数理解析研究所講究録 1216 巻 2001 年 59-69

A GEOMETRIC APPROACH WITH APPLICATIONS TO PERIODICALLY FORCED DYNAMICAL SYSTEMS IN THE PLANE

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1. Second order equations and planar systems

We are interested in the problem of the existence of T-periodic solutions (for some $T > 0$) of the scalar nonlinear second-order ordinary differential equation

$$\ddot{x} + F(x, \dot{x}) = e(t), \tag{1}$$

where

$F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous*

and

$e : \mathbb{R} \to \mathbb{R}$ *is continuous and T-periodic.*

We recall that sometimes equation (1) can be thought like a generalized Liénard equation, having the form of

$$\ddot{x} + \phi(x, \dot{x})\dot{x} + g(x) = e(t). \tag{2}$$

Work performed under the auspices of GNAFA (project "Sistemi Dinamici, Reti Neurali e Applicazioni").
The present article is based on the talk delivered by the first author at the "RIMS Symposium on Qualitative Theory of Functional Equations and its Applications to Mathematical Science", Kyoto, November 6–10, 2000.
Gabriele Villari would like to express his gratitute to Professors H. Aikawa, J. Sugie and T. Hara for their help to his visit to Japan. He was invited to Japan by Grant-in-Aid for Scientific Research (A) No.11304008 and (B) No.12440040, Japanese Ministry of Education, Science, Sports and Culture.

Indeed, if we split the term F as

$$F(x,y) = \phi(x,y)y + F(x,0), \quad \text{with } \phi(x,y) = \frac{F(x,y) - F(x,0)}{y},$$

and $\phi(x,\cdot)$ can be continuously defined at $y = 0$, then, from (1) we obtain (2) for $g(x) = F(x,0)$.

We are interested in the study of a true non-autonomous equation and hence we assume that $e(\cdot) \not\equiv 0$. Moreover, possibly subtracting the value

$$\bar{e} := \frac{1}{2}(\sup e(t) - \inf e(t))$$

to both the sides of (1), we can assume that $e(\cdot)$ changes sign and there is a (minimal) constant $E > 0$, such that

(i_1) $|e(t)| \leq E$, *for all* $t \in [0,T]$.

With respect to $F(x,y)$, the following condition will be assumed throughout:

(i_2) *there is a constant* $d > 0$ *such that* $F(s,0) < -E$ *for all* $s \leq -d$ *and* $F(s,0) > E$, *for all* $s \geq d$.

As a consequence of assumptions (i_1) and (i_2), we have that if we write equation (1) like a system of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -F(x,y) + e(t) \end{cases} \tag{3}$$

in the phase-plane, then the trajectories of (3) which lie outside any rectangle of the form $[-d,d] \times [-r,r]$ move in the plane around the origin in the clockwise sense.

In order to perform some phase-plane analysis on system (3), it is often convenient to analyze the behavior or the trajectories of the comparison systems

$$\begin{cases} \dot{x} = y \\ \dot{y} = -F(x,y) - E \end{cases} \tag{4}$$

and

$$\begin{cases} \dot{x} = y \\ \dot{y} = -F(x,y) + E, \end{cases} \tag{5}$$

respectively.

Remark 1. An observation about the direction of the vector fields associated to the systems (3), (4) and (5) shows that the trajectories of all those systems move from the left to the right in the open upper half-plane ($y > 0$) and from the right to the left in the lower half-plane ($y < 0$). Moreover, from a comparison of the corresponding vector fields, it is possible to see that, for $y > 0$, the trajectories of (3) are "deviated" toward the left with respect to those of (4) and to the right with respect to those of (5), while the contrary happens for $y < 0$.

Remark 2. Although we have confined ourselves to the study of equation (1), we point out that all the results we are going to present are still valid for the equation

$$\ddot{x} + F(x, \dot{x}) = e(t, x, \dot{x}),$$

if $e(\dots)$ is a *bounded* function which is T-periodic in the t-variable.

