数理解析研究所講究録 1215 巻 2001 年 182-194

Iterated Elimination and No Trade Theorem

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Abstract

"No Trade Theorem," presented by Milgrom and Stokey (1982), implies that purely speculative trades are impossible. The basic idea is that, if the participating agents are rational enough, each of them can reason that "the trade is acceptable for all of them only if the trade is acceptable for all of them only if $\cdots$," and they will conclude that the trade is possible if and only if all of agents are indifferent between accepting and rejecting the trade. The rationality of the agents is usually represented by the assumption that the acceptability of the trade is "common knowledge" between the agents.

However, the concept of common knowledge does not exactly correspond to the iterated reasoning above. Common knowledge is knowledge without uncertainty while the agent who reasons iteratedly "the trade is acceptable only if $\cdots$" need not know whether the trade is actually acceptable or not. In this paper, we will introduce notions of the iterated reasoning and the acceptability of a trade instead of common knowledge, and show that our results contain usual No Trade Theorem.

1 Introduction

Milgrom and Stokey (1982) show the impossibility of the purely speculative trade. The main assumption used to derive their result is that it is common knowledge between the participating agents that all of them expect some gain from trading. In this paper, we will formulate the problem in another way, and generalize "No Trade Theorem" for the case in which it may not be common knowledge that all the agents prefer a given speculative trade. The basic idea is that while it is not common knowledge that the trade is accepted by all, the iterated reasoning of the type "if each agent still wants to trade even when he knows that I want to trade when I know that he wants to trade when $\cdots$" may make it common knowledge. This is illustrated by the following example, which is essentially the same as one in Milgrom and Stokey (1982).

Suppose two agents to be risk-neutral. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ be a set of the states of the world. Each agent gets private information, which has the following structure:

$$\phi_1 = \{(\omega_1,\ \omega_2),\ (\omega_3)\},$$
$$\phi_2 = \{(\omega_1),\ (\omega_2,\ \omega_3)\}.$$

For example, if ω_2 is realized, then agent 1 gets $\phi_1(\omega_2) = (\omega_1,\ \omega_2)$ and agent 2 $\phi_2(\omega_2) = (\omega_2,\ \omega_3)$ before the realized state is revealed. Assume that the probability measure is uniform and that the following bet is proposed: if $\omega = \omega_1$ agent 2 pays one dollar to agent 1, if $\omega = \omega_3$ agent 1 pays one dollar to agent 2, and if $\omega = \omega_2$ the bet is drawn. The following table illustrates this bet.

	ω_1	ω_2	ω_3
Agent1	1	0	−1
Agent2	−1	0	1

The event "the trade is profitable for agent 1" is $\{\omega_1,\ \omega_2\}$, and the event "the trade is profitable for agent 2" is $\{\omega_2,\ \omega_3\}$. Therefore, the event "the trade is profitable to both of them" is $\{\omega_2\} = \{\omega_1,\ \omega_2\} \cap \{\omega_2,\ \omega_3\}$. It is, however, not common knowledge because each agent cannot know $\{\omega_2\}$ from his private information: agent 1, who gets private information $\phi_1(\omega_2) = \{\omega_1,\ \omega_2\}$, does not know which state has realized. Since it is not common knowledge that the trade is profitable to both of them, we cannot apply "No Trade Theorem" to this bet.

As long as both of the agents are rational, however, they can reason whether the bet is profitable or not. For example, agent 1 will reason as follows: "If $\omega = \omega_1$, agent 2 will refuse the bet. Therefore if my private information is (ω_1, ω_2) and agent 2 accepts the bet, then the state is ω_2, at which I am indifferent between accepting and rejecting the bet." Agent 2 uses a similar reasoning, therefore the bet is accepted by both only at ω_2.

