;
- $P_E : \Omega \longrightarrow 2^\Omega$ is defined by $P_E(\omega) \underset{\text{def}}{:=} \bigcup_{i=1,2} P_i(\omega)$;
- A_i is a set of available actions for player i;
- $V_i : A_1 \times A_2 \times \Omega \longrightarrow \mathbb{R}$ is i's utility function with the property that $V_i(a_i, a_{-i}; \omega)$ is injective on A_i for each $a_{-i} \in A_{-i}$ and $\omega \in \Omega$.

Example 2. A tuple $\mathcal{V} = \langle \Omega, (P_i)_{i=1,2}, (A_i)_{i=1,2}, (V_i)_{i=1,2} \rangle$ given as below is a game with belief:

- $\Omega = \{\omega_1, \omega_2\}$
- $P_i : \Omega \longrightarrow 2^\Omega$ is given by $P_1(\omega_1) := \{\omega_1\}, P_1(\omega_2) := \{\omega_2\}, P_2(\omega_1) := \{\omega_2\}$, and $P_2(\omega_2) := \{\omega_1\}$;
- $A_1 = A_2 = \{a_1, a_2\}$;
- $V_i : A_1 \times A_2 \times \Omega \longrightarrow \mathbb{R}$ is defined by Table 2.

TABLE 2

Player 2

	$(V_1(\cdot,\cdot;\omega_1), V_2(\cdot,\cdot;\omega_1))$			$(V_1(\cdot,\cdot;\omega_2), V_2(\cdot,\cdot;\omega_2))$		
		a_1	a_2		a_1	a_2
Player 1	a_1	1, 1	3, 0	a_1	2, 2	0, 3
	a_2	0, 3	2, 2	a_2	3, 0	1, 1

□

3. SYSTEM

Let us consider a system of multi-modal logic as follows.

3.1. **Syntax.** The *language* of the system consists of the symbols, the terms and the sentences as follows:

- *Symbols*:

Non-modal operators :	$\top, \neg, \rightarrow, \wedge, \cdots;$	
Modal operators :	$\Box_1, \Box_2, \Box_E;$	
Variables:	$\mathbf{a}_1^1, \mathbf{a}_2^1, \cdots, \mathbf{a}_n^1, \cdots$	(Actions for players 1)
	$\mathbf{a}_1^2, \mathbf{a}_2^2, \cdots, \mathbf{a}_n^2, \cdots$	(Actions for players 2)
Predicates:	$=$	(Equality on the actions)
	$\mathrm{dom}_1, \mathrm{dom}_2.$	(Dominant actions)

- *Terms* and *Sentences*:

  (i) The variables are *terms*;

  (ii) If s and t are two terms then $\mathrm{s} = \mathrm{t}$ and $\mathrm{dom}_1(\mathrm{s}), \mathrm{dom}_2(\mathrm{t})$ are *atomic* sentences;

  The *sentences* of the language form the least set containing all *atomic* sentences $\mathbf{P}_m (m = 0, 1, 2, \dots)$ closed under the following operations:

  - nullary operators for *falsity* $\bot$ and for *truth* $\top$;
  - unary and binary syntactic operations for *negation* $\neg$, *conditionality* $\rightarrow$ and *conjunction* $\wedge$, respectively;
  - three unary operations for *modality* $\Box_1, \Box_2$, and $\Box_E$.

The intended interpretation of $\Box_i\varphi$ is the sentence that 'player i believes a sentence φ,' and $\Box_E\varphi$ is that 'everybody believes φ.' The sentence $\mathrm{dom}_i(\mathbf{a}_k)$ is interpreted as '$\mathbf{a}_k$ is a dominant action for i.'

3.2. **System of utility maximizers.** By this we mean a set of sentences, denoted by L,

- containing a set of all *tautologies* and closed under *substitution* and *modus ponens*;
- has the following *inference rules* and *axioms*:

(N) $\quad \Box_*\top \quad$ for $* = 1, 2, E$

(RE$_\Box$) $$\frac{\varphi \longleftrightarrow \psi}{\Box_*\varphi \longleftrightarrow \Box_*\psi} \quad \text{for } * = 1, 2, E;$$

(Def$\Box_E$) $\quad \Box_E\varphi \longleftrightarrow \Box_1\varphi \wedge \Box_2\varphi;$

(RE$_{\mathbf{dom}}$) $\quad \mathbf{a}_k^i = \mathbf{a}_l^i \wedge \mathrm{dom}_i(\mathbf{a}_k^i) \longrightarrow \mathrm{dom}_i(\mathbf{a}_l^i) \quad$ for $i = 1, 2$.