2. Geometrical methods

A classical geometric approach for problem

$$(\mathcal{P}) \qquad \begin{cases} \ddot{x} + F(x, \dot{x}) = e(t) \\ x(t+T) = x(t), \quad \forall\, t \in I\!R \end{cases}$$

is based on the Brouwer fixed point theorem.

In this light, assuming the uniqueness of the solutions for the associated Cauchy problems, we can try to construct a flow-invariant region in the plane for system (3). Usually, in the applications, such positively invariant region is a compact set having as boundary a simple closed curve which, in turns, is made by the union of a finite number of trajectories of some comparison equations. If this is the case, the flow-invariant region is homeomorphic to a closed disc and therefore it possesses the fixed point property.

The existence of a T-periodic solution is thus proved by obtaining a fixed point for the associated Poincaré map $w \mapsto z(\cdot; 0, w)$, where $z(\cdot; t_0, w) = z(t) = (x(t), y(t))$ denotes the solution of (3) with $z(t_0) = w$.

A first kind of result in order to apply this approach is that of viewing system (3) like a perturbation of an autonomous equation which describes a global center in the phase-plane (see [28] and the references therein).

As pointed out in Remark 1, a comparison of the respective slopes shows that trajectories of system (3) are "guided" by those of the autonomous systems (5) and (4) in the upper ($y > 0$), respectively lower ($y < 0$), half-plane. Hence, in order to produce the desired positively invariant region, we need some appropriate geometrical behavior of the trajectories of the associated autonomous systems. Usually, at least for large orbits, the qualitative properties of the autonomous comparison systems are the same like those of system

$$(6) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -F(x, y). \end{cases}$$

With this respect, the following definition, which was stated in [3], plays a crucial rôle.

Definition. *System* (6) *has property* (B) *if there is a point* $P(x_0, y_0)$ *with* $y_0 \neq 0$, *such that the positive semi-trajectory* $\gamma^+(P)$ *passing through* P *intersects the* x-*axis, while the negative one* $\gamma^-(P)$ *does not.*

The following result was proved in the above mentioned paper:

Proposition 2.1. *System* (6) *has property* (B) *in the upper half-plane, if and only if there exists a differentiable function* $\varphi(x)$ *and some* $\bar{x} > 0$ *such that* $\varphi(x) > 0$ *for* $x < \bar{x}$, $\varphi(\bar{x}) = 0$, *and* $-F(x, \varphi(x)) \leq \varphi'(x)\varphi(x)$ *for every* $x < \bar{x}$.

Clearly, this result may be proved in a similar way in the lower half-plane.

The desired flow-invariant region may be easily constructed if both systems (4) and (5) have property (B), or if one has property (B) for one of the systems and we can prove that trajectories of the other one intersect the x -axis for x large enough.

However, we observe that, in order to apply the previous result, it is crucial to produce a suitable function $\varphi(x)$. In general, this is not easy, and, for this reason, sometimes one may use a different approach, based on a comparison method. For instance, if we can split F as $F(x, y) = \psi(x, y)y + g(x)$ (similarly like in (2)) and $\psi(x, y) \geq h(x)$ for $y > 0$, then we can have the property (B) satisfied for system (6), if we have an appropriate trajectory $y = \varphi(x)$ for the Liénard system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -h(x)y - g(x). \end{cases}$$

See [3] for a more complete discussion in this direction.

We also observe that, in order to apply this kind of approach, one needs to have available some auxiliary results guaranteeing that the solutions of an autonomous system intersect (or do not intersect) the x-axis (which is the vertical isocline for (6)). In this connection, we recall that for the Liénard systems, classical results about the intersection/non-intersection property with the vertical isocline were obtained in the fifties and sixties by Filippov and Opial. More recent ones can be found in [7], [8], [9], [27], [29] and the references therein.

