Order three symmetry of a vertex operator algebra

Hiromichi Yamada

Department of Mathematics
Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan
yamada@math.hit-u.ac.jp

In this note we shall report an attempt to find a vertex operator algebra which possesses an order three symmetry. We are actually interested in a subalgebra having this property of a vertex operator algebra associated with a lattice \( L = \sqrt{2}A_2 \). The work is not completed yet. We shall show some results so far obtained.

1 Notation and Setting

Let \( \{\alpha_1, \alpha_2\} \) be the set of simple roots of type \( A_2 \), so that \( \langle \alpha_i, \alpha_i \rangle = 2 \) and \( \langle \alpha_1, \alpha_2 \rangle = -1 \). We shall consider three automorphisms of the root lattice \( \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 \). First,

\[
\sigma : \alpha_1 \mapsto \alpha_2 \quad \alpha_1 + \alpha_2 \mapsto \alpha_1
\]

is an automorphism of order three. Exchange of \( \alpha_1 \) and \( \alpha_2 \) induces an order two automorphism \( \rho \) of the root lattice. Finally, let \( \theta \) be the order two automorphism \( \alpha \mapsto -\alpha \) as usual. Note that \( \rho \sigma \rho = \sigma^{-1} \). Let \( \tau_i \) be the reflection with respect to \( \alpha_i \). Then \( \tau_1 \tau_2 = \sigma \) and \( \tau_1 \tau_2 \tau_1 = \rho \theta \). Hence

\[
\langle \tau_1, \tau_2, \theta \rangle = \langle \sigma, \rho, \theta \rangle \cong S_3 \times \mathbb{Z}_2
\]

Let \( L = \mathbb{Z}\sqrt{2}\alpha_1 + \mathbb{Z}\sqrt{2}\alpha_2 \) be \( \sqrt{2} \) times the root lattice of type \( A_2 \) and \( V_L \) be the vertex operator algebra associated with the lattice \( L \) as defined in [3]. The vertex operator algebra \( V_L \) was first studied in [5] and several applications were developed in [4], [5], [8]. We shall use the same notation as in [5]. The three automorphisms \( \sigma, \rho, \theta \) can be extended to automorphisms of \( V_L \). We shall denote these automorphisms of \( V_L \) by the same symbols.

We want to know the vertex operator subalgebra

\[
(V_L)^\sigma = \{ v \in V_L | \sigma v = v \}
\]

and also its irreducible modules.
We need some other subalgebras. Let
\[ V_L^\pm = \{ v \in V_L \mid \theta v = \pm v \}, \]
\[ V_L^k = \{ v \in V_L \mid \sigma v = \zeta^k v \}, \]
\[ V_L^{k,\pm} = \{ v \in V_L \mid \sigma v = \zeta^k v, \quad \theta v = \pm v \}, \]
where \( \zeta = \exp(2\pi\sqrt{-1}/3) \) is a primitive cubic root of unity. Similar notations will be used for the homogeneous subspace
\[ (V_L)_{(m)} = \{ v \in V_L \mid \mathrm{wt} v = m \} \]
of weight \( m \). For example,
\[ (V_L^{k,\pm})_{(m)} = V_L^{k,\pm} \cap (V_L)_{(m)}. \]
Since \( \rho \sigma \rho = \sigma^{-1} \) and since \( \theta \) commutes with \( \rho \) and \( \sigma \), it follows that \( \rho(V_L^{0,\pm}) = V_L^{0,\pm} \) and \( \rho(V_L^{1,\pm}) = V_L^{2,\pm}. \)

