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Periodic Solutions of Linear Differential Equations

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Abstract

We deal with autonomous linear differential equations of the form $dx/dt = Ax(t)+f(t)$ in a Banach space $\mathbb{X}$, where A is the generator of a C_0-semigroup $U(t)$ on $\mathbb{X}$. In this paper we give fixed point theorems on an affine linear map, which are closely related to mean ergodic theorems. As applications, criteria on the existence of periodic solutions of the equation are obtained.

1 Introduction

Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$. We consider criteria for the existence of periodic solutions of inhomogeneous linear differential equations of the form

$$\frac{dx}{dt}=Ax+f(t), \quad t\in\mathbb{R}, \quad x\in\mathbb{X}, \tag{1}$$

and the corresponding homogeneous linear differential equation

$$\frac{dx}{dt}=Ax. \tag{2}$$

In the present paper it is assumed that $A:\mathcal{D}(A)\subset\mathbb{X}\to\mathbb{X}$ is the generator of a C_0-semigroup $U(t)$ on $\mathbb{X}$, and $f:\mathbb{R}\to\mathbb{X}$ is a nontrivial τ-periodic continuous function on $\mathbb{R}$.

In 1974, Chow-Hale [2] obtained the following fixed point theorem on an affine linear map, from which the present paper is motivated.

Theorem 1.1 *Let T be a bounded linear operator on $\mathbb{X}$ and $b(\neq 0)\in\mathbb{X}$ be fixed. Put $Vx=Tx+b, x\in\mathbb{X}$. Assume that the range $\mathcal{R}(I-T)$, I being the identity, is a closed and that there is an $x_0\in\mathbb{X}$ such that*

$$\{x_0, Vx_0, \cdots, V^n x_0, \cdots\} \tag{3}$$

is bounded. Then V has a fixed point in $\mathbb{X}$; that is, the equation

$$(I-T)x=b \tag{4}$$

has a solution.

In the present paper we will consider the solvability (the existence of fixed points on the affine linear map V) for the equation (4) in connection with Theorem 1.1 and mean ergodic theorems. As applications, criteria on the existence of periodic solutions to the equation (1) are obtained.

Using Theorem 1.1, the Massera theorem [13] on the existence of periodic solutions in finite dimensional spaces is generalized to the case of infinite dimensional spaces as follows.

Theorem 1.2 *Assume that $\mathcal{R}(I - U(\tau))$ is closed. Then the equation* (1) *has τ-periodic solutions if and only if it has a bounded solution on* $[0, \infty)$.

More recently, the Massera type theorems for various equations in Banach spaces have been investigated by many authors : for example, Shin-Naito [17], Shin-Naito-Minh [18], Hino-Murakami-Yoshizawa [8], Li-Lim-Li [10] and Li-Cong-Lin-Liu [11]. However, in general, it is not easy to show the existence of bounded solutions. Other directions for criteria for the existence of periodic solutions have been considered by Hatvani-Kristin [6], Hino-Naito-Minh-Shin [7],Naito-Minh [14] and Goldstein [5] in finite or infinite dimensional spaces. The problem to guarantee the existence of periodic solutions is to solve the equation (4). The problem to solve the equation (4) have been investigated in connection with mean ergodic theorems by many authors ; Dotson [3], Lin-Sine [12], Shaw [16] and others.

In this paper we will consider the problem to solve the equation (4) in connection with Theorem 1.1 and mean ergodic theorems in Section 2, and as applications, give criteria for the existence of periodic mild solutions to the equation (1) in Section 3.

2 Fixed point theorems

In this section we consider the solvability of the equation

$$(I - T)x = b, \tag{5}$$

where $T : \mathbb{X} \to \mathbb{X}$ is a bounded linear operator and $b(\neq 0) \in \mathbb{X}$ is fixed. Put $Vx = Tx + b$. The averages $A_n(T)$ of the operator T are defined as follows :

$$A_n(T) = \frac{1}{n} S_n(T), \quad S_n(T) = \sum_{k=0}^{n-1} T^k.$$

2.1 The case where T is weakly ergodic.

First, we consider the case where a bounded linear operator T is weakly ergodic. Denote by $w - \lim_{n\to\infty} x_n$ the weak convergence of a sequence $\{x_n\} \subset \mathbb{X}$. A bounded linear operator T is called weakly ergodic if $w - \lim_{n\to\infty} A_n(T)x$ exists for all $x \in \mathbb{X}$. Denote the dual space of $\mathbb{X}$ by $\mathbb{X}^*$ and set $N^* = \{z^* : (I - T)^* z^* = 0\} \subset \mathbb{X}^*$ for the equation (5).