3. Topological methods

Various approaches based on the use of topological degree and its applications to the periodic boundary value problems for ODEs have been developed in the past years and can be found in the literature. Here, we just sketch a few of them which can find useful applications in the study of planar (or higher dimensional systems).

Keeping the notation of the previous section, we set

$$z(t; w) := z(t; t_0, w),$$

for some $t_0 \in [0, T[$ fixed in advance (the most common choice is usually $t_0 = 0$). Even if with this position we implicitly assume the uniqueness of the solutions to the initial value problems associated to (3), we note that this assumption is not needed in the results below (to do this, use mollifiers to approximate the given vector field with some smooth vector fields which are arbitrarily close to the given one).

First, as an application of the Poincaré - Bohl theorem, we state the following:

Lemma 3.1. *Let $A \subset I\!R^2$ be an open bounded set containing the origin $0 = (0,0)$ and such that all the solutions of* (3) *with initial value in $\overline{A}$ are defined for all $t \in [t_0, t_0 + T]$. Suppose also that*

$$z(t_0 + T; w) \neq \mu w, \quad \forall\, w \in \partial A,\ \forall\, \mu > 1.$$

Then, equation (1) *has at least one T-periodic solution with $(x(t_0), \dot{x}(t_0)) \in \overline{A}$.*

In the applications, one usually is led to pass to the polar coordinates and therefore we define by $\theta(t; w)$ the angular component of a solution starting from the initial point w at the time $t = t_0$. In this case, the main assumption of this result is satisfied if $(\theta(t_0 + T; w) - \theta(t_0; w))/2\pi$ is not an integer, for all $w \in \partial A$. Various examples where this approach is applied to second order scalar equations can be found in [33].

The next result that we recall here is due to M.A. Krasnosel'skiĭ [11] and uses a different condition on the boundary. It also requires the verification of a suitable condition on the Brouwer degree for the vector field of the differential system for $t = t_0$ (which corresponds to the non-vanishing of the index [12] of the field along the boundary of A). By (i_1) and (i_2), such a condition is always satisfied (and therefore, we don't need to mention it anymore) if A contains the segment $[-d, d] \times \{0\}$.

Lemma 3.2. *Let $A \subset I\!R^2$ be an open bounded set containing the segment $[-d, d] \times \{0\}$ and such that all the solutions of* (3) *with initial value in $\overline{A}$ are defined for all $t \in [t_0, t_0 + T]$. Suppose also that*

$$z(t; w) \neq w, \quad \forall\, w \in \partial A,\ \forall\, t \in]t_0, t_0 + T[\,.$$

Then, equation (1) *has at least one T-periodic solution with $(x(t_0), \dot{x}(t_0)) \in \overline{A}$.*

Another classical result that we would like to recall here is a consequence of a continuation theorem due to Mawhin (see, e.g., [18]), which is based on a functional-analytic approach and therefore it avoids the requirement of continuability of the solutions along the interval $[t_0, t_0 + T]$. The main assumptions for this result require a suitable "transversality" condition on the boundary of A and a degree condition on the averaged vector field of (3). Fortunately, like in Lemma 3.2, we don't need to recall explicitly the hypothesis on the degree, as it is always satisfied when (i_1) and (i_2) hold and the set A contains the segment $[-d, d] \times \{0\}$.

Lemma 3.3. *Let $A \subset I\!R^2$ be an open bounded set containing the segment $[-d, d] \times \{0\}$ and suppose that there is no T-periodic solution $z(t) = (x(t), y(t))$ of the system*

$$\left\{ \begin{array}{l} x' = \lambda y \\ y' = -\lambda(F(x, y) + e(t)), \end{array} \right. \tag{7}$$

(for some $\lambda \in]0, 1[$), such that $z(t) \in \overline{A}$ for all $t \in I\!R$ and $z(\hat{t}) \in \partial A$ for some $\hat{t} \in I\!R$. Then, equation (1) *has at least one T-periodic solution with $(x(t), \dot{x}(t)) \in \overline{A}$, for all $t \in I\!R$.*

Various applications of this and related continuation theorems (even for higher order systems) can be found in [18] and [19]. Applications to some classes of plane systems related to the Liénard equation are given in [21] and [22], as well.

