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# Practical Stability of Hopfield-type Neural Networks

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## 1 Introduction

The asymptotic, global and exponential stability properties of Hopfield-type neural networks have been extensively studied since Hopfield announced his results in the early 1980s (for a review of recent results, see Guan et al. [1]). This reflects the importance of Hopfield-type neural networks as applied to associative memory, pattern recognition and optimization problems.

This paper considers a seemingly less important stability concept to neural networks. Historically termed *practical stability* and first proposed by LaSalle and Lefschetz [2], it offers a very general notion that may indicate any one of these: asymptotic or global types of stability; total stability or stability under persistent disturbances; instability or boundedness of solutions. It is neither weaker nor stronger than ordinary stability, and it does not imply stability or convergence of trajectories. This may explain the negligible volume of literature devoted so far to practical stability of neural networks.

The theory of practical stability, developed intensively in the 1980s and early 1990s ([3]—[6]), is important in certain engineering applications, some of which are cited in [7]— [9]. Essentially, these applications have one common problem, namely, the existence of external inputs or disturbances, possibly random, time-varying or unbounded in time, that cause instability and tend to produce oscillations. In such a situation, if the system trajectories oscillate around a mathematically unstable course, then the next best course of action would be to ensure that the performance of the system in question is still acceptable in a practical sense. Specifically, a concrete system will be considered stable if, in case the initial values and/or the external disturbances are bounded by suitable constraints, the deviations of the motions from the equilibrium remain within certain bounds *determined* by the physical situation. LaSalle and Lefschetz [2], clearly summed up the underlying issue:

> Before one can speak of practical stability one must decide on: (a) how near the desired state it is necessary to have the system operate; (b) the magnitude of the perturbations to be expected; and (c) how well the initial conditions can be controlled. After this has been decided, it is possible to speak of practical stability.

In this article, we reconsider practical stability as applied to neural networks — since it was first briefly discussed (without compelling reasons, however) in 1993 by Köksal and Sivasundaram [10] — given the obvious importance of the role of external inputs in neural networks, such as setting the general level of excitability of the network through constant biases, or providing direct parallel inputs to drive specific neurons [11]. In the control of chaos in neural networks, external inputs are a means of "pinning" the state of a few neurons [12]. In the study of oscillatory neurocomputers, the external inputs, either constant, quasiperiodic or chaotic, impose a dynamic connectivity [13]. In the design of cerebellar model articulation controllers (CMAC), the aim is to obtain tolerable solutions, not desirable solutions [14]. Perhaps, a more telling situation involves networks which are fed via external inputs, possibly time-varying, and then run without resetting the initial conditions. Hence, as indicated by Guzelis and Chua [15], having a bounded input that assures a bounded output - that is, *input-output stability* — is of importance in such applications. Using a feedback configuration and the *finite gain stability* concept ([16], [17]), — the standard techniques for input-output stability analysis — Guzelis and Chua designed a neural network system which is $L_p$-stable, basically meaning that an external input in $L_p$ space produces an output in $L_p$ space, $p = 2$ and $p = \infty$.[1]

Hence, there is indeed some merit in looking at the effects of external inputs. This paper offers a simple but rigorous method of doing so, namely, via the concept of practical stability. The Hopfield-type neural network, due to its well-understood functions, is analyzed for its practical stability properties. The overall emphasis in this paper is on the effects of time-varying external inputs. We shall not consider external disturbances that depend both on time and system variables, given that well-known results, one of which is Malkin's Theorem [2], have established stability under such disturbances.

We begin by showing that if an external input in $L_2$ is applied to an exponentially stable system, then the system maintains convergence of system trajectories to fixed-point attractors. This result is obtained without recasting the network into a feedback configuration. In the absence of convergence, we provide a practical stability criterion, the main focus of this paper. To establish practical stability, we use the *comparison principle*. The interested reader in this simple, but effective method, may consult Yoshizawa [19], or, for a more recent reference, Kaszkurewicz and Bhaya [20], who showed that the use of the comparison principle leads to *diagonal stability* [21].

The preliminary sections (Sections 2 and 3) list the definitions of stability and appropriate theorems to be applied, and provide an outline of the Hopfield-type model. The main results are in Section 4.

## 2 Useful Stability Results

In this paper, we use the definitions of Lyapunov stability and exponential stability found in Sastry [17].

We will need the following important lemma proved in Appendix A. It will play the key role

[1]This questions Haykin's general statement ([18], page 537) that BIBO stability analysis was irrelevant in neural networks.

in proving convergence of system trajectories in the presence of time-varying external inputs. It will also be useful in establishing our practical stability criterion.