(RU$_{\mathbf{dom}}$) $\quad \mathrm{dom}_i(\mathbf{a}_k^i) \wedge \mathrm{dom}_i(\mathbf{a}_l^i) \longrightarrow \mathbf{a}_k^i = \mathbf{a}_l^i \quad$ for $i = 1, 2$.

Let $\mathcal{S}$ be the set of all sentences in a system L, Γ a subset of $\mathcal{S}$ and let φ be a sentence. Then φ is *L-deducible*, $\Gamma \vdash_L \varphi$, if there exists $\chi_1, \chi_2, \cdots \chi_m \in \Gamma$ such that the sentence of form $\chi_1 \wedge \chi_2 \wedge \cdots \wedge \chi_m \longrightarrow \varphi$ is in Γ. φ is an *L-theorem* (or simply a *theorem*), $\vdash_L \varphi$, if $\varphi \in L$.

3.3. **Semantics.** A *model* $\mathcal{M}$ for a system L is a tuple $\langle \mathcal{V}, v_{\mathcal{M}}, \pi_{v_{\mathcal{M}}} \rangle$ with the following structures:

- $\mathcal{V} = \langle \Omega, (P_*)_{*=1,2,E}, (A_i)_{i=1,2}, (V_i)_{i=1,2} \rangle$ is a game with belief such that
  - (i) $A_i := \{a_1^i, a_2^i, \cdots, a_n^i, \cdots\}$ for $i = 1, 2$;
  - (ii) $V_i(a_i, a_{-i}; \omega)$ is injective on A_i for each $a_{-i} \in A_{-i}$ and $\omega \in \Omega$.
- $v_{\mathcal{M}}^i : \{\mathbf{a}_k^i \mid k = 1, 2, \cdots, n, \cdots\} \to A_i$ is a mapping of i's variables into i's available actions;
- $v_{\mathcal{M}} : \bigcup_{i=1,2}\{\mathbf{a}_k^i \mid k = 1, 2, \cdots, n\} \to \bigcup_{i=1,2} A_i$ is the *valuation* of variables into available actions defined by

$$v_{\mathcal{M}}(\mathbf{a}_k^i; \omega) := v_{\mathcal{M}}^i(\mathbf{a}_k^i; \omega);$$

- $\pi_{v_{\mathcal{M}}} : \{\mathbf{P}_m \mid m = 0, 1, 2, \ldots\} \times \Omega \longrightarrow \{\mathsf{true}, \mathsf{false}\}$ is a truth assignment such that, for all $\omega \in \Omega$,
  - (i) $\pi_{v_{\mathcal{M}}}(\mathbf{a}_k^i = \mathbf{a}_l^i, \omega) = \mathsf{true}$ if and only if $v_{\mathcal{M}}(\mathbf{a}_k^i; \omega) = v_{\mathcal{M}}(\mathbf{a}_l^i; \omega)$;
  - (ii) (a) $\pi(\mathrm{dom}_1(\mathbf{a}_k^1), \omega) = \mathsf{true}$ if and only if $V_1(v_{\mathcal{M}}(\mathbf{a}_k^1), c; \omega) \geq V_1(b, c; \omega)$ for all $b \in A_1, c \in A_2$;
    (b) $\pi(\mathrm{dom}_2(\mathbf{a}_l^2), \omega) = \mathsf{true}$ if and only if $V_2(a, v_{\mathcal{M}}(\mathbf{a}_l^2); \omega) \geq V_2(a, b; \omega)$ for all $a \in A_1, b \in A_2$.