Although the existence of a priori bounds for the solutions is not required, the topological lemmas recalled here can find useful application in problems where the a priori bounds for the solutions are available. In recent years, other approaches have been considered, in order to deal with some situations where the possibility of a sequence of unbounded solutions cannot be avoided. Due to the lack of space, we can only briefly recall here some other directions which were pursued and quote the corresponding results in [4], [13], [14], [15] and [23].

4. A DIFFERENT APPROACH

We present now a different approach which combines the topological methods with some geometrical features of the trajectories of some associated autonomous differential system. We focus our attention on those autonomous systems which possess a separatrix which lies in the lower half-plane. Some preliminary results in this direction have been recently proposed in [30]. Therein, it is possible to find various applications to the periodic boundary value problem for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = e(t). \tag{8}$$

In a recent forthcoming article [31], dealing with equations (1) and (2), we obtain further developments in this direction. In fact, we exploit some time-mapping properties of the solutions "near" the separatrix in order to produce suitable a priori bounds and hence the existence of T-periodic solutions using a degree-theoretic method.

Our main tool is the following lemma which follows from [5, Corollary 6] (the details will be given in [31]).

Lemma 4.1. *Assume that there is a constant $R > 0$ such that*

$$F(s,0)s > 0, \quad \forall\, s \in \mathbb{R}, \text{ with } |s| \geq R \tag{9}$$

and that the a priori bounds

$$||x||_\infty < R \quad \text{and} \quad ||x'||_\infty < R, \tag{10}$$

hold for each $x(\cdot)$, which is a T-periodic solution of

$$\ddot{x} + F(x,\dot{x}) = \lambda e(t), \tag{11}$$

for some $\lambda \in]0,1[$. Then, (1) has at least one T-periodic solution.

Observe that (i_2) makes (9) always satisfied (at least for R sufficiently large, say $R \geq d$). Hence, we can concentrate ourselves on the search of the a priori bounds for the T-periodic solutions of (11).

We shall also make suitable comparison between the trajectories of system

$$(12)\qquad \begin{cases} x' = y \\ y' = -F(x,y) + \lambda e(t) \end{cases}$$

with $\lambda \in]0,1[$ and those of systems (5) and (4), respectively.

Condition (A'). *We say that system* (5) *satisfies property* (A') *if there is a separatrix* Γ *for* (5), *with* Γ *contained in the open third quadrant* $(x < 0, y < 0)$ *and such that the projection of* Γ *into the* x*-axis is an unbounded interval.*

Condition (B'). *We say that system* (5) *satisfies property* (B') *if every trajectory of* (5) *departing from the* x*-axis at a point* $(x_0, 0)$ *with* $x_0 < 0$ *and* $|x_0|$ *large enough, intersects again the* x*-axis at some point* $(x_1, 0)$ *for some positive* x_1.

Condition (B') avoids the existence of a separatrix Γ which is contained in the upper half-plane and crosses the negative x-axis. If we have a separatrix in the region $y > 0$ which has the same property like (A'), then we could manage this case as well, just arguing in a symmetric manner with respect to what will be done below, or adapting the argument from [3], previously discussed in Section 2.

The next results are auxiliary lemmas (some of them, just stated without proof) which allow to simplify the search of the a priori bounds for the solutions of (11), provided that we are able to bound only a part of the solutions. All the missing details will be given in [31].

Accordingly, from now on, and in order to avoid unnecessary repetitions, we suppose in the rest of this section that $u(\cdot)$ is a T-periodic solutions of (11), for some $\lambda \in]0,1[$.