**LEMMA 1** Let $x(t) \geq 0$ satisfy the differential inequality

$$x'(t) \leq -\alpha x(t) + \sigma(t)\,,\ x(0) = x_0\,,\ t \geq 0\,.$$

Suppose $\alpha > 0$ and that $\sigma(t)$ is bounded on $[0, \infty)$ and $\sigma(t) \to 0$ as $t \to \infty$. Then $x(t) = x(t; x_0) \to 0$ as $t \to \infty$.

The definitions of practical stability concepts are as in Lakshmikantham *et al.* [4], page 9. For these, consider the system

$$\boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}),\ \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0\,,\ \ t_0 \geq 0\,, \tag{2.1}$$

where $\boldsymbol{f} \in C[\boldsymbol{R}_+ \times \boldsymbol{R}^n, \boldsymbol{R}^n]$. Suppose that the function $\boldsymbol{f}$ is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $\boldsymbol{x}(t) = \boldsymbol{x}(t; t_0, \boldsymbol{x}_0)$ of (2.1).

**Definition 1** System (2.1) is said to be

(PS1) *practically stable* if given $(\lambda, A)$ with $0 < \lambda < A$, we have $\|\boldsymbol{x}_0\| < \lambda$ implies that $\|\boldsymbol{x}(t)\| < A$, $t \geq t_0$ for some $t_0 \in \boldsymbol{R}_+$;

(PS2) *uniformly practically stable* if (PS1) holds for every $t_0 \in \boldsymbol{R}_+$;

(PS3) *uniformly practically quasi stable* if given $(\lambda, B, T) > 0$, we have $\|\boldsymbol{x}_0\| < \lambda$ implies that $\|\boldsymbol{x}(t)\| < B$, $t \geq t_0 + T$, for every $t_0 \in \boldsymbol{R}_+$;

(PS4) *strongly uniformly practically stable* if (PS2) and (PS3) hold simultaneously.

The following comparison principle for practical stability, where $K = \{b \in C[\boldsymbol{R}_+, \boldsymbol{R}_+] : b(u)$ is strictly increasing in $u$ and $b(u) \to \infty$ as $u \to \infty\}$, and $S(\rho) = \{\boldsymbol{x} \in \boldsymbol{R}^n : \|\boldsymbol{x}\| < \rho\}$, is also from [4], page 60:

**THEOREM 1** Assume that

*1.* $\lambda$ and $A$ are given such that $0 < \lambda < A$;

*2.* $V \in C[\boldsymbol{R}_+ \times \boldsymbol{R}^n, \boldsymbol{R}_+]$ and $V(t, \boldsymbol{x})$ is locally Lipschitzian in $\boldsymbol{x}$ ;

*3.* for $(t, \boldsymbol{x}) \in \boldsymbol{R}_+ \times S(A)$, $b_1(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq b_2(\|\boldsymbol{x}\|)$, $b_1, b_2 \in K$ and $D^+V(t, \boldsymbol{x})_{(2.1)} \leq g(t, V(t, \boldsymbol{x}))\,,\ \ g \in C[\boldsymbol{R}_+^2, \boldsymbol{R}]$;

*4.* $b_2(\lambda) < b_1(A)$ holds.

Then the practical stability properties of the scalar differential equation $z'(t) = g(t, z)$, $z(t_0) = z_0 \geq 0$, imply the corresponding practical stability properties of the system (2.1).

# 3 The Hopfield-type Model

The Hopfield-type model [11] is of the type

$$x' = Ax + h(x) + u(t)\,. \tag{3.1}$$

Here, $x = (x_1, \ldots, x_n)^T$ where $x_i$ denotes the activation potential of the $i$-th neuron, $i = 1, \ldots, n$; $A = \operatorname{diag}(-a_1, \ldots, -a_n)$, where

$$a_i = \frac{1}{C_i}\left(\frac{1}{R_i} + \sum_{j=1}^{n}\left|\frac{1}{R_{ij}}\right|\right) > 0\,,$$