Lemma 2.1 $< x^*, b > = < x^*, A_n(T)b >$ *for all* $n \in \mathbb{N}$ *and* $x^* \in N^*$.

Proof For every $x^* \in N^*$ the relation $x^* = T^*x^*$ holds, from which it follows that

$$\begin{aligned} < x^*, b > &= < T^*x^*, b > = < x^*, Tb > \\ &= < x^*, T^2 b > \\ &\cdots \\ &= < x^*, T^k b >, \quad k = 1, 2, \cdots . \end{aligned}$$

This relation implies that

$$\begin{aligned} < x^*, A_n(T)b > &= < x^*, \frac{1}{n}\sum_{k=0}^{n-1} T^k b > \\ &= \frac{1}{n}\sum_{k=0}^{n-1} < x^*, b > \\ &= < x^*, b > . \end{aligned}$$

Therefore the proof of the lemma is completed.

Denote by $S(b)$ the closed linear space generated from $\{b, Tb, T^2b, \cdots\}$. Lemma 2.1 implies that if $b \in \mathcal{R}(I-T)$ and $\mathcal{R}(I-T)$ is closed, then $S(b) \subset \mathcal{R}(I-T)$. The following two theorems are the main results on the existence of fixed points in this section.

Theorem 2.1 *Assume that $\mathcal{R}(I-T)$ is closed. Then the following statements are equivalent.*

1) *The equation* (5) *has a solution.*

2) $< x^*, b > = 0$ *for all* $x^* \in N^*$.

3)

$$< x^*, A_n(T)b > = 0 \quad \textit{for all } n \in \mathbb{N} \textit{ and } x^* \in N^*. \tag{6}$$

4) *There exists an $x_0 \in \mathbb{X}$ such that the set*

$$\{x_0, Vx_0, \cdots, V^n x_0, \cdots\}$$

is bounded.

Proof It is well known that the equivalence between the assetions 1) and 2) are the Fredholm alternative theorem. The equivalence between the assertion 1) and the assertion 4) follows from Theorem 1.1. The remainder is deduced from the above lemma 2.1. Therefore the proof is completed.

Theorem 2.2 *Assume that there exists an $\alpha > 0$ such that $\|T^n\| \le \alpha < \infty$ for all $n \in \mathbb{N}$ in the equation* (5). *Then the statements are equivalent each other.*

1) *There exists an $x_0 \in \mathbb{X}$ such that the set*

$$\{x_0, Vx_0, \cdots, V^n x_0, \cdots\}$$

is bounded.

2)

$$\limsup_{n\to\infty} \|S_n(T)b\| < \infty.$$

Proof Assume that the assertion 1) holds. Since

$$V^n x_0 = T^n x_0 + S_n(T)b,$$

we have

$$\|nA_n(T)b\| \leq \|T^n x_0 + S_n(T)b\| + \|T^n x_0\| \leq \|V^n x_0\| + \alpha\|x_0\| < \infty.$$

This implies that $\limsup_{n\to\infty} \|S_n(T)b\| < \infty$.

Conversely, assume that for any $x \in \mathbb{X}$, $\limsup_{n\to\infty} \|V^n x\| = \infty$. Note that

$$\|V^n x\| \leq \|T^n x\| + \|S_n(T)b\| \leq \alpha\|x\| + \|S_n(T)b\|.$$

Taking the upper limit to both sides of the above inequality, it follows from the assertion 2) and the assumption that

$$\infty = \limsup_{n\to\infty} \|V^n x\| \leq \alpha\|x\| + \limsup_{n\to\infty} \|S_n(T)b\| < \infty.$$

This is a contradiction. Therefore the proof is complete.

Remark 2.2 *A bounded linear operator $F : \mathbb{X} \to \mathbb{X}$ is called a semi-Fredholm operator if $\mathcal{R}(F)$ is closed and if $\mathcal{N}(F)$ is of finite dimension. If T is a compact linear operator or a semi-Fredholm operator, the closedness of the range $\mathcal{R}(T - I)$ in Theorem 2.1 is satisfied.*

The following corollaries are immediate results of Theorem 2.1, but we will give other proofs.