A first consequence of (i_1) and (i_2) is the following:

Lemma 4.2. *Under* (i_1) *and* (i_2), *there is some* $\hat{t} \in [0,T]$ *such that* $|u(\hat{t})| < d$.

In order to obtain property (B'), we could take advantage of some more or less standard results which can be found in the literature or that can be adapted from know facts about second order scalar equations having a simpler form than (1). For instance, we can have:

Lemma 4.3. *Assume that*

$$(13)\qquad F(x,y) \geq F(x,0), \quad \forall x \in I\!R \text{ and } y \geq 0.$$

Then, property (B') *holds.*

Or, more generally,

Lemma 4.4. *Assume that*

$$(14)\qquad F(x,y) - F(x,0) \geq -|M(x)|y, \quad \forall x \leq 0 \text{ and } y \geq 0$$

and

$$(15)\qquad F(x,y) - F(x,0) \geq \phi(x)y, \quad \forall x \geq 0 \text{ and } y \geq 0,$$

where ϕ *satisfies assumptions on the line of* [29]. *Then, property* (B') *holds.*

Let $\xi \in I\!R$ be a given real number. Let us denote by $\underline{u}_\xi$ the minimum of the intersections of $(u(t), \dot{u}(t))$ with the line $x = \xi$. In particular, $\underline{u}_0$ will denote the minimum of the intersections of $(u(t), \dot{u}(t))$ with the y-axis.

Lemma 4.5. *Assume that system* (5) *has a separatrix* Γ *which lies in the third quadrant and crosses the negative* y*-axis. If the time along the separatrix from* $x = 0$ *to the point at infinity is larger than* T*, then there is* $R > 0$ *such that*

$$\min u(t) > -R, \quad \underline{u}_0 > -R.$$

Proof. (sketched) Consider the separatrix Γ for system (5). By the assumptions, Γ is the graph of a continuously differentiable function $y = -a(x)$, with $a : (-\infty, 0] \to]0, +\infty)$.

A comparison between the slopes of the trajectories of systems (12) with those of (5) shows that those of the former system are directed outward with respect to those of the second one. In particular, if $u(t_0) \leq -a(0) < 0$, then $\dot{u}(t) \leq -a(u(t))$, for all $t \geq t_0$, that is, if a solution of (12) crosses the negative y-axis below the separatrix, then it must stay below it for all the future time. Hence, we have that since $u(\cdot)$ is a periodic function and therefore, the associated trajectory $(u, \dot{u})$ must intersects the y-axis at some point $(0, \dot{u}(t_0))$, we see that it must be $\dot{u}(t_0) > -a(0)$.

Now, let t_1 be such that $u(t_1) = 0$ and $\dot{u}(t_1) < 0$. By the above observation, we have that $-a(0) < u(t_1) < 0$. Let $t_2 > t_1$ be the first time after t_1 when the trajectory $(u(t), \dot{u}(t))$ meets the negative x-axis and note also that $\dot{u}(t) > -a(u(t))$, for all $t \in [t_1, t_2]$. Then, dividing by $-a(u(t))$, we can write

$$1 > \frac{\dot{u}(t)}{-a(u(t))}$$

and integrating between t_1 and t_2, we have

$$\begin{aligned} T &> t_2 - t_1 > \int_{t_1}^{t_2} \frac{\dot{u}(t)}{-a(u(t))}\, dt \\ &= \int_{u(t_2)}^{u(t_1)} \frac{du}{a(u)} = \int_{u(t_2)}^{0} \frac{du}{a(u)} \\ &= \int_{-K}^{0} \frac{du}{a(u)}, \quad \text{where we have set } u(t_2) := -K. \end{aligned}$$

The last integral turns out to be the time Δt along the separatrix Γ for the orbit path between $(0, -a(0))$ and $(-K, -a(-K))$.