$C_i > 0$ is the input capacitance, $R_i > 0$ is the input resistance and $R_{ij} \in \boldsymbol{R} = (-\infty, \infty)$ is the input connecting resistance (no assumption is made on its symmetricity); $h(x) = (h_1(x), \ldots, h_n(x))^T$ $h_i(x) = \sum_{j=1}^{n} B_{ij}F_j(\varphi_j x_j)$, where $B_{ij} = 1/(C_i R_{ij})$ and $F_i : \boldsymbol{R} \to (-1, 1)$ is a nonlinear activation function not necessarily monotonically increasing, with gain constant $\varphi_i$; and $u(t) = (u_1, \ldots, u_n)^T$, $u_i(t) = I_i(t)/C_i$, where $I_i : \boldsymbol{R}_+ \to \boldsymbol{R}$ is an external input current, and $u_i(t)$ is defined almost everywhere in $[0, \infty)$. In this paper, we shall refer to $u_i(t)$ as an *external input* and $u(t)$ as the *external input vector*. The $i$-th component of system (3.1) is

$$x_i' = -a_i x_i + \sum_{j=1}^{n} B_{ij}F_j(\varphi_j x_j) + u_i(t)\,.$$

When the external input vector is zero, the nonautonomous system (3.1) reduces to the autonomous system

$$x' = Ax + h(x)\,. \tag{3.2}$$

For this, we assume that $x^*$ is an equilibrium point, so that $Ax^* + h(x^*) = \mathbf{0}$. By translating the origin, $\mathbf{0}$, to this equilibrium point, we can make $\mathbf{0}$ an equilibrium point. In this case, $h(\mathbf{0}) \equiv \mathbf{0}$. Since this is of great notational help, we will henceforth consider $\mathbf{0}$ as an equilibrium point of (3.2). Finally, we require that $h(x)$ has continuous first partial derivatives in $x$.

# 4 Main Results

## 4.1 Convergence of System Trajectories in the Presence of External Inputs in $L_2$-Space

For the purpose of illustrating persistence of convergence in the presence of time-varying external inputs, we need, for the autonomous system (3.2), a simple exponential stability criterion, which may not necessarily be the best compared with established results. Some of these results, applicable to autonomous systems with constant external input vectors, are proposed in Fang and Kincaid [22] and Yi et al. [23].

Define

$$D(x) = [d_{ij}]_{n\times n} = \begin{cases} \left[\displaystyle\int_0^1 \frac{\partial h_i(sx)}{\partial(sx_j)}ds\right]_{n\times n}, & x \neq \mathbf{0}, \\ \mathbf{0}, & x = \mathbf{0}\,. \end{cases}$$

Then, given the differentiable function $f = f(sx + (1-s)y)$, the result

$$\int_0^1 \frac{d}{d(sx + (1-u)y)}[f(sx + (1-s)y)]ds = \frac{f(x) - f(y)}{x - y},$$

which can be easily verified by the fundamental theorems of calculus, yields $\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{D}(\boldsymbol{x})\boldsymbol{x}$. Hence, assuming $\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0}$, system (3.1) can be written as

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{D}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{u}(t), \tag{4.1}$$

or, componentwise,

$$x_i' = m_{ii}(\boldsymbol{x})x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} m_{ij}(\boldsymbol{x})x_j + u_i(t), \tag{4.2}$$

where $m_{ii}(\boldsymbol{x}) = -a_i + d_{ii}(\boldsymbol{x})$ and $m_{ij}(\boldsymbol{x}) = d_{ij}(\boldsymbol{x})$ when $i \neq j$. Define

$$-r_i(\boldsymbol{x}) = m_{ii}(\boldsymbol{x}) + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n}(|m_{ij}(\boldsymbol{x})| + |m_{ji}(\boldsymbol{x})|). \tag{4.3}$$

**THEOREM 2** Assume that $\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{h} \in C^1[\boldsymbol{R}^n, \boldsymbol{R}^n]$. If for $1 \leq i \leq n$ and $\boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{x} \neq \boldsymbol{0}$, there exists a constant $c$ such that $0 < c \leq r_i(\boldsymbol{x})$, then the equilibrium point $\boldsymbol{0}$ of the autonomous system (3.2) is globally exponentially stable.

**Proof** Using $V = \sum_{i=1}^n x_i^2/2$, we have

$$V'_{(3.2)} = \sum_{i=1}^{n} x_i \left( m_{ii}(\boldsymbol{x})x_i + \sum_{\substack{j=1 \\ i\neq j}}^{n} m_{ij}(\boldsymbol{x})x_j \right) \leq -\sum_{i=1}^{n} r_i(\boldsymbol{x})x_i^2 \leq -2cV.$$

Hence, $V(t, \boldsymbol{x}(t; \boldsymbol{x}_0, t_0)) \leq V(t_0, \boldsymbol{x}_0)e^{-2c(t-t_0)}$, $t \geq t_0 \geq 0$, implying global exponential stability of the trivial solution of system (3.2).