3.4. **Validity.** *Truth* $\models_\omega^{\mathcal{M}} \varphi$ at ω in $\mathcal{M}$ is inductively defined as follows:

(i) $\models_\omega^{\mathcal{M}} \mathbf{P}_m$ if and only if $\pi(\mathbf{P}_m, \omega) = \mathsf{true}$ for each atomic sentence $\mathbf{P}_m$;

(ii) $\models_\omega^{\mathcal{M}} \top$;

(iii) $\models_\omega^{\mathcal{M}} \varphi \longrightarrow \psi$ if and only if $\models_\omega^{\mathcal{M}} \varphi$ implies $\models_\omega^{\mathcal{M}} \psi$;

(iv) $\models_\omega^{\mathcal{M}} \Box_i \varphi$ if and only if $P_i(\omega) \subseteq \{\xi \in \Omega | \models_\xi^{\mathcal{M}} \varphi\}$, for $i = 1, 2$;

(v) $\models_\omega^{\mathcal{M}} \Box_E \varphi$ if and only if $P_E(\omega) \underset{\mathrm{def}}{:=} \bigcup_{i=1,2} P_i(\omega) \subseteq ||\varphi||^{\mathcal{M}}$,

where $||\varphi||^{\mathcal{M}}$ denotes $\{\xi \in \Omega | \models_\xi^{\mathcal{M}} \varphi\}$. We say that φ is *valid* in $\mathcal{M}$ and write $\models^{\mathcal{M}} \varphi$ if φ is true for every $\omega \in \Omega$.

4. Logic of Utility Maximizers

4.1. We concern with proving that a system of utility maximizers is determined by the class of models.

Definition. The *logic of utility maximizers* is the smallest system of utility maximizers, denoted by L^{um}, and $\mathbf{M}_{L^{um}}$ denotes the class of all models for L^{um}.

4.2. Let Σ be a subset of the set of all sentences $\mathcal{S}$ in a system L. We say that $\mathcal{M}$ is a *model for* Σ if every member of Σ is true in $\mathcal{M}$. Let $\mathbf{C}$ be a class of models. We denote $\models_{\mathbf{C}} \varphi$ to mean that φ is valid in every model of $\mathbf{C}$. A system of utility maximizers L is *sound with respect to* $\mathbf{C}$ if every member of $\mathbf{C}$ is a model for L. It is *complete with respect to* $\mathbf{C}$ if every sentence valid in all members of $\mathbf{C}$ is a theorem of L. We say that L is *determined by* $\mathbf{C}$ if L is sound and complete with respect to $\mathbf{C}$. We say a model for L to be *finite* if its state-space is a finite set.

4.3. **Soundness and Completeness for L^{um}.** The following theorems are our main results:

Theorem 1. *The logic of utility maximizers is sound with respect to the class* $\mathbf{M}_{L^{um}}$ *of all models.*

Proof. immediately follows from the definition of the model. □

Theorem 2. *The logic of utility maximizers is complete with respect to the class* $\mathbf{M}_{L^{um}}$ *of all models.*

Theorem 3. *The logic of utility maximizers has the finite model property.*

The proofs of the theorems will be found in the later section (4.9).

4.4. **Lindenbaum's lemma.** Let Γ be a set of sentences in a system L and φ a sentence. A set Γ is *L-consistent* if there exists at least one sentence not L-deducible from Γ, and *L-inconsistent* otherwise. A set Γ is *L-maximal*, $\mathrm{Max}_L\Gamma$ if Γ is L-consistent and for each sentence φ either $\varphi \in \Gamma$ or $\neg\, \varphi \in \Gamma$. We denote by Ω_L the class of L-maximally consistent sets. For a sentence φ, $|\varphi|_L := \{\mathrm{Max}_L\Gamma \in \Omega_L \mid \varphi \in \mathrm{Max}_L\Gamma\}$. For a set Λ of sentences, $|\Lambda|_L := \{\mathrm{Max}_L\Gamma \in \Omega_L \mid \Lambda \subseteq \mathrm{Max}_L\Gamma\}$.

To prove the theorem we need the following result, which proof can be found in [4].

Lemma 1. *Let L be a system of utility maximizers.*

(i) *Every L-consistent set of sentences has an L-maximal extension;*

(ii) $\Gamma \vdash_L \varphi$ *if and only if* $|\Gamma|_L \subseteq |\varphi|_L$.

We record the immediate consequences of the lemma as follows:

Remark. (i) A sentence in L is deducible from a set of sentences Γ if and only if it belongs to every L-maximally consistent set of Γ;

(ii) A sentence is a theorem of L if and only if it is a member of every L-maximally consistent set of sentences.

(iii) A sentence of form $\varphi \to \psi$ is a theorem of L if and only if $|\varphi|_L \subseteq |\psi|_L$.

