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Arithmetical properties of functions satisfying linear q-difference equations: a survey

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0. Why q-difference equations?

Analytically the beautiful and now so classical transcendence results of Hermite, Lindemann and Weierstraß on the exponential function depend heavily on the so-called formulae of Hermite. And behind these is essentially the simple differential equation $Df = f$ of exp.

As it is well-known, all analytic transcendence methods apply to special entire, meromorphic and locally holomorphic functions f satisfying certain types of functional equations, often an algebraic differential equation

$$F(z, f(z), Df(z), \ldots, D^m f(z)) = 0, \tag{1}$$

F being a polynomial in $m+2$ variables. The situation for algebraic independence or even only for irrationality or linear independence is quite similar to that of transcendence.

Having all these facts in mind it is plausible that one shall encounter new arithmetical problems as follows. Replace the differential operator D in (1) by the q-difference operator Δ_q, introduced by Jackson in 1908 and defined by

$$\Delta_q f(z) := \frac{f(qz) - f(z)}{(q-1)z}$$

for complex $q \neq 1$. Then the differential equation (1) changes into an algebraic q-difference equation

$$\phi(z, f(z), f(qz), \ldots, f(q^m z)) = 0 \tag{2}$$

where, after clearing denominators, ϕ is again a polynomial. From the analytical point of view, the linear case of (2), i.e.

$$f(q^m z) = R_0(z) f(q^{m-1} z) + \ldots + R_{m-1}(z) f(z) + R_m(z) \tag{3}$$

with polynomials $R_0, \ldots, R_m$ has been studied intensively since 1890 by Poincaré and others. Clearly, the special case $m = 1$ of (3), i.e.

$$f(qz) = R_0(z) f(z) + R_1(z) \tag{4}$$

is understood even better, compare [22].

1. Irrationality

We now proceed to ask arithmetical questions concerning functions satisfying q-difference equations of type (2), (3) or (4). According to our above remarks we may start with a *q-analogue of the exponential function.* To do so we consider

$$\Delta_q f(z) = f(z) \iff f(z) = \left(1 + \frac{(q-1)z}{q}\right) f\left(\frac{z}{q}\right)$$

with the additional condition $f(0) = 1$.

Supposing $|q| > 1$ for the whole paper,

this initial value problem is solved exactly by the product $\Pi_{j\geq 1}(1+(q-1)q^{-j}z)$, which we finally normalize to

$$E_q(z) := \prod_{j=1}^{\infty} \left(1 + \frac{z}{q^j}\right).$$

This is an entire transcendental function satisfying $E_q(qz) = (1+z)E_q(z)$ and therefore has Taylor series[1]

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod\limits_{\nu=1}^{n}(q^\nu - 1)} \tag{5}$$

about the origin. Historically, this was the *second* example of a solution of a Poincaré equation of type (4) which was studied arithmetically, by Lototsky [16] in 1943, compare our Corollary 2 below.

The *first* such example was

$$T_q(z) := \sum_{n=0}^{\infty} z^n q^{-n(n-1)/2} \tag{6}$$

satisfying $T_q(qz) = qz\,T_q(z) + 1$. Here, using Hermite's method of Padé type approximations of the first kind, Tschakaloff [23] got, in 1921, irrationality and linear independence results, compare Corollary 1 and Theorem 4 below. Clearly, the interest in the function T_q stems from its close connection with the "right half" of Jacobi's theta series $\Sigma_{n\in\mathbb{Z}}\, z^n\, q^{-n^2}$ which equals $T_{q^2}(z/q)$.

The first result we quote precisely is the following Schneider-Lang type criterion.

[1] Empty products are always defined to be 1.

Theorem 1 [4]. *Let K be either $\mathbb{Q}$ or an imaginary quadratic number field, and let $q \in O_K$, the ring of integers of K. Let $f(z) = \Sigma_{n=0}^{\infty} b_n z^n$ be an entire transcendental function such that there exist*

(i) *a sequence $(B_n)_{n=0,1,\ldots}$ in $K^\times$ with $|B_n| \leq |q|^{\beta n^2 + o(n^2)}$ for some fixed real $\beta \geq 0$ such that $B_n b_\nu \in O_K$ for $\nu = 0, \ldots, n$,*

(ii) *an $\alpha \in K^\times$ such that $f(\alpha q^{-m}) \in K$ for any $m \in \mathbb{N}_0 := \{0, 1, \ldots\}$,*