Corollary 2.3 *Assume that $\mathcal{R}(I - T)$ is closed. If*

$$w\text{-}\lim_{n\to\infty} A_n(T)b = 0, \tag{7}$$

then the equation (5) *has a solution.*

Proof It is sufficient to see that $b \in \mathcal{R}(I-T)$. Now, assume that $b \notin \mathcal{R}(I-T)$. Since $\mathcal{R}(I - T)$ is a closed linear subspace of $\mathbb{X}$, it follows from the Hahn-Banach theorem that there is a $x^* \in \mathbb{X}^*$ such that

$$< x^*, b > \neq 0, \quad < x^*, (I - T)y >= 0 \quad (\forall y \in \mathbb{X}).$$

The second condition implies that

$$< x^*, (I - T)y >=< (I - T)^* x^*, y >= 0 \quad (\forall y \in \mathbb{X});$$

and so, $(I - T)^* x^* = 0$. Namely, $x^* \in N^*$. ¿From Lemma 2.1 and the first condition, we have

$$< x^*, A_n(T)b >=< x^*, b > \neq 0.$$

Combining this relation with the assumption, we can obtain

$$0 = \lim_{n\to\infty} < x^*, A_n(T)b >=< x^*, b > \neq 0,$$

which yields a contradiction. Therefore, $b \in \mathcal{R}(I - T)$; it means that the equation (5) has a solution.

The following result is slightly different from Theorem 2.1.

Corollary 2.4 *Assume that* $w-\lim_{n\to\infty} A_n(T)x$ *exists for every* $x \in \mathbb{X}$ *or* $w-\lim_{n\to\infty} \frac{T^n}{n}x = 0$ *for every* $x \in \mathbb{X}$. *If*

$$w-\lim_{n\to\infty} A_n(T)b \neq 0, \tag{8}$$

then the equation (5) *has no solutions.*

Proof Let now the equation (5) have a solution x_0. Then

$$x_0 = Tx_0 + b.$$

Multiplying the operator $A_n(T)$ to both sides, we have

$$(I-T)A_n(T)x_0 = A_n(T)b.$$

Namely,

$$\frac{I-T^n}{n}x_0 = A_n(T)b.$$

If $w-\lim_{n\to\infty} \frac{T^n}{n}x = 0$ for every $x \in \mathbb{X}$, then $w-\lim_{n\to\infty} A_n(T)b = 0$. This yields a contradiction.

Next, we assume that $w-\lim_{n\to\infty} A_n(T)x$ exists for every $x \in \mathbb{X}$. Since

$$\frac{T^n}{n}x_0 = \frac{n+1}{n}A_{n+1}(T)x_0 - A_n(T)x_0,$$

it follows that

$$\frac{x_0}{n} - \frac{n+1}{n}A_{n+1}(T)x_0 + A_n(T)x_0 = A_n(T)b. \tag{9}$$

Since $w - \lim_{n\to\infty} A_n(T)x_0 := z$ exists, $\lim_{n\to\infty} < x^*, A_n(T)x_0 > = < x^*, z >$ for all $x^* \in \mathbb{X}^*$. Hence, taking the weak limit to the right hand side of the relation (9), we have

$$\begin{aligned}
&\lim_{n\to\infty}\{\frac{1}{n} < x^*, x_0 > - < x^*, \frac{n+1}{n}A_{n+1}(T)x_0 - A_n(T)x_0 >\} \\
&= -\lim_{n\to\infty}\frac{n+1}{n} < x^*, A_{n+1}(T)x_0 > + \lim_{n\to\infty} < x^*, A_n(T)x_0 > \\
&= - < x^*, z > + < x^*, z > \\
&= < x^*, 0 > = 0.
\end{aligned}$$

Therefore, the weak limit to the left hand side of the relation (9) becomes

$$\lim_{n\to\infty} < x^*, A_n(T)b > = 0 \text{ for all } x^* \in \mathbb{X}^*.$$

This is a contradiction, which completes the proof.

The following remark is concerned with the uniqueness of solution of the equation (5).

Remark 2.5 *Assume that $\mathcal{R}(I-T)$ is closed. Then the equation* (5) *has a unique solution if $w-\lim_{n\to\infty} A_n(T)b = 0$ and $\mathcal{N}(I-T) = \{0\}$.*

Sufficient conditions to guarantee the existence of the limit $w-\lim_{n\to\infty} A_n(T)x$ for all x are given in the following lemmas, cf. [4, pp. 595-597, pp. 660-662].