4.5. **Canonical Model.** Let L be a system of utility maximizers. The *canonical model* $\mathcal{M}_L$ for L is the tuple $\langle \mathcal{V}_L, v_L, \pi_{v_L} \rangle$ for L with the game with belief $\mathcal{V}_L = \langle \Omega_L, (P_i^L)_{i=1,2}, (A_i^L)_{i=1,2}, (V_i^L)_{i=1,2} \rangle$, which consists of:

- Ω_L is the set of all the L-maximally consistent sets of sentences;

- $P_*^L(\omega) := \{ \xi \in \Omega_L \mid \Box_* \varphi \in \omega \Rightarrow \varphi \in \xi \}$ for $\omega \in \Omega_L$ and $* = 1, 2, E$.
- $\langle (A_i^L)_{i=1,2}, (V_i^L)_{i=1,2} \rangle$ is defined by Table 3:

TABLE 3

		Player 2	
	$V_1^L(\cdot\,; \omega), V_2^L(\cdot\,; \omega)$	a_1^2	a_2^2
Player 1	a_1^1	1, 1	1, 0
	a_2^1	0, 1	0, 0

for every $\omega \in \Omega_L$.

- $v_L^i : \{\mathbf{a}_k^i \mid k = 1, 2, \cdots, n, \cdots\} \times \Omega_L \to A_i$ is the mapping defined by

$$v_L^i(\mathbf{a}_k^i; \omega) := \begin{cases} a_1^i & \text{if } \mathrm{dom}_i(\mathbf{a}_k^i) \in \omega; \\ a_2^i & \text{otherwise;} \end{cases}$$

- $v_L : \bigcup_{i=1,2}\{\mathbf{a}_k^i \mid k = 1, 2, \cdots, n, \cdots\} \times \Omega_L \to \bigcup_{i=1,2} A_i$ is defined by the same way as above.
- $\pi_L : \{\mathbf{P}_m \mid m = 0, 1, 2, \ldots\} \times \Omega_L \longrightarrow \{\mathsf{true}, \mathsf{false}\}$ is the truth assignment such that

(i) $\pi_L(\mathbf{a}_k^i = \mathbf{a}_l^i, \omega) = \mathsf{true}$ if and only if $v_L(\mathbf{a}_k^i; \omega) = v_L(\mathbf{a}_l^i; \omega)$;

(ii) (a) $\pi_L(\mathrm{dom}_1(\mathbf{a}_k^1), \omega) = \mathsf{true}$ if and only if
$V_1^L(v_L(\mathbf{a}_k^1), b; \omega) \geq V_1^L(a, b; \omega)$ for all $a \in A_1, b \in A_2$;

(b) $\pi_L(\mathrm{dom}_2(\mathbf{a}_l^2), \omega) = \mathsf{true}$ if and only if
$V_2^L(a, v_L(\mathbf{a}_l^2); \omega) \geq V_2^L(a, b; \omega)$ for all $a \in A_1, b \in A_2$.

Proposition 1. *The canonical model is actually a model for a system of utility maximizers.*

Proof. immediately follows from the definition of the canonical model. □

4.6. The important result about the canonical model is the following:

Basic theorem. *Let $\mathcal{M}_L$ be the canonical model for a system L of utility maximizers. Then for every sentence φ,*

$$\models^{\mathcal{M}_L} \varphi \quad \textit{if and only if} \quad \vdash_L \varphi \,.$$

In other words,

$$||\varphi||^{\mathcal{M}_L} = |\varphi|_{\mathcal{M}_L}.$$

Proof. By induction on the complexity of φ. We treat only the case $\varphi = \Box_* \psi$. As an inductive hypothesis we assume that $||\psi||^{\mathcal{M}_L} = |\psi|_{\mathcal{M}_L}$. Then for every $\omega \in \Omega_L$,

$\models_\omega^{\mathcal{M}_L} \Box_* \psi$ if and only if $P_*^L(\omega) \subseteq ||\psi||^{\mathcal{M}_L}$,
by the definition of canonical model ;
if and only if $P_*^L(\omega) \subseteq |\psi|_L$,
by the inductive hypothesis as above;
if and only if $\vdash_L \psi$.

□

4.7. **Filtration.** Let Γ be a set of sentences with the following properties:

(γ_1) Γ is closed under subsentences:
(γ_2) Both $\mathrm{dom}_i(\mathbf{a}_k^i)$ and $\mathrm{dom}_i(\mathbf{a}_l^i)$ belong in Γ whenever $\mathbf{a}_k^i = \mathbf{a}_l^i$.