**Remark 4.1** As remarked earlier, Theorem 2 is a simple, possibly relatively restrictive result, the emphasis in this paper being on the effects of time-varying inputs. Nonetheless, it is applicable and examples can be easily found. It could also be recast to fit under other generalized concepts of stability. As an example, if we write $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{A} + \boldsymbol{D}(\boldsymbol{x})$ in (4.1) and let $\boldsymbol{C}(\boldsymbol{x})$ denote the comparison matrix of $\boldsymbol{M}(\boldsymbol{x})$, where $\boldsymbol{C}(\boldsymbol{x}) = [c_{ij}(\boldsymbol{x}]_{n\times n}$ is defined as $c_{ii}(\boldsymbol{x}) = m_{ii}(\boldsymbol{x})$ and $c_{ij}(\boldsymbol{x}) = |m_{ij}(\boldsymbol{x})|$. Let $\boldsymbol{R} = [\boldsymbol{C}(\boldsymbol{x}) + \boldsymbol{C}^T(\boldsymbol{x})]/2$. Then $r_i(\boldsymbol{x})$ defined in (4.3) is the negative of the $i$-th row of $\boldsymbol{R}$. Requiring that $r_i \geq c > 0$ in Theorem 2 is equivalent to requiring that $\boldsymbol{R}$ is strictly diagonally dominant, so that $\boldsymbol{R}$ has the property of diagonal stability ensuring global asymptotic stability. Then several conditions for global stability, weaker than requiring $\boldsymbol{R}$ to be diagonally stable, are given in Kaszkurewicz and Bhaya [21].

For our main result in this subsection, we recall the definition of functions in the class $L_p$ [16].

**Definition 2** For all constants $t_0 \geq 0$ and $p \in [0, \infty)$, we label as $L_p[t_0, \infty)$, or simply $L_p$, the set consisting of all measurable functions $f(\cdot) : [t_0, \infty) \to \boldsymbol{R}$ such that $\int_{t_0}^{\infty} |f(t)|^p \, dt < \infty$.

**THEOREM 3** Let the conditions of Theorem 2 hold so that the equilibrium point $\mathbf{0}$ of the autonomous system (3.2) is globally exponentially stable. If $u_i(\cdot) \in L_2[t_0, \infty)$ for all $i = 1, \ldots, n$, then every solution $x_i(t)$ given in (4.2) of the nonautonomous system (3.1) tends to zero as $t \to \infty$.

**Proof** For $t \geq t_0 \geq 0$, define

$$W(t, \boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \frac{1}{4\epsilon}\sum_{i=1}^{n}\int_t^{\infty}[u_i(s)]^2 ds\,.$$

Since $u_i(\cdot) \in L_2[t_0, \infty)$, we have

$$\frac{d}{dt}\left[\int_t^{\infty}[u_i(s)]^2 ds\right] = \frac{d}{dt}\left[\int_{t_0}^{\infty}[u_i(s)]^2 ds - \int_{t_0}^{t}[u_i(s)]^2 ds\right] = -[u_i(t)]^2\,,$$

implying therefore the differentiability and hence the existence on $[t_0, \infty)$ of the second term of $W$. Thus, for $\epsilon > 0$ sufficiently small such that $(c - \epsilon) > 0$, we have, along a trajectory of the nonautonomous system (3.1),

$$\begin{aligned} W'_{(3.1)} &\leq -\sum_{i=1}^{n} r_i(\boldsymbol{x})x_i^2 + \sum_{i=1}^{n} x_i u_i(t) - \frac{1}{4\epsilon}\sum_{i=1}^{n}[u_i(t)]^2 \\ &\leq -(c-\epsilon)\sum_{i=1}^{n} x_i^2 + \frac{1}{4\epsilon}\sum_{i=1}^{n}[u_i(t)]^2 - \frac{1}{4\epsilon}\sum_{i=1}^{n}[u_i(t)]^2 \\ &= -(c-\epsilon)\sum_{i=1}^{n} x_i^2 = -2(c-\epsilon)W + \frac{c-\epsilon}{2\epsilon}\sum_{i=1}^{n}\int_t^{\infty}[u_i(s)]^2 ds\,. \end{aligned} \tag{4.4}$$

Let $\sigma(t)$ be the second term in (4.4). Then $\sigma(t)$ is bounded on $[t_0, \infty)$ and $\sigma(t) \to 0$ as $t \to \infty$. Thus, by Lemma 1, $W \to 0$ as $t \to \infty$. Thus, all solutions $\boldsymbol{x}(t) \in \boldsymbol{R}^n$ of system (3.1) tend to $\mathbf{0}$ as $t \to \infty$.