(iii) *a sequence $(C_m)_{m \in \mathbb{N}_0}$ in $K^\times$ with $|C_m| \leq |q|^{\gamma m^2 + o(m^2)}$ for some fixed real $\gamma \geq 0$ such that $C_m f(\alpha q^{-\mu}) \in O_K$ for $\mu = 0, \ldots, m$.*

Then the inequality

$$\rho^*(f) := \limsup_{r \to \infty} \frac{\log |f|_r}{(\log r)^2} \geq \left(\frac{1}{\beta} + \frac{1}{\gamma}\right) \frac{1}{4 \log |q|}$$

holds, where $|f|_r$ denotes the maximum of $|f(z)|$ on $|z| = r$.

As an easy consequence one deduces essentially Tschakaloff's irrationality result [23 I]:

Corollary 1. *If K and q are as in Theorem 1, then $T_q(\alpha) \notin K$ holds for any $\alpha \in K^\times$.*

Remark. Bézivin [2] proved recently for $q \in \mathbb{Z}$ that $\alpha \neq 0$ and $T_q(\alpha)$ cannot both belong to a quadratic number field, and this remains true for certain $q = q_1/q_2 \in \mathbb{Q}$ with sufficiently small value of $(\log |q_2|)/(\log |q_1|)$.

Corollary 2. *If K and q are as in Theorem 1, then $E_q(\alpha) \notin K$ for any $\alpha \in K^\times \setminus \{-q, -q^2, \ldots\}$.*

This is essentially Lototsky's [16] result for which we sketch a *proof*. From (5) we see that $B_n = \Pi_{\nu=1}^{n} (q^\nu - 1)$ and thus $\beta = 1/2$ is a good choice in (i) of Theorem 1. We assume $E_q(\alpha) \in K, = S/T$ say, and we write $\alpha = s/t$ with $S, T, s, t \in O_K$, $Tt \neq 0$. The functional equation of E_q leads to

$$E_q\left(\frac{\alpha}{q^m}\right) = E_q(\alpha) \prod_{j=1}^{m} \left(1 + \frac{\alpha}{q^j}\right)^{-1} = \frac{S t^m q^{m(m+1)/2}}{T \prod_{j=1}^{m} (t q^j + s)}$$

such that we may take $C_m = T \Pi_{j=1}^{m} (t q^j + s)$ and thus $\gamma = 1/2$ in (iii) of Theorem 1. Therefore we get $\rho^*(E_q) \geq 1/(\log |q|)$ contradicting the well-known fact $\rho^*(E_q) = 1/(2 \log |q|)$, compare [4] or [22].

Remarks. 1) The "huge" final contradiction in the proof of Corollary 2 can be used to get the following quantitative refinement.

Theorem 2. *Under the hypotheses of Corollary 2, and for arbitrary $\varepsilon \in \mathbb{R}_+$ one has*

$$|E_q(\alpha) - \frac{P}{Q}| \gg_\varepsilon |Q|^{-\frac{7}{3}-\varepsilon}$$

for any $P, Q \in O_K$, $Q \neq 0$.

This was first proved in [4], and later, with a different method by Popov [19] who replaced the ε in the exponent by $O((\log|Q|)^{-1/2})$. In the case $\alpha = 1$ (and thus $\alpha = q^r$ for any $r \in \mathbb{Z}$) the present author could very recently replace $7/3$ by $13/6$.

2) Euler's formula from 1748, namely

$$\frac{1}{E_q(-1)} = \prod_{j=1}^{\infty}(1-q^{-j})^{-1} = 1 + \sum_{n=1}^{\infty} p(n) q^{-n}, \tag{7}$$

p denoting the partition function, allows to deduce arithmetical informations on the series in (7) from those on the E_q function.

Of course, one may also consider infinite products of the shape

$$F_q(z) := \prod_{j=1}^{\infty}\left(1 + \frac{z}{q^j} + c_2\frac{z^2}{q^{2j}} + \cdots + c_\ell \frac{z^\ell}{q^{\ell j}}\right)$$

generalizing E_q, with $c_2, \ldots, c_\ell \in K$, $c_\ell \neq 0$. Clearly, F_q satisfies the functional equation

$$F_q(qz) = (1 + z + c_2 z^2 + \cdots + c_\ell z^\ell) F_q(z)$$

of type (4). In the case $\ell = 2$, $c_2 \in K^\times$, first considered by Zhou and Lubinsky [25], we have