Now, by the assumption on the time along the separatrix, we know there is $R > 0$, such that the time along the separatrix from $x = 0$ to $x = -R$ is larger than T and from this, we we can easily conclude that $\min u(t) = u(t_2) > -R$. □

We remark that the choice of the y-axis here is merely conventional. The following variant is obviously true:

Lemma 4.6. *Assume that system* (5) *has a separatrix* Γ *which lies in the third quadrant and crosses the line* $x = \xi$ *at a negative point. If the time along the separatrix from* $x = \xi$ *to the point at infinity is larger than* T, *then there is* $R > 0$ *such that*

$$\min u(t) > -R, \quad \underline{u}_\xi > -R.$$

At this point, we have found conditions in order to bound the minimum of $u(t)$, the maximum of $\dot{u}(t)$ and the maximum of $u(t)$. It remains to prove a bound for $\min \dot{u}(t)$, provided that $|u(t)|$ and $\max \dot{u}(t)$ are (a priori) bounded.

Here, for instance, we could invoke some already known assumptions which, for a second order equation, provide a bound for $|\dot{x}|$, whenever a bound for $|x|$ is given. It was M. Nagumo in 1942 [20] who obtained a classical result in this direction. It was proved in [20] (see also [10]), that if $|F(x,y)|$ (when $|x|$ bounded) grows less than $\omega(|y|)$, with $\int^{+\infty} \frac{s}{\omega(s)}\, ds = +\infty$, then the existence of a uniform bound for x implies the existence of a uniform bound for $\dot{x}$. Conditions of this kind are named as *Bernstein-Nagumo conditions*, with reference to the pioneering work of S. Bernstein [1], [2], as well.

A special, but interesting case in which the Nagumo condition is satisfied, is given by the Liénard equations of the type (8). Indeed, here the nonlinear function $F(x,y) = f(x)y + g(x)$ has linear growth in y and therefore, if we bound x, we obtain a bound for $\dot{x}$ as well.

In [16] J. Mawhin, extended this concept, to a wider class of equations, by introducing the definition of a *Nagumo equation* (with respect to the periodic boundary value problem), as that of a second order equation where if we have a priori bounds for the T-periodic solutions in the $\|\cdot\|_\infty$-norm then we can find bounds for their derivatives. The fact that we can restrict (according to Mawhin [16]) our consideration only to the T-periodic solutions, allows us to extend the class of equations for which this argument can be applied. In particular, for some equations, like the Rayleigh equation

$$\ddot{x} + f(\dot{x}) + g(x) = e(t),$$

this generalized form of the Nagumo condition is always valid, without any need of a growth condition in $f(y)$. For a general discussion about this topic, also with respect to different boundary conditions, see also [17].

Here, modifying the definition in [16], we could introduce the following

Definition. We say that (1) is a *generalized Nagumo equation with respect to the T-periodic problem*, if for any $r > 0$, there is $\eta(r) > 0$ such that if $u(\cdot)$ is any T-periodic solution of (11), for some $\lambda \in]0,1[$, such that

$$|u(t)| \le r \quad \text{and} \quad \dot{u}(t) \le r, \quad \forall\, t \in [0,T],$$

then

$$u(t) \ge -\eta(r), \quad \forall\, t \in [0,T].$$

Now, we are in position to give our main result for equation (1).

Theorem 4.1. *Suppose that* (1) *is a generalized Nagumo equation with respect to the T-periodic problem with $e(t)$ and $F(x,y)$ satisfying* (i_1) *and* (i_2). *Assume, moreover, that* (5) *satisfies conditions* (A') *and* (B').

If the time along the separatrix from a point $P \in \Gamma$ to infinity is larger than T, then (1) *has at least one T-periodic solution.*

If we apply Theorem 4.1 to the Liénard equation (8), we can re-estabilish various results recently obtained in [30]. On the other hand, our theorem also allows a wide range of applications to some generalized Liénard equations in the form of (2), as well as to other significant classes of second order ODEs. It will be possible to find the corresponding results and applications in the forthcoming paper [31].

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