**Remark 4.2** Since

$$\begin{aligned} \left(\int_t^{t+1}|u_i(s)|\,ds\right) &\leq \left(\int_t^{t+1}|u_i(s)|^2\,ds\right)^{1/2}\left(\int_t^{t+1}1\,ds\right)^{1/2} \\ &\leq \left(\int_t^{\infty}|u_i(s)|^2 ds\right)^{1/2} \to 0 \quad \text{as} \quad t \to \infty\,, \end{aligned}$$

Theorem 3 includes the external input $u_i$ such that $\int_t^{t+1}|u_i(s)|\,ds \to 0$ as $t \to \infty$. A stronger condition, namely that $u_i$ be uniformly continuous on $[t_0, \infty)$ gives us inputs of the form $u_i \to 0$ as $t \to \infty$ by Barbalat's Lemma [17].

**Remark 4.3** In Theorem 3, if $|u_i(t)| = k_i$, for some constant $k_i > 0$ and for all $t \geq t_0 \geq 0$, then we still have an autonomous system. For this, an interesting result by Kaszkurewicz and Bhaya [24], who utilized the concept of diagonal stability, ensured persistence of global asymptotic stability of system (3.1) under perturbations in the nonlinear activation functions, assumed to have satisfied certain conditions. Improved results, also where the external input vector is constant, can be found in Arik and Tavsanoglu [25] and Guan et al. [1]. If $|u_i(t)| < k_i$, then it is a mistake to use the well-known Malkin's Theorem to conclude total stability or stability under persistent disturbances since $\boldsymbol{u}$ does not depend on $\boldsymbol{x}$. In fact, to conclude this, it is best to use a practical stability criterion since it gives more than a mere statement of the existence of the bounds of the disturbances and initial conditions that maintain bounded outputs. We provide such a criterion next.

## 4.2 Practical Stability

**THEOREM 4** Let $\boldsymbol{h}(\mathbf{0}) = \mathbf{0}$. Assume that $|u_i(t)| \leq k_i$ for some constant $k_i \geq 0$ and for all $t \geq t_0 \geq 0$, or $u_i(\cdot) \in L_2[t_0, \infty)$, $i = 1, \ldots, n$. Then system (3.1) is strongly uniformly practically stable.

**Proof** This is given in the Appendix B.

**Remark 4.4** In Köksal and Sivasundaram [10], it is not always easy to satisfy the given practical stability criteria. Moreover, even if the criteria are applicable, they allow only the analysis of Hopfield-type neural networks whose autonomous components are globally exponentially stable. To apply Theorem 4, we need not have an asymptotically stable autonomous system.

**Remark 4.5** Theorem 4 proves conclusively that, in addition to having bounded activation functions, we must have at least an external input vector that is constant, or time-varying but bounded or in $L_2$ to ensure practical stability. That is, it shows that in the presence of such disturbances, it is possible to pre-assign the bounds of the initial states and the neural network outputs using only the parameters $B_{ij}$ of the system. One way to do this is to use the estimates, (4.7) and (4.8), or (4.9) and (4.10) shown in the proof, noting that one can get different sufficient conditions for practical stability if different norms are used.

**EXAMPLE 1** Practical stability concepts could add an extra dimension to the control of chaos in neural networks, given that controlling chaos usually consists in forcing a system out of a chaotic attractor by using external inputs [12]. This extra dimension involves the determination of the output bound of a chaotic system given the bounds of the initial state and the external inputs. As a simple illustration, consider the following two-neuron system analyzed by De Wilde [26]:

$$\begin{aligned} x_1' &= -x_1 + \tanh x_1 + \tanh x_2 \,, \\ x_2' &= -x_2 - 100 \tanh x_1 + 2 \tanh x_2 \,. \end{aligned}$$

The system exhibits a periodic attractor at $(0,0)$. The behaviour of this system becomes more complex if an external input is added, and becomes chaotic if a periodic input such as the sine or cosine function is added. Nevertheless, in such cases, for example,

$$\left.\begin{aligned} x_1' &= -x_1 + \tanh x_1 + \tanh x_2 + a \sin t, & x_1(t_0) &= x_{10} \\ x_2' &= -x_2 - 100 \tanh x_1 + 2 \tanh x_2 + b \cos t, & x_2(t_0) &= x_{20} \,. \end{aligned}\right\} \tag{4.5}$$

where the external inputs are bounded by $k_1 = |a|$ and $k_2 = |b|$, the solutions oscillate about $(0,0)$ and remain bounded by Theorem 4. Indeed, by the results given in the proof of the theorem, if $n = 2$, $a_1 = a_2 = 1$, $|u_1(t)| \le 1 = k_1$, $|u_2(t)| \le 2 = k_2$, and $\epsilon_1 = \epsilon_2 = 0.5$, then $\alpha_* = 2\min\{a_1 - \epsilon_1, a_2 - \epsilon_2\} = 1$, and

$$\beta_* = \frac{(|B_{11}| + |B_{12}| + k_1)^2}{4\epsilon_1} + \frac{(|B_{21}| + |B_{22}| + k_2)^2}{4\epsilon_2} = \frac{10825}{2}.$$

Thus, (4.7) yields $\max\{\lambda^2,\ 10825\} < A^2$. Hence, for example, if we fix $A = \sqrt{10825} + \varepsilon$, $\varepsilon > 0$, then we can fix any $\lambda < A$, and every chaotic trajectory of system (4.5) starting within $\lambda$ stays within $A$.