By [1] it is known that there are three mutually orthogonal conformal vectors \( \omega^1, \omega^2, \omega^3 \) with central charges \( \frac{1}{2}, \frac{7}{10}, \frac{4}{5} \) respectively in \( V_L \). We shall recall their definition. For convenience, set
\[ x(\alpha) = e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \quad w(\alpha) = \frac{1}{2}\alpha (\alpha - 1)^2 - x(\alpha). \]
Now let
\[ s^1 = \frac{1}{4}w(\alpha_1), \]
\[ s^2 = \frac{1}{5}(w(\alpha_1) + w(\alpha_2) + w(\alpha_1 + \alpha_2)), \]
\[ \omega = \frac{1}{6}(\alpha_1 (-1)^2 + \alpha_2 (-1)^2 + (\alpha_1 + \alpha_2)(-1)^2). \]
Then \( \omega \) is the Virasoro element of \( V_L \). The conformal vectors \( \omega^i \) are defined by
\[ \omega^1 = s^1, \quad \omega^2 = s^2 - s^1, \quad \omega^3 = \omega - s^2. \]
Denote by \( \text{Vir}(\omega^i) \) the subalgebra of \( V_L \) generated by \( \omega^i \). Then
\[ \text{Vir}(\omega^i) \cong L(c_i, 0), \quad i = 1, 2, 3, \]
with \( c_1 = \frac{1}{2}, c_2 = \frac{7}{10}, c_3 = \frac{4}{5} \). Since \( \omega^i \)'s are mutually orthogonal, the subalgebra \( T \) generated by \( \omega^1, \omega^2, \) and \( \omega^3 \) are isomorphic to a tensor product of \( \text{Vir}(\omega^i) \)'s:
\[ T \cong \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \otimes \text{Vir}(\omega^3) \]
\[ \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{4}{5}, 0\right). \]
As a $T$-module $V_L$ is completely reducible and in fact it is a direct sum of 8 irreducible $T$-modules. Each irreducible $T$-module is of the form (see [2])

$$L(\frac{1}{2}, h_1) \otimes L(\frac{7}{10}, h_2) \otimes L(\frac{4}{5}, h_3).$$

The following is the list of $(h_1, h_2, h_3)$ of the irreducible direct summands in $V_L$:

$$(0, 0, 0), \quad (0, \frac{3}{5}, \frac{2}{5}), \quad (\frac{1}{2}, \frac{1}{10}, \frac{2}{5}), \quad (0, \frac{3}{5}, \frac{7}{5}), \quad (\frac{1}{2}, \frac{1}{10}, \frac{7}{5}), \quad (\frac{1}{2}, \frac{3}{2}, 0), \quad (0, 0, 3), \quad (\frac{1}{2}, \frac{3}{2}, 3).$$

More precisely, $(0, 0, 0), (0, \frac{3}{5}, \frac{7}{5}), (\frac{1}{2}, \frac{1}{10}, \frac{7}{5}), (\frac{1}{2}, \frac{3}{2}, 0)$ are the direct summands in $V_L^+$ and the remaining four are those in $V_L^-$. We note that $\omega^i$'s are $\theta$-invariant and that $s^2 = \omega^1 + \omega^2$ and $\omega^3$ are $\sigma$-invariant. However, $\omega^1$ is not $\sigma$-invariant. Although $T$ is not invariant under $\sigma$, it contains a subalgebra

\[ \text{Vir}(s^2) \otimes \text{Vir}(\omega^3) \cong L(\frac{6}{5}, 0) \otimes L(\frac{4}{5}, 0), \]

which is fixed by $(\sigma, \theta)$, and $V_L^{k, \pm}$ is a module for $\text{Vir}(s^2) \otimes \text{Vir}(\omega^3)$. As a $\text{Vir}(s^2) \otimes \text{Vir}(\omega^3)$-module $V_L^{k, \pm}$ is completely reducible and each irreducible direct summand is of the form $L(\frac{6}{5}, h) \otimes L(\frac{4}{5}, h')$. Hence it is natural to ask:

**Problem** Determine the decomposition of $V_L^{k, \pm}$ into a direct sum of irreducible $\text{Vir}(s^2) \otimes \text{Vir}(\omega^3)$-modules.

## 2 Some Calculations

An irreducible direct summand isomorphic to $L(\frac{6}{5}, h) \otimes L(\frac{4}{5}, h')$ is generated by a highest weight vector $v = v(h, h')$ as a module for $\text{Vir}(s^2) \otimes \text{Vir}(\omega^3)$. Recall that a highest weight vector $v = v(h, h')$ is a vector which satisfies the conditions

$$\begin{align*}
(s^2)_1 v &= hv, \\
(\omega^3)_1 v &= h'v, \\
(s^2)_n v &= (\omega^3)_n v = 0 & \text{for}& \ n \geq 2.
\end{align*}$$

Here we denote by $u_n$ the component operator of the vertex operator $Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$.