Thus, the rational reasoning reveals that a trade is not strictly profitable if the trade is accepted. Even for a more complicated trade, the iterated reasoning can eliminate states at which the bet will be rejected. For example, suppose the following bet.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
Agent1	1	−1	1	−1	2	3	−5
Agent2	−1	1	−1	1	−2	−3	5

Then, the sequence of eliminated states is

$$\omega_7 \quad \Rightarrow \quad \omega_5 \text{ and } \omega_6 \quad \Rightarrow \quad \omega_4 \quad \Rightarrow \quad \omega_3,$$

and ω_1 and ω_2 survive. As a result, the expected profits of the trade on (ω_1, ω_2) are 0 for both of the agents.

In the case of the following bet, all of the states are eliminated, and the trade will be rejected.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
Agent1	1	−1.1	1	−1	2	3	−5
Agent2	−1	1.1	−1	1	−2	−3	5

Therefore, we can derive the "no-trade" result from the iterated reasoning even if the assumption of common knowledge is not satisfied. In this paper, we will introduce a notion of the acceptability of a trade by using the iterated reasoning. We also show that our result is more general than the usual No Trade Theorem. Volij (2000) also formulates the iterated reasoning, but it assumes unlimited communication among agents, while our definition does not need any communication.

The paper is organized as follows. Section 2 gives some basic notions, such as common knowledge, an economy with asymmetric information, and the original description of No Trade Theorem. Section 3 gives our main result, and the acceptability of trades is formulated.

2 Common Knowledge and No Trade Theorem – Preliminary Results

In this section, we give some basic concepts, such as information functions, knowledge operators, and common knowledge. All propositions and lemmas are described without proofs. For more detail, see Fudenberg and Tirole (1991), Geanakoplos (1994), or Osborne and Rubinstein (1996).

Definition 2.1 (Information Functions) *: Let Ω be a finite set and X a random variable on $(\Omega, 2^{\Omega})$. Ω represents the set of states of the world, and X a private signal.* ***An information function*** *$\phi : \mathbf{\Omega} \to \mathbf{2^{\Omega}}$ is a map defined by*

$$\phi(\omega) := X^{-1} \circ X(\omega) = \{\rho \in \Omega \,|\, X(\rho) = X(\omega)\} \quad (\omega \in \Omega). \tag{1}$$

An information function ϕ gives a partition of Ω: both

$$\textit{either} \quad \phi(\omega) \cap \phi(\omega') = \emptyset \quad \textit{or} \quad \phi(\omega) = \phi(\omega') \quad ({}^{\forall}\omega, \omega' \in \Omega)$$

and

$$\bigcup_{\omega \in \Omega} \phi(\omega) = \Omega$$

are always satisfied.

Definition 2.2 (Knowledge Operators) *: A map* $\mathbf{K} : \mathbf{2^{\Omega}} \to \mathbf{2^{\Omega}}$ *is a* ***knowledge operator*** *defined by*

$$K(E) := \{\omega \in \Omega \,|\, \phi(\omega) \subset E\} \quad (E \in 2^{\Omega}). \tag{2}$$

An agent knows event $E \in 2^{\Omega}$ at $\omega \in \Omega$ if he knows that the true state surely lies in E, that is, if $\phi(\omega) \subset E$. Since any information function must satisfy $\omega \in \phi(\omega)$, E is always true when an agent knows E.

Proposition 2.3 (Basic Properties of Knowledge Operators) *: For each $A, B \in 2^{\Omega}$,*

1. $K(\Omega) = \Omega$
2. $K(A) \cap K(B) = K(A \cap B)$
3. $A \subset B \;\Rightarrow\; K(A) \subset K(B)$
4. $K(A) \subset A$

5. $K \circ K(A) = K(A)$

6. $K(A)^c = K(K(A)^c)$.

Definition 2.4 (Self-Evident) : *$E \in 2^{\Omega}$ is **self-evident** if $K(E) = E$.*

It is easy to see that E is self-evident if and only if

$$\omega \in E \Rightarrow \phi(\omega) \subset E \tag{3}$$

is satisfied. Moreover,

Proposition 2.5 : *An event $E \in 2^{\Omega}$ is self-evident if and only if E satisfies*

$$E = \bigcup_{\omega \in E} \phi(\omega). \tag{4}$$

Definition 2.6 : *Let $\mathcal{I} = \{1, \cdots, I\}$ be a set of agents. The event **"everyone knows E"** is a set*

$$K_{\mathcal{I}}(E) := \bigcap_{i \in \mathcal{I}} K_i(E) \tag{5}$$

where K_i is the knowledge operator of agent i, whose information function is ϕ_i. An event $E \in 2^{\Omega}$ is self-evident for every agent $i \in \mathcal{I}$ if $K_{\mathcal{I}}(E) = E$.