**Appendix A : Proof of Lemma 1**

By standard manipulation, we have

$$x(t) \le e^{-\alpha t}x_0 + e^{-\alpha t}\int_0^t e^{\alpha s}\sigma(s)ds. \tag{4.6}$$

The first term of (4.6) goes to 0 as $t \to \infty$. By assumption on $\sigma$, we have that for every $\epsilon > 0$, there exists a $T > 0$ such that $|\sigma(t)| < \epsilon$ for all $t \ge T$. Hence, on letting $\|\sigma\|_\infty = \sup_{t\ge 0}|\sigma(t)|$, the second integral term of (4.6) is estimated as

$$\begin{aligned}
&\left|e^{-\alpha t}\int_0^t e^{\alpha s}\sigma(s)ds\right| \le e^{-\alpha t}\int_0^t e^{\alpha s}|\sigma(s)|ds \\
&\le e^{-\alpha t}\left(\int_0^T + \int_T^t\right)e^{\alpha s}|\sigma(s)|ds \le e^{-\alpha t}\left(\int_0^T \|\sigma\|_\infty e^{\alpha s}ds + \epsilon\int_0^t e^{\alpha s}ds\right) \\
&\le e^{-\alpha t}\left(\|\sigma\|_\infty\frac{e^{\alpha T}}{\alpha} + \epsilon\frac{e^{\alpha t}}{\alpha}\right) \le \frac{\|\sigma\|_\infty}{\alpha}e^{-\alpha(t-T)} + \frac{\epsilon}{\alpha}.
\end{aligned}$$

Since for every $\epsilon > 0$, there exists $S > 0$ such that $e^{-\alpha t} < \epsilon$ for all $t \ge S$, we have, for all $t \ge T + S$,

$$\left|e^{-\alpha t}\int_0^t e^{\alpha s}\sigma(s)ds\right| \le \frac{\epsilon}{\alpha}(1 + \|\sigma\|_\infty).$$

This proves $x(t) = x(t; x_0) \to 0$ as $t \to \infty$ because we can choose $\epsilon$ as small as we wish.

**Appendix B : Proof of Theorem 4**

1. Case where $|u_i(t)| \le k_i$.

   Using $V_i(t,x) = x_i^2/2$ we have, recalling that $F_i : \boldsymbol{R} \to (-1, 1)$,

$$\begin{aligned}
V'_{i(3.1)} &= x_i[-a_ix_i + h_i(x) + u_i(t)] = -a_ix_i^2 + x_i\left(\sum_{j=1}^n B_{ij}F_j(\varphi_jx_j) + u_i(t)\right) \\
&\le -a_ix_i^2 + |x_i|\left(\sum_{j=1}^n |B_{ij}| + k_i\right).
\end{aligned}$$

   Since we can always find a constant $\epsilon_i > 0$ sufficiently small such that $(a_i - \epsilon_i) > 0$, we have

$$V'_{i(3.1)} \le -(a_i - \epsilon_i)x_i^2 + \frac{1}{4\epsilon_i}\left(\sum_{j=1}^n |B_{ij}| + k_i\right)^2.$$

Let $V = \sum_{i=1}^n V_i = \|\boldsymbol{x}\|^2/2$, and $\epsilon_i \in (0, a_i)$ be a constant. Define $\alpha_* = 2\min\{a_1 - \epsilon_1, \ldots, a_n - \epsilon_n\} > 0$, and

$$\beta_* = \sum_{i=1}^n \frac{1}{4\epsilon_i}\left(\sum_{j=1}^n |B_{ij}| + k_i\right)^2 .$$