Lemma 2.6 1) *If $w-\lim_{n\to\infty} A_n(T)x$ exists for every $x \in \mathbb{X}$, then*

$$\mathbb{X} = \overline{\mathcal{R}(I-T)} \oplus \mathcal{N}(I-T). \tag{10}$$

2) *If $w-\lim_{n\to\infty} T^n x/n = 0$ for every $x \in X$, and $\{A_n(T)x\}$ is weakly sequentially compact, then $w-\lim_{n\to\infty} A_n(T)x$ exists for every $x \in \mathbb{X}$.*

3) *If the sequence $\{A_n(T)\}$ is bounded, then $\{A_n(T)x\}$ converges for every $x \in \mathbb{X}$ if and only if $T^n x/n$ converges to zero for every $x \in \mathbb{X}$ and $\{A_n(T)x\}$ is weakly sequentially compact for every $x \in \mathbb{X}$.*

4) *Let $\mathbb{X}$ be a reflexive Banach space. Then the sequence $\{A_n(T)x\}$ converges for every $x \in \mathbb{X}$ if and only if for every $x \in \mathbb{X}$ it is bounded and $\lim_{n\to\infty} T^n x/n = 0$.*

Using the avobe lemma, we can obtain sufficient conditions to satisfy the conditions in Corollary 2.3 and Corollary 2.4. Throughout the paper we will use some notations below. If T is a linear operator on $\mathbb{X}$, then $\mathcal{R}(T)$ and $\mathcal{N}(T)$ stand for its range and its null space, respectively. As usual, $\sigma(T), \rho(T), \sigma_p(T)$, and $\sigma_c(T)$ are the notations of the spectrum, the resolvent set, the point spectrum and the continuous spectrum of the operator T, respectively.

We will state results on the uniqueness of solutions to the equation (5).

Theorem 2.3 *Assume that that $\mathcal{R}(I-T)$ is closed and $w-\lim_{n\to\infty} A_n(T)x$ exists for all $x \in \mathbb{X}$. Then the following statements are equivalent.*

1) $1 \in \rho(T)$.

2) $w-\lim_{n\to\infty} A_n(T)x = 0$ for all $x \in \mathbb{X}$.

3) *For each $b \in \mathbb{X}$, the equation* (5) *has a unique solution.*

4) *For each $b \in \mathcal{R}(I-T)$, the equation* (5) *has a unique solution.*

Proof First, we will show the equivalence between the assertion 2) and the assertion 3). If the assertiion 2) is satisfied, then the existence of solutions of the equation (5) is ensured by Corollary 2.3. To prove the uniqueness of solutions, one assume that there exist two solutions $x_0, y_0, z_0 := x_0 - y_0$ to the equation (5). Then $Tz_0 = z_0$, from which it follows that $A_n(T)z_0 = z_0$. Hence for any $x^* \in \mathbb{X}^*$,

$$< x^*, A_n(T)z_0 >=< x^*, z_0 > .$$

With the assumption 2), we have $< x^*, z_0 >= 0$, which means that $z_0 = 0$; that is, $x_0 = y_0$. The converse follows from Corollary 2.4.

Next, we will derive the assertion 3) from the assertion 4). If the assertion 4) is satisfied, then $\mathcal{N}(I-T) = \{0\}$. Hence $\mathbb{X} = \mathcal{R}(I-T)$. Indeed, by the decomposition (10) in Lemma 2.6 and the closedness of $\mathcal{R}(I-T)$, we see that $\mathbb{X} = \mathcal{R}(I-T)$. We note that for each $b \in \mathbb{X}$, the equation (5) has a unique solution if and only if $1 \in \rho(T)$. Therefore the remainder is obvious.

Proposition 2.7 *Assume that* $w - \lim_{n\to\infty} A_n(T)x$ *exists for all* $x \in \mathbb{X}$ *and that* $b \in \mathcal{R}(I-T)$ *and* $\mathcal{R}(I-T) \subsetneq \mathbb{X}$. *Then the equation* (5) *has a unique solution if and only if* $1 \in \sigma_c(T)$.