Proposition 2.7 : *For all $i \in \mathcal{I}$,*

$$K_{\mathcal{I}}(E) = E \iff K_i(E) = E \iff E = \bigcup_{\omega \in E} \phi_i(\omega).$$

Definition 2.8 (Common Knowledge) : *Let $\{K_{\mathcal{I}}^n(E)\}_{n \in N}$ be a decreasing sequence of events defined inductively by*

$$K_{\mathcal{I}}^1(E) := K_{\mathcal{I}}(E), \quad K_{\mathcal{I}}^n(E) := K_{\mathcal{I}} \circ K_{\mathcal{I}}^{n-1}(E) \qquad (n = 1, 2, 3, \cdots),$$

*and $K_{\mathcal{I}}^{\infty}(E) := \bigcap_{n=1}^{\infty} K_{\mathcal{I}}^n(E)$. An event $E \in 2^{\Omega}$ is **common knowledge at** $\omega \in \Omega$ if*

$$\omega \in K_{\mathcal{I}}^{\infty}(E). \tag{6}$$

Note that $K^2_{\mathcal{I}}(E) = K_{\mathcal{I}}(K_{\mathcal{I}}(E))$ means that "everyone knows that everyone knows E," and $K^{\infty}_{\mathcal{I}}(E)$ is the event that "everyone knows that everyone knows that everyone knows $\cdots$ E."

It is important that, if $E \in 2^{\Omega}$ is common knowledge at $\omega \in \Omega$, then $\omega \in E$ must hold. This means that any events which are not true cannot be common knowledge.

Lemma 2.9 : *$E \in 2^{\Omega}$ is common knowledge at $\omega \in \Omega$ if and only if there exits an event $F \in 2^{\Omega}$ which satisfies both $K_{\mathcal{I}}(F) = F$ and $\omega \in F \subset E$.*

Definition 2.10 (An Economy with Asymmetric Information) : *Consider a pure exchange economy with I agents in an uncertain environment $(\Omega, 2^{\Omega}, \boldsymbol{P})$. There are l commodities in each state of the world. Let L^l_+ (or L^l) be the set of the $\boldsymbol{R}^l_+$-valued (or $\boldsymbol{R}^l$-valued) random variables on $(\Omega, 2^{\Omega}, \boldsymbol{P})$ and we assume that the consumption set is L^l_+.*

A pure exchange economy with asymmetric information *$\mathcal{E}^{\phi}$ is described by*

$$\mathcal{E}^{\phi} = \{\ (E[U_i], e_i, \phi_i)_{i \in \mathcal{I}}\ ,\ \mathcal{Y}\ \}\ , \tag{7}$$

where

- *$\mathcal{Y} \subset (L^l)^I$ is the convex feasible set,*
- *$\mathcal{I} = \{1,\ 2,\ \cdots,\ I\}$ is the set of agents,*
- *$U_i\ :\ \boldsymbol{R}^l_+ \to \boldsymbol{R}$ is the utility function of agent $i \in \mathcal{I}$,*
- *$e_i \in L^l_+$ is the initial endowment of agent i, and*
- *$\phi_i\ :\ \Omega \to 2^{\Omega}$ is the information function of agent i.*

A net trade $Y = (y_i)_{i \in \mathcal{I}} \in (L^l)^I$ is feasible if $Y \in \mathcal{Y}$. Usually, the feasible set $\mathcal{Y}$ is specified by

$$\mathcal{Y} = \left\{ Y = (y_1, \cdots, y_I) \in (L^l)^I \,\middle|\, \sum_{i \in \mathcal{I}} y_i \leq 0 \right\}\ .$$