Then, $V'_{(3.1)} \leq -\alpha_* V + \beta_* \stackrel{\text{def}}{=} g(t, V)$. Hence, the comparison scalar differential equation is

$$z' = g(t,z) = -\alpha_* z + \beta_*, \quad z(t_0) = z_0, \quad z(t) \geq 0 \; \forall\, t \geq t_0 \geq 0\,,$$

the solution, of which, is

$$z(t; t_0, z_0) = \left(z_0 - \frac{\beta_*}{\alpha_*}\right) e^{-\alpha_*(t-t_0)} + \frac{\beta_*}{\alpha_*}\,,$$

so that $z(t; t_0, z_0) \leq \max\{z_0, \beta_*/\alpha_*\}$ and $\limsup_{t\to\infty} z(t) \leq \beta_*/\alpha_*$, implying therefore the boundedness of the solutions of system (3.1). Now, let $(\lambda, A, B, T) > 0$ be given such that $\lambda < A$, $B < A$, $t \geq t_0 + T$,

$$\begin{aligned}
&\left(z_0 - \frac{\beta_*}{\alpha_*}\right) e^{-\alpha_*(t-t_0)} + \frac{\beta_*}{\alpha_*} \leq \max\left\{\left(z_0 - \frac{\beta_*}{\alpha_*}\right) e^{-\alpha_* T} + \frac{\beta_*}{\alpha_*}\,,\; \frac{\beta_*}{\alpha_*}\right\} \\
&\leq \max\left\{b_2(\lambda) e^{-\alpha_* T} + \frac{\beta_*}{\alpha_*}\left(1 - e^{-\alpha_* T}\right)\,,\; \frac{\beta_*}{\alpha_*}\right\} < b_1(B)\,,
\end{aligned}$$

and $\max\{z_0, \beta_*/\alpha_*\} \leq \max\{b_2(\lambda), \beta_*/\alpha_*\} < b_1(A)$, where $b_1$ and $b_2$ are as defined in Theorem 1. Let $b_1(\|\boldsymbol{x}\|) = b_2(\|\boldsymbol{x}\|) = V(t, \boldsymbol{x})$. Then $b_1(A) = A^2/2$, $b_1(B) = B^2/2$, $b_2(\lambda) = \lambda^2/2$,

$$\max\left\{\frac{\lambda^2}{2}, \frac{\beta_*}{\alpha_*}\right\} < \frac{A^2}{2}\,, \tag{4.7}$$

and

$$\max\left\{\frac{\lambda^2}{2} e^{-\alpha_* T} + \frac{\beta_*}{\alpha_*}\left(1 - e^{-\alpha_* T}\right)\,,\; \frac{\beta_*}{\alpha_*}\right\} < \frac{B^2}{2}\,. \tag{4.8}$$

Hence, system (3.1) is strongly uniformly practically stable since $V$ satisfies the conditions of the comparison principle Theorem 1.

2. Case where $u_i(\cdot) \in L_2[t_0, \infty)$.

   Let $q_i = \sum_{j=1}^n |B_{ij}|$. Again, with $V_i(t, \boldsymbol{x}) = x_i^2/2$, we have,

$$\begin{aligned}
V'_{i(3.1)} &= -a_i x_i^2 + x_i\left(\sum_{j=1}^n B_{ij} F_j(\varphi_j x_j) + u_i(t)\right) \\
&\leq -a_i x_i^2 + q_i |x_i| + |u_i(t)||x_i|\,.
\end{aligned}$$

Let $\epsilon_i > 0$ and $\epsilon'_i > 0$, $i = 1, \ldots, n$, be constants such that $(\epsilon_i + \epsilon'_i) \in (0, a_i)$ and define $\alpha = 2\min\{(a_1 - \epsilon_1 - \epsilon'_1), \ldots, (a_n - \epsilon_n - \epsilon'_n)\} > 0$, $\epsilon_* = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$,

$$\tau = \max\left\{\frac{(a_1 - \epsilon_1 - \epsilon'_1)}{2\epsilon_1}, \ldots, \frac{(a_n - \epsilon_n - \epsilon'_n)}{2\epsilon_n}\right\} \quad \text{and} \quad \beta = \sum_{i=1}^n \frac{q_i^2}{4\epsilon'_i}\,.$$

Then we can always find $\epsilon_i > 0$ and $\epsilon_i' > 0$ sufficiently small such that $(a_i - \epsilon_i - \epsilon_i') > 0$ and

$$\begin{aligned} V'_{i(3.1)} &\leq -a_i x_i^2 + \frac{1}{4\epsilon_i'} q_i^2 + \epsilon_i' x_i^2 + \frac{1}{4\epsilon_i}[u_i(t)]^2 + \epsilon_i x_i^2 \\ &\leq -(a_i - \epsilon_i - \epsilon_i') x_i^2 + \frac{1}{4\epsilon_i'} q_i^2 + \frac{1}{4\epsilon_i}[u_i(t)]^2 . \end{aligned}$$

Now, for $t \geq t_0 \geq 0$, define

$$W_i(t, \boldsymbol{x}) = V_i + \frac{1}{4\epsilon_i} \int_t^\infty [u_i(s)]^2 ds .$$