Proof If the equation (5) has a unique solution, then $\mathcal{N}(I-T) = \{0\}$. By Lemma 2.6 we see that $\mathbb{X} = \overline{\mathcal{R}(I-T)}$. Since $\mathcal{R}(I-T) \subsetneq \mathbb{X}$, we see that $1 \in \sigma_c(T)$. The converse is clear.

2.2 The special cases.

Next, we consider the case where a bounded linear operator T is strongly ergodic ; that is, $\lim_{n\to\infty} A_n(T)x$ exists for all $x \in \mathbb{X}$. To do so, the following lemma is needed, cf. [4, p. 662].

Lemma 2.8 *When the limit* $Px := \lim_{n\to\infty} A_n(T)x$ *exists for all* $x \in \mathbb{X}$ *it is a projection of* $\mathbb{X}$ *upon the linear space* $\mathcal{N}(I-T)$ *and the complementary projection has* $\overline{\mathcal{R}(I-T)}$.

Proposition 2.9 *Assume that* $\mathcal{R}(I-T)$ *is closed and* $Px := \lim_{n\to\infty} A_n(T)x$ *exists for all* $x \in \mathbb{X}$. *Then the following statements hold true.*

1) *The following assertions are equivalent.*
   (1) $Pb = 0$.
   (2) *The equation* (5) *has a solution.*
2) *The following assertions are equivalent.*
   (1) $1 \in \rho(T)$.
   (2) $Px = 0$ *for all* $x \in \mathbb{X}$.
   (3) *For each* $b \in \mathcal{R}(I-T)$, *the equation* (5) *has a unique solution.*
3) *The following assertions are equivalent.*
   (1) $Pb \neq 0$.
   (2) *The equation* (5) *has no solutions.*

Proof The proof follows easily from the properties of the projection P, Lemma 2.8 and Theorem 2.3.

Finally, we consider the case where a bounded linear operator T is uniformly ergodic. A bounded linear operator T is called uniformly ergodic if $A_n(T)$ converges uniformly (or in the uniform operator topology). It has the following properties(see Theorem 2.1 and Theorem 2.7 in [9]).

Lemma 2.10 1) *Assume that* $\|T^n/n\| \to 0$ *as* $n \to \infty$. *Then* T *is uniformly ergodic if and only if* $\mathcal{R}(I-T)$ *is closed.*

2) *Assume that there exists an* $\alpha > 0$ *such that* $\|T^n\| \leq \alpha < \infty (n = 1, 2.\cdots)$. *Then* T *is uniformly ergodic if and only if either* $1 \in \rho(T)$ *or* 1 *is a pole of first order of the resolvent* $R(\lambda, T)$.

Using the above result, Corollaries 2.3 and 2.4 we can obtain the following result.

Proposition 2.11 *Assume that there exists an* $\alpha > 0$ *such that* $\|T^n\| \le \alpha < \infty (n = 1, 2. \cdots)$ *and that* T *is uniformly ergodic. Then the equation* (5) *has a solution if and only if the relation*

$$\lim_{n \to \infty} A_n(T)b = 0$$

holds.

3 Periodic solutions of linear equations

In this section we consider criteria on the existence of periodic solutions of the inhomogeneous linear differential equation (1) by using the fixed point theorems in the previous sections. Put $P_\tau(\mathbb{X}) = \{f : \mathbb{R} \to \mathbb{X} \text{ is a } \tau - \text{periodic continuous function}\}$. Note that $f \equiv 0 \in P_\tau(\mathbb{X})$. In this section we assume that $f \not\equiv 0 \in P_\tau(\mathbb{X})$ and A is the generator of a C_0-semigroup $U(t)$ on the Banach space $\mathbb{X}$. It is called to be a mild solution (in shot, solution) of the equation (1) if it is a continuous solution satisfying the equation

$$x(t) = U(t)x(0) + \int_0^t U(t-s)f(s)ds, \quad t \ge 0. \tag{11}$$

An operator $V : \mathbb{X} \to \mathbb{X}$ is defined as follows :

$$Vz = U(\tau)z + \int_0^\tau U(\tau - s)f(s)ds.$$

The following lemma is obvious.

Lemma 3.1 *The following statements are equivalent.*
1) *The equation* (1) *has a* τ*-periodic solution.*
2) V *has a fixed point* ; *that is, the following equation*

$$(I - U(\tau))z = \int_0^\tau U(\tau - s)f(s)ds \tag{12}$$

has a solution.
3)

$$\int_0^\tau U(\tau - s)f(s)ds \in \mathcal{R}(I - U(\tau)) \tag{13}$$

holds.