The expected utility function of agent i $E[U_i(\,\cdot\, + e_i)] : L^l \to \boldsymbol{R}$ *is assumed to be increasing for all i, and simply denoted by* $E[U_i](\,\cdot\,)$. *As a usual assumption, all the agents have a common prior. Then, the value of* $E[U_i](\,\cdot\,)$ *is calculated by*

$$E[U_i](y_i) = \sum_{\omega \in \Omega} U_i\left(y_i(\omega) + e_i(\omega)\right) \boldsymbol{P}(\omega) \quad (y_i \in \mathcal{Y}).$$

Theorem 2.11 (No Trade Theorem, Milgrom and Stokey (1982)) : *At a pure exchange economy with asymmetric information* $\mathcal{E}^{\phi} = \{(E[U_i], e_i, \phi_i)_{i \in \mathcal{I}},\ \mathcal{Y}\}$, *suppose that all the agents are weakly risk-averse (all the agents' utility functions are concave) and that a net trade* $\hat{Y} = (\hat{y}_i)_{i \in \mathcal{I}}$ *is Pareto-optimal ex ante.*

If it is common knowledge at ω^* *that each agent weakly prefers a feasible net trade* $Y = (y_i)_{i \in \mathcal{I}} \in \mathcal{Y}$ *to* $\hat{Y}$, *then every agent is indifferent between* Y *and* $\hat{Y}$.

Proof: *See Milgrom and Stokey (1982) or Fudenberg and Tirole (1991).*

【*Q.E.D.*】

The event "each agent weakly prefers a feasible net trade $Y = (y_i)_{i \in \mathcal{I}} \in \mathcal{Y}$ to $\hat{Y}$" is given as follows:

$$\Pi = \bigcap_{i \in \mathcal{I}} \left\{ \omega \in \Omega \;\middle|\; E[U_i \,|\, \phi_i(\omega)](y_i) \geq E[U_i \,|\, \phi_i(\omega)](\hat{y}_i) \right\}.$$

The assumption that Π is common knowledge at ω^* implies $\omega^* \in \Pi$, namely, all of the agents actually prefer Y to $\hat{Y}$ at ω^*.

3 Iterated Elimination – Main Results

Theorem 2.11 assumes that the weak profitability of the trade is common knowledge. This assumption implies that all of the agents know the trade is actually profitable for all. We show this assumption can be relaxed since, as mentioned in section 1, the rational agent can reason whether the trade is acceptable or not for him without knowing the trade is profitable for the others. To this end, we give an formal definition of the iterated reasoning process. First, we introduce the desirability, which is used to judge whether the trade is profitable or not.

Second, information functions and knowledge operators are extended on a family of sets. Finally, the rational-reasoning process is formulated as a sequence of iteratedly-refined knowledge operators.

Definition 3.1 (Desirability) : *Suppose that agent i is faced with two trades, $Y = (y_i)_{i\in\mathcal{I}}$ and $\hat{Y} = (\hat{y}_i)_{i\in\mathcal{I}}$.*

*At an event $A \in 2^{\Omega}$, Y is **desirable** for agent i relative to $\hat{Y}$ if*

$$E\,[U_i\,,\,A]\,(y_i) \geq E\,[U_i\,,\,A]\,(\hat{y}_i).$$

A set of all events which make Y desirable for agent i is denoted by $\mathcal{A}_i(y_i, \hat{y}_i)$, that is,

$$\mathcal{A}_i(y_i, \hat{y}_i) := \left\{A \in 2^{\Omega} \;\middle|\; E\,[U_i\,,\,A]\,(y_i) \geq E\,[U_i\,,\,A]\,(\hat{y}_i)\right\}. \tag{8}$$

Definition 3.2 (Knowledge Operators on a Family of Sets) : *A map $\mathcal{K}_i$ is Agent i's **knowledge operator on a family of sets** defined by*

$$\mathcal{K}_i[\mathcal{A}] := \{\omega \in \Omega \mid \phi_i(\omega) \in \mathcal{A}\} \quad ({}^{\forall}\mathcal{A} \subset 2^{\Omega}),$$

where ϕ_i is the information function of agent i.