It seems that $V_L^{k, \pm}$ is a direct sum of infinitely many irreducible $\text{Vir}(s^2) \otimes \text{Vir}(\omega^3)$-modules. However, only a few irreducible direct summands are known. In fact, by a direct calculation we can determine the highest weight vectors $v(h, h')$ such that $h + h' \leq 3$. From this result we have
Lemma 2.1 (1) $V_{L}^{0,+}$ contains three irreducible direct summands whose highest weights are at most 3. They are isomorphic to

\[ L(\frac{6}{5}, 0) \otimes L(\frac{4}{5}, 0), \quad L(\frac{6}{5}, \frac{8}{5}) \otimes L(\frac{4}{5}, \frac{7}{5}), \quad L(\frac{6}{5}, 3) \otimes L(\frac{4}{5}, 0). \]

The automorphism $\rho$ acts as 1 on the first one and $-1$ on the other two irreducible direct summands.

(2) For $k = 1, 2$, $V_{L}^{k,+}$ contains two irreducible direct summands whose highest weights are at most 3. They are isomorphic to

\[ L(\frac{6}{5}, \frac{3}{5}) \otimes L(\frac{4}{5}, \frac{7}{5}), \quad L(\frac{6}{5}, 2) \otimes L(\frac{4}{5}, 0). \]

(3) $V_{L}^{0,-}$ contains three irreducible direct summands whose highest weights are at most 3. They are isomorphic to

\[ L(\frac{6}{5}, \frac{8}{5}) \otimes L(\frac{4}{5}, \frac{2}{5}), \quad L(\frac{6}{5}, \frac{13}{5}) \otimes L(\frac{4}{5}, \frac{2}{5}), \quad L(\frac{6}{5}, 0) \otimes L(\frac{4}{5}, 3). \]

The automorphism $\rho$ acts as 1 on the first one and $-1$ on the other two irreducible direct summands.

(4) For $k = 1, 2$, $V_{L}^{k,-}$ contains only one irreducible direct summand whose highest weight is at most 3. It is isomorphic to

\[ L(\frac{6}{5}, \frac{3}{5}) \otimes L(\frac{4}{5}, \frac{2}{5}). \]

Next, we shall consider the character of $V_{L}^{\pm}$. As a vector space $V_{L} = M(1) \otimes \mathbb{C}[L]$, where $M(1)$ is the free boson part and $\mathbb{C}[L]$ is the group algebra of the additive group $L$. Thus $\mathbb{C}[L]$ has a basis $\{e^{\alpha} | \alpha \in L\}$ with multiplication $e^{\alpha}e^{\beta} = e^{\alpha+\beta}$.

Let $\mathcal{H}_{n,j}$ be the space of homogeneous polynomials in two variables $\alpha_{1}(-n)$ and $\alpha_{2}(-n)$ of degree $j$. It is of dimension $j + 1$ and

\[ \{\alpha_{1}(-n)^{i-1}\alpha_{2}(-n)^{i} | 0 \leq i \leq j\} \]

forms its basis. Moreover, $\mathcal{H}_{n,j}$ is invariant under $\sigma$ and $\rho$, and $\theta$ acts as $(-1)^{j}$ on $\mathcal{H}_{n,j}$. Note that

\[ M(1) = \bigoplus_{n=1}^{\infty} \bigoplus_{j=0}^{\infty} \mathcal{H}_{n,j} \]

as vector spaces. Set

\[ \mathcal{H}_{n,j}(k) = \{v \in \mathcal{H}_{n,j} | \sigma v = \zeta^{k}v\}, \quad k = 0, 1, 2. \]