Theorem 3 [5]. *Let K and q be as in Theorem 1, and let $\alpha \in K^\times$ satisfy $1 \pm \alpha q^{-j} + c_2\alpha^2 q^{-2j} \neq 0$ for any $j \in \mathbb{N} := \{1, 2, \ldots\}$. Then $F_q(\alpha)$ and $F_q(-\alpha)$ cannot both belong to K, and moreover*

$$\max_{\nu=0,1} |F_q((-1)^\nu \alpha) - \frac{P_\nu}{Q}| \gg |Q|^{-7-\varepsilon}.$$

Zhou and Lubinsky, using explicit formulae for multivariate Padé approximants, could only treat the qualitative case of $q, c_2, \alpha \in \mathbb{Q}_+$. The following Conjecture is proved only if $|q|$ is large in terms of c_2 and α, compare [5].

Conjecture. *$F_q(\alpha) \notin K$ should be true under the hypotheses of Theorem 3, but without any assumption* on $1 - \alpha q^{-j} + c_2\alpha^2 q^{-2j}$.

The case $\ell \geq 3$ was considered only very recently by Väänänen and the present author [9]. Unfortunately, we have no result at all for $\ell \geq 4$, and even if $\ell = 3$ we need very special c_2, c_3, we must restrict K to $\mathbb{Q}$ or the Gaussian field $\mathbb{Q}(i)$, and we can only say that for $\alpha \in K^\times$ (plus some natural non-vanishing condition) at least one of the $F_q(i^\nu\alpha)$, $\nu = 0,1,2,3$, doesn't belong to K and furthermore

$$\max_{\nu=0,1,2,3} |F_q(i^\nu\alpha) - \frac{P_\nu}{Q}| \gg |Q|^{-25-\varepsilon}.$$

2. Linear independence

Using Hermite's analytic method, i.e. Padé approximations of the first kind and a non-vanishing argument for a certain determinant, Tschakaloff [23 II] proved in 1921 the *qualitative* part of

Theorem 4 [6]. *Let K and q be as in Theorem 1, and let $\alpha_1, \ldots, \alpha_\ell \in K^\times$ satisfy $\alpha_i/\alpha_j \notin q^{\mathbb{Z}}$ for $i \neq j$ (if $\ell > 1$). Then, for arbitrary $\varepsilon \in \mathbb{R}_+$, the inequality*

$$(8) \qquad |h_0 + h_1 T_q(\alpha_1) + \cdots + h_\ell T_q(\alpha_\ell)| \gg_\varepsilon H^{-(2\ell-1+(4\ell^2+1)^{1/2})/2-\varepsilon}$$

holds for any $\underline{h} = (h_0, \ldots, h_\ell) \in O_K^{\ell+1} \setminus \{\underline{0}\}$ with $|h_\lambda| \leq H$ for $\lambda = 1, \ldots, \ell$.

Remarks. 1) In (8), the estimate $|\ldots| \gg H^{-\ell}$ would be best possible; the actual exponent there is essentially 2ℓ for large ℓ.

2) By the same method as in [6], Katsurada [15] generalized (8) to

$$(9) \qquad |h_0 + \sum_{\lambda=1}^{\ell} \sum_{\mu=0}^{m} h_{\lambda\mu} T_q^{(\mu)}(\alpha_\lambda)| \gg_\varepsilon H^{-c(\ell,m)-\varepsilon}$$

with some explicit constant $c(\ell, m)$.

3) For $q \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{Q}^\times$, $\alpha_i/\alpha_j \notin q^{\mathbb{Z}}$ for $i \neq j$, the pure linear independence of 1 and the $T_q^{(\mu)}(\alpha_\lambda)$ over $\mathbb{Q}$ was yet shown by Skolem [20] in 1949. He used Hilbert's method, as devised in 1893 for the transcendence of e and π, which is very arithmetic in nature and based on divisibility considerations.

Both above-mentioned methods, Hermite's and Hilbert-Skolem's, depend on direct constructions of appropriate diophantine approximations. In contrast to this, Bézivin [1] developed in the late 1980's a more function-theoretic method for linear independence of values of certain entire functions generalizing T_q. Key point in his argument is a criterion à la Kronecker or Borel-Dwork for the rationality of a suitable auxiliary function. But, at least until now, no quantitative version of Bézivin's linear independence method became available.