Proposition 3.3 : *At $\omega^* \in \Omega$, agent i weakly prefers Y to $\hat{Y}$ if and only if $\omega^* \in \mathcal{K}_i[\mathcal{A}_i(y_i\,,\,\hat{y}_i)]$.*

Proof:

$$\begin{aligned}
\omega^* \in \mathcal{K}_i[\mathcal{A}_i(y_i\,,\,\hat{y}_i)] \quad &\Longleftrightarrow \quad \phi_i(\omega^*) \in \mathcal{A}_i(y_i\,,\,\hat{y}_i) \\
&\Longleftrightarrow \quad E\,[U_i\,,\,\phi_i(\omega^*)]\,(y_i) \geq E\,[U_i\,,\,\phi_i(\omega^*)]\,(\hat{y}_i).
\end{aligned}$$

[*Q.E.D.* **]**

Suppose that an event $\Sigma \in 2^{\Omega}$ is revealed to all the agents. Then, each agent can refine his private information. The refined information of agent i is defined by the cap product of his private information $\phi_i(\,\cdot\,)$ and the revealed information Σ, that is, $\phi_i(\,\cdot\,) \cap \Sigma$.

Definition 3.4 (Refinement of Knowledge Operators) *: **A knowledge operator refined by an event** $\Sigma \in 2^{\Omega}$ is the restriction of $\mathcal{K}_i$ on Σ, and denoted by $\mathcal{K}_i|_{\Sigma}$, that is,*

$$\mathcal{K}_i\Big|_{\Sigma}[\mathcal{A}] := \{\omega \in \Sigma \mid \phi_i(\omega) \cap \Sigma \in \mathcal{A}\} \qquad ({}^{\forall}\mathcal{A} \subset 2^{\Omega}).$$

Note that $\mathcal{K}_i|_{\emptyset}[\mathcal{A}]$ is empty for any $\mathcal{A} \subset 2^{\Omega}$ by the definition.

Definition 3.5 (Iterated Elimination) *: A sequence of sets $\{\Omega^n\}_{n \in N}$ is **the iterated-eliminating sequence** if Ω^n is recursively defined as follows:*

$$\begin{cases} \Omega^0 := \Omega \\ \Omega^{n+1} := \bigcap_{i \in \mathcal{I}} \mathcal{K}_i|_{\Omega^n}[\mathcal{A}_i(y_i, \hat{y}_i)] \qquad (n = 0, 1, 2, \cdots). \end{cases} \tag{9}$$

By Proposition 3.3, the event Ω^1 equals to the event "Y is desirable for all the agents relative to $\hat{Y}$." The event Ω^2 means "Y is desirable for all the agents even after their refining of their information functions by Ω^1," and so on. The states eliminated at each step of reasoning are $\Omega \backslash \Omega^1$, $\Omega^1 \backslash \Omega^2$, $\Omega^2 \backslash \Omega^3, \cdots$.

It is important that Definition 3.5 needs no communication between the participating agents. Each agent can refine hypothetically his information function, and decide whether he accepts the trade or not.

Definition 3.6 (Acceptability) *: At a state $\omega \in \Omega$, a trade $Y = (y_i)_{i \in \mathcal{I}}$ is **acceptable** for agent i relative to $\hat{Y} = (\hat{y}_i)_{i \in \mathcal{I}}$ if*

$$\omega \in \mathcal{K}_i\Big|_{\Omega^n}[\mathcal{A}_i(y_i, \hat{y}_i)] \qquad (n = 0, 1, 2, \cdots). \tag{10}$$

If (10) holds for all $i \in \mathcal{I}$, that is,

$$\omega \in \bigcap_{i \in \mathcal{I}} \mathcal{K}_i\Big|_{\Omega^n}[\mathcal{A}_i(y_i, \hat{y}_i)] \ \left(= \Omega^{n+1}\right) \qquad (n = 0, 1, 2, \cdots), \tag{11}$$

then Y is acceptable for all the agents relative to $\hat{Y}$ at ω.