First, we consider the case where

$$b := \int_0^\tau U(\tau - s)f(s)ds \neq 0. \tag{14}$$

The following result is the main theorem for criteria on the existence of periodic solutions of the equation (1).

Theorem 3.1 *Assume that* $\int_0^\tau U(\tau - s)f(s)ds \neq 0$. *Then the following statements hold true.*

1) *Assume that* $\mathcal{R}(I - U(\tau))$ *is closed. If*

$$w\text{-}\lim_{n\to\infty} A_n(U(\tau)) \int_0^\tau U(\tau - s)f(s)ds = 0,$$

then the equation (1) *has a* τ*-periodic solution.*

2) *Assume that* $w\text{-}\lim_{n\to\infty} A_n(U(\tau))x$ *exists for every* $x \in \mathbb{X}$, *or* $w\text{-}\lim_{n\to\infty} \frac{T^n}{n} x = 0$ *for every* $x \in \mathbb{X}$. *If*

$$w\text{-}\lim_{n\to\infty} A_n(U(\tau)) \int_0^\tau U(\tau - s)f(s)ds \neq 0,$$

then the equation (1) *has no* τ*-periodic solutions.*

3) *Assume that* $\mathcal{R}(I - T)$ *is closed and that*

$$w - \lim_{n\to\infty} A_n(U(\tau))x = 0 \quad \text{for all } x \in \mathbb{X}.$$

Then the equation (1) *has a unique* τ*-periodic solution.*

Proof Let b be as in (14). Then the equation (12) becomes $(I - U(\tau))x = b$. Thus, the proof easily follows from the assumption, Corollaries 2.3, 2.4, Theorem 2.3 and Lemma 3.1.

Theorem 3.2 *Assume that there exists an* $M > 0$ *such that* $\|U(t)\| \leq M < \infty, t \geq 0$, *and that* $\mathcal{R}(I - U(\tau))$ *is closed. Then*

$$\limsup_{n\to\infty} \|S_n(U(\tau)) \int_0^\tau U(\tau - s)f(s)ds\| < \infty,$$

if and only if the equation (1) *has a* τ*-periodic solution.*

The proof follows from Theorem 2.1 and Theorem 2.2.

Proposition 3.2 *Assume that* $w - \lim_{n\to\infty} A_n(U(\tau))x$ *exists for all* $x \in \mathbb{X}$ *and that* $\int_0^\tau U(\tau - s)f(s)ds \in \mathcal{R}(I - U(\tau))$ *and* $\mathcal{R}(I - U(\tau)) \subsetneq \mathbb{X}$. *Then the equation* (1) *has a unique* τ*-solution if and only if* $1 \in \sigma_c(U(\tau))$.

The proof follows from Proposition 2.7.

Proposition 3.3 *Let* b *be as in* (14) *and* $Px := \lim_{n\to\infty} A_n(U(\tau))x$ *exists for all* $x \in \mathbb{X}$. *If* $\mathcal{R}(I - U(\tau))$ *is closed, then the following statements hold true.*

1) *The following assertions are equivalent.*
   (1) $Pb = 0$.
   (2) *The equation* (1) *has a* τ*-periodic solution.*
2) *The following assertions are equivalent.*
   (1) $1 \in \rho(U(\tau))$.
   (2) $Px = 0$ *for all* $x \in \mathbb{X}$.
3) *If* $Px = 0$ *for all* $x \in \mathbb{X}$, *then the equation* (1) *has a unique* τ*-periodic solution.*
4) *The following assertions are equivalent.*
   (1) $Pb \neq 0$.
   (2) *The equation* (1) *has no* τ*-periodic solutions.*

Proof The proof follows from Proposition 2.9 and Theorem 3.1.

Next, we consider the case where $b := \int_0^\tau U(\tau - s)f(s)ds = 0$. To do so, we present criteria of the existence of periodic solutions to the homogeneous linear equations (2) by using spectral mapping theorems [15].

Lemma 3.4 *The following statements are equivalent.*
1) *The equation* (2) *has nontrivial* τ*-periodic solutions.*
2) $\sigma_p(A) \cap \frac{i2\pi\mathbb{Z}}{\tau} \neq \emptyset$.
3) $1 \in \sigma_p(U(\tau))$.