Using a number field version of Hilbert-Skolem's reasoning Wallisser and the author [12] proved recently the following quantitative linear independence result à la Bézivin.

Theorem 5. *Let the subsequent hypotheses be satisfied.*

(i) $q_1, \ldots, q_r \in \overline{\mathbb{Q}}^\times$ *are one or several sets of conjugates of integers, multiplicatively independent, and one dominating in absolute value;*

(ii) *there is a prime ideal* $\mathfrak{p} \subset O_K$, *where* $K := \mathbb{Q}(q_1, \ldots, q_r)$, *which divides all principal ideals* $(q_1), \ldots, (q_r)$;

(iii) *with* $\beta_1, \ldots, \beta_r \in \mathbb{Q}^\times$ *let* $A(\nu) := \beta_1 q_1^\nu + \cdots + \beta_r q_r^\nu \neq 0$ *for* $\nu \in \mathbb{N}$;

(iv) *for* $\alpha_1, \ldots, \alpha_\ell \in K^\times$ *no quotient* α_i/α_j *with* $i \neq j$ *(if* $\ell > 1$*) belongs to the subgroup* $\langle q_1, \ldots, q_r \rangle$ *of* $K^\times$ *generated by* $q_1, \ldots, q_r$;

(v) $\alpha_i/A(\nu) \in \mathbb{Q}$ *for any* $i = 1, \ldots, \ell$ *and* $\nu \in \mathbb{N}$.

Then, for the entire transcendental function

$$f(z) := \sum_{n=0}^{\infty} \frac{z^n}{\prod_{\nu=1}^{n} A(\nu)}$$

the following estimate holds

$$\log |h_0 + h_1 f(\alpha_1) + \cdots + h_\ell f(\alpha_\ell)| \gg -(\log H)^{2r/(r+1)}. \tag{10}$$

Remarks. 1) If $r = 1$, then $q := q_1 \in \mathbb{Z}$; choosing $\beta := \beta_1 = 1/q$ we get $A(\nu) = q^{\nu-1}$, thus $\Pi_{\nu=1}^{n} A(\nu) = q^{n(n-1)/2}$, and therefore our f reduces to the Tschakaloff function T_q from (6). Since the exponent $2r/(r+1)$ is 1, we get back essentially the linear independence measure (8) of Theorem 4, of course now with an unspecified exponent of H.

2) We could include derivatives of f, too, as Bézivin [1] did as well as Katsurada, compare (9).

3) It should be pointed out that we could prove very recently [12 II] Theorem 5, with $2(r+1)/(r+2)$ as new exponent of $\log H$ in (10), replacing condition (iii) by the following more general one.

(iii') *with* $R_j \in K[X] \setminus \{0\}$ *for* $j = 1, \ldots, r$ *let* $A(\nu) := R_1(\nu) q_1^\nu + \cdots + R_r(\nu) q_r^\nu \neq 0$ *for* $\nu \in \mathbb{N}$;

4) Concerning applications, mainly to functions f where $A(.)$ is connected with so-called PV-numbers, the reader is referred to [12], [12 II].

Since the case $A(\nu) = q^\nu - 1$ is covered by Bézivin's results [1], but not by Theorem 5, we cannot apply this quantitative result to $f = E_q$. But using Padé approximations of the second kind and some Siegel-Shidlovsky type arguments, Väänänen [24] found very recently quite general linear independence measures which we quote here only for the function E_q and for K and q as in Theorem 1.

Theorem 6. *If $\alpha_1, \dots, \alpha_\ell \in K^\times \setminus \{-q, -q^2, \dots\}$ satisfy $\alpha_i/\alpha_j \notin q^{\mathbb{Z}}$ for $i \neq j$, then $1, E_q(\alpha_1), \dots, E_q(\alpha_\ell)$ are linearly independent over K, and moreover the inequality*

$$|h_0 + h_1 E_q(\alpha_1) + \cdots + h_\ell E_q(\alpha_\ell)| \gg_\varepsilon H^{-c-\varepsilon} \tag{11}$$

holds with an explicitly computed constant $c > 0$.

Remark. Here c is asymptotically $c_1 \ell^3$ for large ℓ, where $c_1 > 0$ doesn't depend on ℓ. It should be pointed out that Väänänen and the author [10], using explicit approximations, got very recently a result which, in the particular case of the function E_q, reads as follows

$$|h_0 + h_1 E_q(\alpha) + h_2 E_q(-\alpha)| \gg_\varepsilon H^{-8-\varepsilon},$$

if $K = \mathbb{Q}$ and $\alpha \in \mathbb{Q}^\times \setminus \{-q, -q^2, \dots\}$.