The acceptability means that "all the agents don't refuse Y even though each knows that all the agents don't refuse Y even though each knows that all the agents don't refuse $Y \cdots$." By the definition, it is clear that the sequence $\{\Omega^n\}_{n\in \boldsymbol{N}}$ is decreasing. Therefore (11) equals to $\omega \in \Omega^\infty (= \bigcap_{n=0}^{\infty} \Omega^n)$.

Note that agent i needs only his private information $\phi_i(\omega^*)$ in order to see that the trade Y is acceptable at ω^*. In other words, he cannot predict that the trade Y will be really accepted even when he knows the trade is acceptable.

Theorem 3.7 (No Trade Theorem under Acceptability) : *At a pure exchange economy with asymmetric information $\mathcal{E}^\phi = \{(E[U_i], e_i, \phi_i)_{i\in\mathcal{I}}\ \ \mathcal{Y}\}$, suppose that all the agents are weakly risk-averse and that a net trade $\hat{Y} = (\hat{y}_i)_{i\in\mathcal{I}}$ is Pareto-optimal ex ante.*

If Y is acceptable for all the agents relative to $\hat{Y}$ at ω^*, then every agent is indifferent between Y and $\hat{Y}$.

Proof: *Since Y is acceptable, $\omega^* \in \Omega^n$ holds for all $n \in \boldsymbol{N}$. It is clear that $\Omega = \Omega^0 \supset \Omega^1 \supset \Omega^2 \supset \cdots\cdots \ni \omega^*$. Since Ω is finite, there is $\tilde{n} \in \boldsymbol{N}$ for which*

$$\Omega^{\tilde{n}} = \Omega^{\tilde{n}+1} = \Omega^{\tilde{n}+2} = \cdots \ni \omega^*. \tag{12}$$

Let $\Omega^* := \Omega^{\tilde{n}}$ and $\phi_i^* := \phi_i \cap \Omega^*$. Then Ω^* satisfies

$$\Omega^* = \bigcap_{i\in\mathcal{I}} \mathcal{K}_i\big|_{\Omega^*}[\mathcal{A}_i(y_i, \hat{y}_i)] = \bigcap_{i\in\mathcal{I}} \left\{\omega \in \Omega^* \;\middle|\; \phi_i^*(\omega) \in \mathcal{A}_i(y_i, \hat{y}_i)\right\}. \tag{13}$$

By Proposition 3.3,

$$E[U_i(y_i),\ \phi_i^*(\omega)] \geq E[U_i(\hat{y}_i),\ \phi_i^*(\omega)] \tag{14}$$

for all $\omega \in \Omega^*$ and $i \in \mathcal{I}$.

Suppose that the inequality in (14) is strict for agent j at $\omega^* \in \Omega^*$. Since Ω^* satisfies

$$\Omega^* = \bigcup_{\omega\in\Omega^*} \phi_i^*(\omega) \quad (i \in \mathcal{I})$$

by Proposition 2.5,

$$\begin{aligned} E\left[U_i(y_i), \Omega^*\right] &= E\left[U_i(y_i)\,,\ \bigcup_{\omega\in\Omega^*} \phi_i^*(\omega)\right] \\ &= \sum_{\phi_i^*(\omega)} E\left[U_i(y_i), \phi_i^*(\omega)\right] \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{\phi_i^*(\omega)} E\left[U_i(\hat{y}_i), \phi_i^*(\omega)\right] \\
&= E\left[U_i(\hat{y}_i), \Omega^*\right]
\end{aligned} \tag{15}
$$

holds for any i, and the inequality is strict for j.