Then,

$$W'_{i(3.1)} \leq -2(a_i - \epsilon_i - \epsilon_i') W_i + \frac{(a_i - \epsilon_i - \epsilon_i')}{2\epsilon_i} \int_t^\infty [u_i(s)]^2 ds + \frac{1}{4\epsilon_i'} q_i^2 .$$

Let $W(t, \boldsymbol{x}) = \sum_{i=1}^n W_i$, $\sigma(t) = \tau \int_t^\infty \|\boldsymbol{u}(s)\|^2 ds$ and $Q = \int_{t_0}^\infty \|\boldsymbol{u}(s)\|^2 ds$. Then $W'_{(3.1)} \leq -\alpha W + \sigma(t) + \beta \stackrel{\text{def}}{=} g(t, W)$. Hence, we have the comparison scalar equation

$$z' = g(t, z) = -\alpha z + \sigma(t) + \beta, \quad z(t_0) = z_0, \quad z(t) \geq 0 \ \ \forall t \geq t_0 \geq 0 .$$

Solving this, we have, for $t \geq t_0 \geq 0$,

$$\begin{aligned} z(t; t_0, z_0) &= \left(z_0 - \frac{\beta}{\alpha}\right) e^{-\alpha(t - t_0)} + \frac{\beta}{\alpha} + e^{-\alpha t} \int_{t_0}^t e^{\alpha s} \sigma(s) ds \\ &\leq \left(z_0 - \frac{\beta}{\alpha}\right) e^{-\alpha(t - t_0)} + \frac{\beta}{\alpha} + \tau Q e^{-\alpha t} \int_{t_0}^t e^{\alpha s} ds \\ &= \left(z_0 - \frac{\beta}{\alpha}\right) e^{-\alpha(t - t_0)} + \frac{\beta}{\alpha} + \frac{\tau Q}{\alpha}(1 - e^{-\alpha(t - t_0)}) \\ &= \left(z_0 - \frac{\beta}{\alpha} - \frac{\tau Q}{\alpha}\right) e^{-\alpha(t - t_0)} + \frac{\beta}{\alpha} + \frac{\tau Q}{\alpha} , \end{aligned}$$

so that $z(t; t_0, z_0) \leq \max\{z_0, (\beta + \tau Q)/\alpha\}$. Moreover, as a consequence of Lemma 1, $\limsup_{t \to \infty} z(t) \leq \beta/\alpha$, implying therefore the boundedness of the solutions of system (3.1). We now have strong uniform practical stability if $(\lambda, A, B, T) > 0$ are given such that $\lambda < A$, $B < A$, $t \geq t_0 + T$,

$$\begin{aligned} &\left(z_0 - \frac{\beta}{\alpha} - \frac{\tau Q}{\alpha}\right) e^{-\alpha(t - t_0)} + \frac{\beta}{\alpha} + \frac{\tau Q}{\alpha} \\ &\leq \max\left\{ b_2(\lambda) e^{-\alpha T} + \frac{\beta + \tau Q}{\alpha}(1 - e^{-\alpha T}), \ \frac{\beta + \tau Q}{\alpha} \right\} < b_1(B) , \end{aligned}$$

and $\max\{z_0, \beta + \tau Q/\alpha\} \leq \max\{b_2(\lambda), (\beta + \tau Q)/\alpha\} < b_1(A)$, where $b_1$ and $b_2$ are defined as in the comparison principle Theorem 1. Let $b_1(\|\boldsymbol{x}\|) = \|\boldsymbol{x}\|^2/2$ and $b_2(\|\boldsymbol{x}\|) = \|\boldsymbol{x}\|^2/2 + Q/(4\epsilon_*)$, noting that $b_1 \leq W \leq b_2$. Then, $b_1(A) = A^2/2$, $b_1(B) = B^2/2$, $b_2(\lambda) = \lambda^2/2 + Q/(4\epsilon_*)$,

$$\max\left\{ \left(\frac{\lambda^2}{2} + \frac{Q}{4\epsilon_*}\right) e^{-\alpha T} + \frac{\beta + \tau Q}{\alpha}(1 - e^{-\alpha T}), \ \frac{\beta + \tau Q}{\alpha} \right\} < \frac{B^2}{2} ,$$

and

$$\max\left\{\frac{\lambda^2}{2}+\frac{Q}{4\epsilon_*},\ \frac{\beta+\tau Q}{\alpha}\right\}<\frac{A^2}{2}. \tag{4.10}$$

Thus, by the comparison principle Theorem 1, system (3.1) is strongly uniformly practically stable.

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