3. Dimension estimates

Here the main problem is as follows. Given $\underline{\omega} = (\omega_1, \dots, \omega_m) \in \mathbb{R}^m \setminus \{\underline{0}\}$ with $m \geq 2$, find conditions for non-trivial lower bounds for

$$D_{\mathbb{Q}}(\underline{\omega}) := \dim_{\mathbb{Q}} \mathbb{Q}\omega_1 + \cdots + \mathbb{Q}\omega_m.$$

Clearly, $D_{\mathbb{Q}}(\underline{\omega}) = m$ holds if and only if $\omega_1, \dots, \omega_m$ are linearly independent over $\mathbb{Q}$.

With Töpfer, we [7] found an axiomatization of a method of Nesterenko [17] to estimate $D_{\mathbb{Q}}(\underline{\omega})$ from below, based on linear elimination theory. But whereas Nesterenko's result leads only to qualitative statements, our generalization gives quantitative results, too. As *one* handy consequence of our (rather cumbersome) main assertion we get back Nesterenko's

Theorem 7. *Suppose $k_0 \in \mathbb{N}$ and $\tau_1, \tau_2 \in \mathbb{R}_+$. Let $\phi : \mathbb{N} \to \mathbb{R}_+$ be monotonically increasing and unbounded, and let $(\Lambda_k(\underline{X}) := \lambda_{k1}X_1 + \cdots + \lambda_{km}X_m)_{k=k_0,k_0+1,\dots}$ be a sequence of linear forms over $\mathbb{Z}$ satisfying*

(i) $\limsup_{k\to\infty} \phi(k+1)/\phi(k) \leq 1$, *and for* $k \in \mathbb{N}, k \geq k_0$

(ii) $\frac{1}{2}\log(\lambda_{k1}^2 + \cdots + \lambda_{km}^2) \leq \phi(k)$,

(iii) $-\tau_1\phi(k) \leq \log|\Lambda_k(\underline{\omega})| \leq -\tau_2\phi(k)$.

Then $D_{\mathbb{Q}}(\underline{\omega}) \geq (1+\tau_1)/(1+\tau_1-\tau_2)$.

Remark. It should be noted that Töpfer [21] generalized our joint results in [7] from $\mathbb{Q}$ to arbitrary algebraic number fields.

From Theorem 7 we deduced with Väänänen [8] the following result concerning the function E_q.

Theorem 8. *If* $q \in \mathbb{Z}$ *and* $\alpha \in \mathbb{Q}^\times \setminus \{-q, -q^2, \ldots\}$, *then the inequality*

$$D_{\mathbb{Q}}(E_q(\alpha), E_q'(\alpha), \ldots, E_q^{(m-1)}(\alpha)) \geq \frac{m(m+1)}{2m + 6\pi^{-2}(m-1)}$$

holds for $m = 1, 2, 3$, *and a similar estimate is true for any* $m \geq 4$. *In particular,* $E_q'(\alpha)/E_q(\alpha)$ *is irrational.*

Our *proof* uses (a bit more than) integrals of the shape

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{E_q(z)dz}{\prod_{\kappa=0}^{k}(z-\alpha q^\kappa)^m} = \sum_{\mu=0}^{m-1} R_\mu(\alpha, q; k)E_q^{(\mu)}(\alpha) \tag{12}$$

with, by the residue theorem, explicit $R_\mu \in \mathbb{Q}(\alpha, q)$ to get an approximation sequence (Λ_k) as needed for Theorem 7. Note that the integrals (12) can be asymptotically evaluated, using Popov's procedure from [19].
The particular case $m = 2$ of Theorem 8 can be restated as follows. Consider the meromorphic function

$$L_q(z) := \frac{E_q'(z)}{E_q(z)} = \sum_{j=1}^{\infty} \frac{1}{q^j + z}$$

which can be regarded as a *q-analogue of the logarithm* since $(q-1)zL_q(z)$, tending to $\log(1+z)$ as $q \to 1$, solves the initial value problem $\Delta_q f(z) = 1/(1+z)$, $f(0) = 0$.