Consider a new trade $Y^* = (y_i^*)_{i \in \mathcal{I}}$ defined by

$$
y_i^* = y_i \mathbf{1}_{\Omega^*} + \hat{y}_i \mathbf{1}_{(\Omega^*)^c} \qquad (i \in \mathcal{I}), \tag{16}
$$

where $\mathbf{1}_{\Omega^*}$ is the indicator function, that is,

$$
\mathbf{1}_{\Omega^*}(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega^* \\ 0 & \text{if } \omega \notin \Omega^*. \end{cases}
$$

Then, for any i,

$$
\begin{aligned}
E\left[U_i(y_i^*)\right] &= E\left[U_i(y_i), \Omega^*\right] + E\left[U_i(\hat{y}_i), (\Omega^*)^c\right] \\
&\geq E\left[U_i(\hat{y}_i), \Omega^*\right] + E\left[U_i(\hat{y}_i), (\Omega^*)^c\right] = E\left[U_i(\hat{y}_i)\right]
\end{aligned} \tag{17}
$$

follows from (15), and the inequality is strict for j. This contradicts to our hypothesis about the ex ante Pareto optimality of $\hat{Y}$.

[*Q.E.D.* **]**

In Theorem 3.8, we will show that acceptability is a weaker condition than common knowledge. Therefore, our result Theorem 3.7 is a generalization of the usual No Trade Theorem.

Theorem 3.8 *: If it is common knowledge at ω^* that each agent weakly prefers Y to $\hat{Y}$, then Y is acceptable for all the agents relative to $\hat{Y}$.*

Proof: *Observe first that the event " each agent weakly prefers Y to $\hat{Y}$ " is equivalent to*

$$
\Omega^1 = \bigcap_{i \in \mathcal{I}} \mathcal{K}_i \Big|_{\Omega^0} [\mathcal{A}_i(y_i\,,\,\hat{y}_i)]. \tag{18}
$$

By Proposition 3.3, $\phi_i(\rho) \in \mathcal{A}_i(y_i\,,\,\hat{y}_i)$ holds for any $\rho \in \Omega^1$ and $i \in \mathcal{I}$.

By Lemma 2.9, there is a set F which satisfies $\omega^* \in F \subset \Omega^1$ and $K_{\mathcal{I}}(F) = F$ since (18) is common knowledge at ω^*. Since F is self-evident, $\rho \in F \Rightarrow \phi_i(\rho) \subset F \subset \Omega^1$ for any $i \in \mathcal{I}$. Thus,

$$\phi_i(\rho) \cap \Omega^1 = \phi_i(\rho) \in \mathcal{A}_i(y_i\,,\,\hat{y}_i)$$

holds for any $i \in \mathcal{I}$ and $\rho \in F$. Moreover,

$$\rho \in \mathcal{K}_i\Big|_{\Omega^1}[\mathcal{A}_i(y_i\,,\,\hat{y}_i)] = \left\{\omega \in \Omega^1 \;\middle|\; \phi_i(\omega) \cap \Omega^1 \in \mathcal{A}\right\}. \tag{19}$$

By Definition 3.5,

$$\omega^* \in F \subset \Omega^2 = \bigcap_{i \in \mathcal{I}} \mathcal{K}_i\Big|_{\Omega^1}[\mathcal{A}_i(y_i\,,\,\hat{y}_i)], \tag{20}$$

which means that Ω^2 is also common knowledge at ω^*.

Thus, $\omega^* \in F \subset \Omega^n$ holds for any $n \in \boldsymbol{N}$. By definition, Y is acceptable for all the agents.

【*Q.E.D*】

4 Conclusion

In this paper, we formulate the iterated-eliminating reasoning and generalize No Trade Theorem. We have shown that the agent can judge whether the newly proposed trade is acceptable or not even when the assumption of common knowledge is not satisfied, and that the trade is accepted by all agents if and only if they are indifferent between trading and not trading.

In order to represent the rationality of the agents, the concept of common knowledge seems to be misleading and confusing. Common knowledge means exact mutual understanding. In betting or the other games with asymmetric information, however, every agent does not have any incentives to communicate and reveal his private information. In such cases, only from his private information, he will reason on his expected gain. The definition of the iterated-eliminating reasoning given in this paper is a little more complicated, but seems to be more straightforward and appropriate than common knowledge.

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