Then $\mathcal{H}_{n,j} = \mathcal{H}_{n,j}(0) \oplus \mathcal{H}_{n,j}(1) \oplus \mathcal{H}_{n,j}(2)$. Note that $\rho(\mathcal{H}_{n,j}(0)) = \mathcal{H}_{n,j}(0)$ and $\rho(\mathcal{H}_{n,j}(1)) = \mathcal{H}_{n,j}(2)$. The dimension of $\mathcal{H}_{n,j}(k)$ is as follows.
Lemma 2.2  (1) If $j \equiv 0 \pmod{3}$, then
\[
\dim \mathcal{H}_{n,j}(0) = j/3 + 1, \quad \dim \mathcal{H}_{n,j}(k) = j/3, \quad k = 1, 2.
\]
(2) If $j \equiv 1 \pmod{3}$, then
\[
\dim \mathcal{H}_{n,j}(0) = (j - 1)/3, \quad \dim \mathcal{H}_{n,j}(k) = (j + 2)/3, \quad k = 1, 2.
\]
(3) If $j \equiv 2 \pmod{3}$, then
\[
\dim \mathcal{H}_{n,j}(k) = (j + 1)/3, \quad k = 0, 1, 2.
\]

From this lemma we know the character of $V_{L}^{k, \pm}$.

Since $L = \sqrt{2}A_{2}$ and $\text{wt} e^{\alpha} = \langle \alpha, \alpha \rangle/2$, the character of $\mathbb{C}[L]$ is nothing but the theta series of the root lattice of type $A_{2}$:
\[
\text{ch} \mathbb{C}[L] = \sum_{\alpha \in L} q^{\text{wt} e^{\alpha}} = \sum_{m,n \in \mathbb{Z}} q^{2m^2 - 2mn + 2n^2} = \theta_{2}(2\tau)\theta_{2}(6\tau) + \theta_{3}(2\tau)\theta_{3}(6\tau),
\]
where $q = \exp(\pi \sqrt{-1} \tau)$ and
\[
\theta_{2}(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2}, \quad \theta_{3}(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}
\]
are Jacobi theta series. Set
\[
\mathbb{C}[L](k) = \{ v \in \mathbb{C}[L] | \sigma v = \zeta^{k}v \}, \quad k = 0, 1, 2.
\]

The automorphism $\sigma$ acts fixed point freely on $L - \{0\}$. Hence
\[
\mathbb{C}[L](0) = \text{span}\{ e^{\alpha} + \sigma e^{\alpha} + \sigma^2 e^{\alpha} \mid 0 \neq \alpha \in L \} \cup \mathbb{C}e^{0},
\]
\[
\mathbb{C}[L](1) = \text{span}\{ e^{\alpha} + \zeta^2 \sigma e^{\alpha} + \zeta \sigma^2 e^{\alpha} \mid 0 \neq \alpha \in L \},
\]
\[
\mathbb{C}[L](2) = \text{span}\{ e^{\alpha} + \zeta \sigma e^{\alpha} + \zeta^2 \sigma^2 e^{\alpha} \mid 0 \neq \alpha \in L \},
\]
and we have

Lemma 2.3 The characters of $\mathbb{C}[L](k)$ are
\[
\text{ch} \mathbb{C}[L](0) = \frac{1}{3} \text{ch} \mathbb{C}[L] + \frac{2}{3},
\]
\[
\text{ch} \mathbb{C}[L](k) = \frac{1}{3} \text{ch} \mathbb{C}[L] - \frac{1}{3}, \quad k = 1, 2.
\]

The character of $V_{L}^{k, \pm}$ follows from the above calculations.
3 Subalgebra $W$

The subalgebra
\[ \{v \in V_L | (s^2)_1 v = 0\} \cong 1_{L(\frac{1}{2}, 0)} \otimes 1_{L(\frac{7}{10}, 0)} \otimes (L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)) \]
is contained in $V_L^0$ (see [5]). Here $1_{L(c, 0)}$ denotes the vacuum vector of $L(c, 0)$. We are interested in the counter part, namely,

\[ W = \{v \in V_L | (\omega^3)_1 v = 0\} \]
\[ = (L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes 1_{L(\frac{4}{5}, 0)}) \oplus (L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}) \otimes 1_{L(\frac{4}{5}, 0)}). \]