Proof The solution $x(t)$ of the equation (2) is expressed as $x(t) = U(t)x(0)$.
Now, if $x(t)$ is a nontrivial τ-periodic solution ; that is, $x(t+\tau) = x(t)$, then $x(\tau) = x(0), x(0) \neq 0$. Hence $(I - U(\tau))x(0) = 0$, and hence, $1 \in \sigma_p((U(\tau))$.
Conversely, if $1 \in \sigma_p(U(\tau))$, then there is an $a(a \neq 0) \in \mathbb{X}$ such that $a = U(\tau)a$. Hence the solution $x(t)$ of the equation (2) through $(0, a)$ is expressed as $x(t) = U(t)a$. This is a τ-periodic solution. Indeed,

$$x(t+\tau) = U(t+\tau)a = U(t)U(\tau)a = U(t)a = x(t).$$

Next, by the spectral mapping theorem [15] ; that is,

$$e^{t\sigma_p(A)} \subset \sigma_p(U(t)) \subset e^{t\sigma_p(A)} \cup \{0\}, \quad t \geq 0,$$

we have

$$1 \in \sigma_p(U(\tau)) \Leftrightarrow 1 \in e^{\tau\sigma_p(A)} \Leftrightarrow \sigma_p(A) \cap \frac{i2\pi\mathbb{Z}}{\tau} \neq \emptyset.$$

Therefore the proof is completed.

Notice that all τ-periodic solutions of the equation (2) are given as the form $x(t) = U(t)a,\ a \in \mathcal{N}(I - U(\tau))$.

Theorem 3.3 *Assume that* $\int_0^\tau U(\tau - s)f(s)ds = 0$. *Then the statements hold true.*
1) *The equation* (1) *has a* τ*-periodic solution.*
2) *If* $1 \in \sigma_p(U(\tau))$ *or* $\sigma_p(A) \cap \frac{i2\pi Z}{\tau} \neq \emptyset$, *then the* τ*-periodic solution of the equation* (1) *is given as*

$$x(t) = U(t)a + \int_0^t U(t-s)f(s)ds,$$

where $a \in \mathcal{N}(I - U(\tau))$.
3) *If* $1 \in \mathbb{R} \setminus \sigma_p(U(\tau))$, *then there exists a unique* τ*-periodic solution of the equation* (1) *and it is given as*

$$x(t) = \int_0^t U(t-s)f(s)ds.$$

Proof Since $b = 0$ in (14), the eqaution (12) becomes

$$(I - U(\tau))x = 0.$$

1) Since $\mathcal{R}(I - U(\tau))$ is a linear space, $b = 0 \in \mathcal{R}(I - U(\tau))$. Thus it follows from Lemma 3.1 that the equation (1) has a τ-periodic solution.

2) The proof follows easily from Lemma 3.4. In fact, if $a \in \mathcal{N}(I - U(\tau))$, it follows from (11) that

$$x(\tau) = U(\tau)a + \int_0^\tau U(\tau - s)f(s)ds = U(\tau)a = a.$$

3) Since $1 \in \mathbb{R} \setminus \sigma_p(U(\tau))$, we see that the inverse $(I - U(\tau))^{-1}$ exists. So, the proof is obvious.

Finally, we state an immediate corollary of the results before.

Corollary 3.5 *Assume that* $\|U(t)\| \leq Me^{-\alpha t}, (M, \alpha > 0), t \geq 0$, *and* $\int_0^\tau U(\tau - s)f(s)ds \neq 0$. *Then the equation* (1) *has a unique* τ*-periodic solution.*

Proof Since

$$\begin{aligned} \|A_n(U(\tau))x\| &\leq \frac{M}{n}(1 + e^{-\alpha\tau} + e^{-2\alpha\tau} + \cdots + e^{-(n-1)\alpha\tau})\|x\| \\ &\leq \frac{M(1 - e^{-n\alpha\tau})}{n(1 - e^{-\alpha\tau})}\|x\|, \end{aligned}$$

we have $\lim_{n\to\infty} \|A_n(U(\tau))\| = 0$. Hence $U(\tau)$ is uniformly ergodic, so that $\mathcal{R}(I - U(\tau))$ is closed. Combining Theorem 3.1 (or Proposition 3.3) with Proposition 2.11 we see that the conclusion of the corollary is true.

Remark 3.6 *The avobe results for the existence of periodic solutions can be extended to a large class of linear equations.*

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