We define the equivalence relation $\equiv_\Gamma$ on Ω_L by

$\omega \equiv_\Gamma \xi$ if and only if for every sentence ψ of Γ,

$$\models_\omega^{\mathcal{M}_L} \psi \iff \models_\xi^{\mathcal{M}_L} \psi \,.$$

We denote by $[\omega]$ the equivalence class of ω and denote by $[X]$ the set of equivalence classes $[\omega]$ for all ω of X whenever X is a subset of Ω_L.

Definition. By the Γ-*filtration* $\mathcal{M}_L^\Gamma$ (or the *filtration of* $\mathcal{M}_L$ *through* Γ), we mean a structure for L

$$\langle \Omega_L^\Gamma, (P_*^{L,\Gamma})_{*=1,2,E}, (A_i^{L,\Gamma})_{i=1,2}, (V_i^{L,\Gamma})_{i=1,2}, v_L^\Gamma, \pi_{v_L}^\Gamma \rangle$$

consists of the following structures: For each $* = 1, 2, E$,

- $\Omega^\Gamma = [\Omega_L]$;
- For every ω, ξ in Ω,
  (i) if $\xi \in P_*^L(\omega)$, then $[\xi] \in P_*^{L,\Gamma}([\omega])$;
  (ii) if $[\xi] \in P_*^{L,\Gamma}([\omega])$, then for every sentence $\varphi \in \Gamma$,
  $$\models_\omega^{\mathcal{M}_L} \Box_* \varphi \quad \text{implies} \quad \models_\xi^{\mathcal{M}_L} \varphi$$
  for $* = 1, 2, E$.
- $A_i^{L,\Gamma} = A_i^L$, and $V_i^{L,\Gamma}(a_1, a_2; [\omega]) := V_i^L(a_1, a_2; \omega)$ for $i = 1, 2$;
- $v_L^{\Gamma i} : \{\mathbf{a}_k^i \mid k = 1, 2, \cdots, n, \cdots\} \times \Omega_L^\Gamma \to A_i$ is the mapping defined by
  $$v_L^{\Gamma i}(\mathbf{a}_k^i; [\omega]) := \begin{cases} a_1^i & \text{if } \mathrm{dom}_i(\mathbf{a}_k^i) \in \omega \cap \Gamma: \\ a_2^i & \text{otherwise;} \end{cases}$$
  $v_L^\Gamma : \bigcup_{i=1,2}\{\mathbf{a}_k^i \mid k = 1, 2, \cdots, n, \cdots\} \times \Omega_L^\Gamma \to \bigcup_{i=1,2} A_i$ is defined by the same way in Subsection (3.3);
- the truth assignment $\pi_L^\Gamma : \{\mathbf{P}_m \mid m = 0, 1, 2, \dots\} \times \Omega_L^\Gamma \longrightarrow$ { true, false} is defined by the same way in Subsection(3).[1]

Remark. We note that $v_L^{\Gamma i}(\cdot; [\omega])$ and $\pi_L^\Gamma(\cdot; [\omega])$ are independent of the choices of representatives in the Γ-equivalence of each ω, because of $(\mathrm{RE}_{\mathbf{dom}})$, $(\mathrm{RU}_{\mathbf{dom}})$ and the definition of Γ-equivalence.

Proposition 2. *The canonical model* $\mathcal{M}_L^\Gamma$ *is actually a member of the class of the models* $\mathbf{M}_L$ *for a system of utility maximizers* L *with the property that*

$$\pi_L^\Gamma(\mathbf{P}_m; [\omega]) = \pi_L(\mathbf{P}_m; \omega)$$

for each $\mathbf{P}_m \in \Gamma$.