The subalgebra $W$ was studied in [8] to construct certain vertex operator algebras associated with $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes. In this note we shall study $W$ as a module for

\[ T' = \text{Vir}(s^2) = L(\frac{6}{5}, 0) \otimes 1_{L(\frac{4}{5}, 0)}. \]

As a $T'$-module, $W$ is completely reducible and each irreducible direct summand is of the form $L(\frac{6}{5}, h)$ for some $h \geq 0$. The characters of irreducible unitary highest weight modules for Virasoro algebras are known (see for example [6], [7], [9]). The characters of those modules appeared above are

\[ \text{ch} L(\frac{1}{2}, 0) = P(q) \sum_{j \in \mathbb{Z}} (q^{(12j+1)} - q^{(3j+1)(4j+1)}), \]
\[ \text{ch} L(\frac{1}{2}, \frac{1}{2}) = q^{\frac{1}{4}} P(q) \sum_{j \in \mathbb{Z}} (q^{(12j+5)} - q^{(3j+2)(4j+1)}), \]
\[ \text{ch} L(\frac{7}{10}, 0) = P(q) \sum_{j \in \mathbb{Z}} (q^{(20j+1)} - q^{(4j+1)(5j+1)}), \]
\[ \text{ch} L(\frac{7}{10}, \frac{3}{2}) = q^{\frac{3}{4}} P(q) \sum_{j \in \mathbb{Z}} (q^{(20j+11)} - q^{(4j+3)(5j+1)}), \]
\[ \text{ch} L(\frac{6}{5}, 0) = P(q)(1 - q), \]
\[ \text{ch} L(\frac{7}{10}, h) = q^h P(q), \quad \text{for} \quad h > 0, \]

where

\[ P(q) = \sum_{n \geq 0} p(n)q^n \]
is the generating function of the partition numbers. The decomposition of $W$ into a direct sum of irreducible $T'$-modules $L(\frac{6}{5}, h)$ will follow if one writes

\[ \text{ch} W = \text{ch} L(\frac{1}{2}, 0) \text{ch} L(\frac{7}{10}, 0) + \text{ch} L(\frac{1}{2}, \frac{1}{2}) \text{ch} L(\frac{7}{10}, \frac{3}{2}) \]
as a linear combination of \( \text{ch} L(\frac{\delta}{5}, h) \)'s. On the other hand, it seems difficult to compute the character of

\[
\{ v \in W \mid \sigma v = \zeta^k v \}, \quad k = 0, 1, 2.
\]

The weight 2 subspace \( W_{(2)} \) is of dimension 3 and spanned by \( \{ w(\alpha_1), w(\alpha_2), w(\alpha_1 + \alpha_2) \} \). Let

\[ v_h = w(\alpha_2) - w(\alpha_1 + \alpha_2). \]

This vector is a highest weight vector in \( L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}) \otimes 1_{L(\frac{4}{5}, 0)} \) for \( L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes 1_{L(\frac{4}{5}, 0)} \), that is,

\[
(\omega^1)_1 v_h = \frac{1}{2} v_h, \quad (\omega^2)_1 v_h = \frac{3}{2} v_h, \\
(\omega^1)_n v_h = (\omega^2)_n v_h = 0 \quad \text{for} \quad n \geq 2.
\]

The vertex operator algebra \( W \) is generated by its weight 2 subspace \( W_{(2)} \). The following property of \( W \) is suggested by Masahiko Miyamoto.

**Proposition 3.1** The vertex operator algebra \( W \) is generated by one vector \( v_h \) or \( u \), where

\[ u = w(\alpha_1) + \zeta^2 w(\alpha_2) + \zeta w(\alpha_1 + \alpha_2) \]

and thus \( \sigma u = \zeta u \). More precisely,

\[
W_{(2)} = \text{span}\{v_h, (v_h)_1 v_h, ((v_h)_1 v_h)_1 ((v_h)_1 v_h)\} = \text{span}\{u, u_1 u, (u_1 u)_1 u\}.
\]

**References**