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[1] $\pi_L^\Gamma : \{\mathbf{P}_m \mid m = 0, 1, 2, \dots\} \times \Omega_L \longrightarrow$ {true, false} is the assignment such that

(i) $\pi_L^\Gamma(\mathbf{a}_k^i = \mathbf{a}_l^i, [\omega]) = \text{true}$ if and only if $v_L^\Gamma(\mathbf{a}_k^i; [\omega]) = v_L^\Gamma(\mathbf{a}_l^i; [\omega])$;

(ii) (a) $\pi_L^\Gamma(\mathrm{dom}_1(\mathbf{a}_k^1), [\omega]) = \text{true}$ if and only if
$V_1^{L,\Gamma}(v_L^\Gamma(\mathbf{a}_k^1), b; [\omega]) \geq V_1^{L,\Gamma}(a, b; [\omega])$ for all $a \in A_1, b \in A_2$;

(b) $\pi_L^\Gamma(\mathrm{dom}_2(\mathbf{a}_l^2), [\omega]) = \text{true}$ if and only if
$V_2^{L,\Gamma}(a, v_L^\Gamma(\mathbf{a}_l^2); [\omega]) \geq V_2^{L,\Gamma}(a, b; [\omega])$ for all $a \in A_1, b \in A_2$.

4.8. By induction on the complexity of a sentence φ we can verify that

Proposition 3. *Let $\mathcal{M}_L$ be the canonical model for a system L and $\mathcal{M}_L^\Gamma$ a Γ-filtration. Then the following two properties are true:*

(i) *For every sentence φ in Γ,*

$$\models^{\mathcal{M}_L} \varphi \quad \textit{if and only if} \quad \models^{\mathcal{M}_L{}^\Gamma} \varphi .$$

(ii) *The model $\mathcal{M}_L^\Gamma$ is finite if so is Γ.*

□

4.9. **Proof of Theorems 2 and 3.** Let us consider the case $L = L^{um}$. It is noted that the canonical model $\mathcal{M}_{L^{um}}$ belongs to the class $\mathbf{M}_{L^{um}}$.

4.9.1. *Proof of Theorem 2.* If a sentence φ is valid in $\mathbf{M}_{L^{um}}$ then φ is true in $\mathcal{M}_{L^{um}}$, and hence φ is a theorem in L^{um} by the basic theorem in Section (4.6).

4.9.2. *Proof of Theorem 3.* Let $\mathbf{M}_{L^{um},FIN}$ denote the subclass of all finite models in $\mathbf{M}_{L^{um}}$. If $\vdash_L \varphi$ then $\models_{\mathbf{M}_{L^{um}}} \varphi$, and so $\models_{\mathbf{M}_{L^{um}FIN}} \varphi$ since $\mathbf{M}_{L^{um}FIN} \subseteq \mathbf{M}_L$. The converse will be shown by contrapositive argument as follows: Suppose that not $\vdash_L \varphi$, so that not $\models^{\mathcal{M}_{L^{um}}} \varphi$ by the basic theorem in in Section (4.6). Let Γ be the least set of subsentences of φ such that $\mathrm{dom}_i(\mathbf{a}_k^i)$ and $\mathrm{dom}_i(\mathbf{a}_l^i) \in \Gamma$ whenever $\mathbf{a}_k^i = \mathbf{a}_l^i \in \Gamma$. By Proposition 3 we conclude that not $\models^{\mathcal{M}_{L^{um}}^\Gamma} \varphi$ and $\mathcal{M}_{L^{um}}^\Gamma \in \mathbf{M}_{L^{um}FIN}$ on noting that Γ is a finite set, so the contradiction follows. □

5. Agreement Theorem of Dominant actions

5.1. Let L be a system of utility maximizers. A model $\mathcal{M} = \langle .., (V_i)_{i=1,2}, ..\rangle$ in $\mathbf{M}_L$is called *symmetric* if $A_1 = A_2$ and $V_1(a,b;\omega) = V_2(b,a;\omega)$ for all $a, b \in A = A_1 = A_2$.

We denote by $\mathbf{M}_L^{sym}$ the class of all symmetric models for L, and write $\models_{\mathbf{M}_L^{sym}} \varphi$ to mean that $\models_\omega^{\mathcal{M}} \varphi$ for all $\mathcal{M} \in \mathbf{M}_L^{sym}$ and for all $\omega \in \mathcal{M}$.

5.2. We will show the agreement theorem on dominant actions:

Proposition 4. *For a system of utility maximizers L we obtain that*

$$\models_{\mathbf{M}_L^{sym}} \Box_E(\mathrm{dom}_1(\mathbf{a}_k^1) \wedge \mathrm{dom}_2(\mathbf{a}_l^2)) \longrightarrow \mathbf{a}_k^1 = \mathbf{a}_l^2.$$

That is: If all players believe that each takes his dominant action then they cannot agree to disagree.

Proof. Let $\mathcal{M} \in \mathbf{M}^{sym}$. Set $d_i : 2^{\Omega} \to 2^A$ by

$$d_i(E) = \{v^i_{\mathcal{M}}(\mathbf{a}^i_k) \in A \mid E \subseteq ||\text{dom}_i(\mathbf{a}^i_k)||^{\mathcal{M}}\}.$$

We can plainly verify the three properties:

(1) $d_i(E) \subseteq d_i(F)$ if $E \supseteq F$. (by definition of d_i)
(2) $|d_i(E)| \leq 1$ if $E \neq \emptyset$, (because $V_i(\cdot, \cdot; \omega)$ is injective.)
(3) $d_1(E) = d_2(E)$, (because $\mathcal{M}$ is symmetric.)

Suppose $\models^{\mathcal{M}}_{\omega} \Box_E(\text{dom}_1(\mathbf{a}^1_k) \wedge \text{dom}_2(\mathbf{a}^2_l))$. Then we obtain

$$\models^{\mathcal{M}}_{\omega} \Box_i \text{dom}_i(\mathbf{a}^i_m) \quad \text{for } m = k, l.$$

It follows from the properties (1), (2), (3) that

$$\begin{aligned}\{v^1_{\mathcal{M}}(\mathbf{a}^1_k)\} &\underset{(1)}{=} d_1(P_1(\omega)) \underset{(1),(2)}{=} d_1(P_E(\omega)) \\ &\underset{(3)}{=} d_2(P_E(\omega)) \underset{(1),(2)}{=} d_2(P_2(\omega)) \underset{(1)}{=} \{v^2_{\mathcal{M}}(\mathbf{a}^2_l)\}.\end{aligned}$$

Thus we obtain that $v^1_{\mathcal{M}}(\mathbf{a}^1_k) = v^2_{\mathcal{M}}(\mathbf{a}^2_l)$, and so $\models^{\mathcal{M}}_{\omega} \mathbf{a}^1_k = \mathbf{a}^2_l$.

□

5.3. **Remarks.**

(i) In view of Example 2 it can be observed that a model $\mathcal{M}$ is not a model of knowledge but a model of belief because it does not satisfy the axiom:

$$\mathbf{T} \qquad B_i(F) \subseteq F.$$

(ii) There is no role of common-belief in Proposition 4.

(iii) It is not true that

$$\models_{\mathbf{M}^{sym}_L} \Box_1 \text{dom}_1(\mathbf{a}^1_k) \wedge \Box_2 \text{dom}_2(\mathbf{a}^2_l) \longrightarrow \mathbf{a}^1_k = \mathbf{a}^2_l.$$

In fact, we can plainly observe that Example 2 gives its counter example.

6. Logic of agreement of dominant actions

6.1. By this we mean the least extension of L^{um}, denoted by L^{ada}, that contains the axiom

$$\text{(ADA)} \quad \Box_E(\text{dom}_1(\mathbf{a}^1_k) \wedge \text{dom}_2(\mathbf{a}^2_l)) \longrightarrow \mathbf{a}^1_k = \mathbf{a}^2_l.$$

It immediately follows from Proposition 4 that

Theorem 4. *The logic L^{ada} is sound with respect to $\mathbf{M}^{sym}_{L^{ada}}$: i.e.,*

$$\vdash_{L^{ada}} \varphi \quad \Longrightarrow \quad \models_{\mathbf{M}^{sym}_{L^{ada}}} \varphi.$$

6.2. **Completeness for L^{ada}.** By the similar argument in Section (4.9) we can prove that:

Theorem 5. *The system L^{ada} is complete with respect to $\mathbf{M}^{sym}_{L^{ada}}$: i.e.,*

$$\vdash_{L^{ada}} \varphi \quad \Longleftarrow \quad \models_{\mathbf{M}^{sym}_{L^{ada}}} \varphi.$$

□

6.3. **Finite model property.** We can also prove that:

Theorem 6. *The system L^{ada} has finite model property; i.e.,*

$$\vdash_{L^{ada}} \varphi \quad \Longleftrightarrow \quad \models_{\mathbf{M}^{sym}_{L^{ada}FIN}} \varphi.$$

□

We will give the detail proofs in the future paper (Matsuhisa and Hirase [7]) with further discussions.

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