Corollary 3. *For* q *and* α *as in Theorem 8,* $L_q(\alpha)$ *is irrational.*

Originally, this was proved by Borwein [3] in 1991. Much earlier, in 1948 Erdös [14] settled the particular case $\alpha = -1, q \in \mathbb{N}$ using the representation $L_q(-1) = \Sigma_{n\geq 1} d(n)q^{-n}$, $d(\,.\,)$ denoting the divisor function.

Our main result with Töpfer [7] was general enough to deduce here even quite good irrationality measures as

$$|h_0 + h_1 L_q(\alpha)| \gg H^{-3.310} \text{ and } |h_0 + h_1 L_q(-1)| \gg H^{-1.508}.$$

4. Transcendence

It is not difficult to ask (open) transcendence questions in the domain under consideration. Suppose $q, \alpha \in \overline{\mathbb{Q}}^{\times}$. Is it true that $T_q(\alpha)$ is transcendental? Are $E_q(\alpha)$, $L_q(\alpha)$ transcendental if $\alpha \neq -q, -q^2, \ldots$?

Certainly, one would first try classical analytic transcendence methods as Gel'fond's or Schneider's. The results, so far obtainable by these two methods concerning entire transcendental solutions of (4), are described in [11]. But they are rather weak and, having transcendence in mind, not promising at all.

With Mahler's method the state of affairs could seem slightly more favorable. To explain why, we describe the main hypotheses of this method very roughly, and show its connection with the above conjecture concerning $T_q(\alpha)$, for instance.

If $\Omega = (\omega_{ij}) \in \mathrm{Mat}(t \times t; \mathbb{N}_0)$ and $\underline{z} = (z_1, \ldots, z_t) \in \mathbb{C}^t$ we denote

$$\Omega\underline{z} := \left(\prod_{j=1}^{t} z_j^{\omega_{1j}}, \ldots, \prod_{j=1}^{t} z_j^{\omega_{tj}} \right).$$

We suppose that f is a complex-valued function, holomorphic in some neighborhood $B(\underline{0})$ of the origin in $\mathbb{C}^t$, with Taylor coefficients about $\underline{0}$ in some fixed algebraic number field, and satisfying a functional equation

$$f(\Omega\underline{z}) = \left(\sum_{\mu=0}^{m} a_\mu(\underline{z}) f(\underline{z})^\mu \right) \Big/ \left(\sum_{\mu=0}^{m} b_\mu(\underline{z}) f(\underline{z})^\mu \right)$$

with all a_μ, b_μ in $\mathbb{C}[\underline{z}]$, a_m, b_m not both 0, and with

$$1 \leq m < r(\Omega). \tag{13}$$

Here $r(\Omega)$ denotes the spectral radius of the matrix Ω, and if λ is an eigenvalue of Ω of absolute value $r(\Omega)$, then $\lambda = r(\Omega)$ must hold. Under these assumptions, $f(\underline{\alpha})$ is transcendental for any $\underline{\alpha} \in B(\underline{0}) \cap (\overline{\mathbb{Q}}^{\times})^t$ (plus a further, but harmless condition).

Let us look at $f(z_1, z_2) := \Sigma_{n=0}^{\infty} z_1^n z_2^{n(n-1)/2}$ in $\mathbb{C} \times U =: B(\underline{0})$, U denoting the unit circle. Obviously, we have $z_1 f(z_1 z_2, z_2) = f(z_1, z_2) - 1$, and therefore our f satisfies

$$f(\Omega\underline{z}) = \frac{f(\underline{z}) - 1}{z_1} \text{ with } \Omega = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

From $f(\alpha, q^{-1}) = T_q(\alpha)$ we could conclude the above conjecture if inequality (13) would not barely missed since (14) implies $1 = m = r(\Omega)$.

In a letter exchange on the $T_q(\alpha)$ problem in the early 1980's with Kurt Mahler, he wrote (in German): "In spite of many investigations, the transcendence of theta functions remains unsolved, and thus every partial result is of great interest... and I guess that the transcendence of $T_q(\alpha)$ will not be solved before the next century."

Nevertheless, *a very big progress* in this area was made recently by Nesterenko [18] who proved even algebraic independence results for certain numbers related to modular functions. It is interesting to note that no classical transcendence method comes nearer to Nesterenko's reasoning than Mahler's method does. Shortly later, Duverney, the two Nishiokas and Shiokawa [13] deduced from Nesterenko's results the transcendence of some particular values of the functions T_q and E_q, namely, for instance, of $T_q(1)$, $T_{q^2}(1/q)$, $E_q(\pm 1)$ and $\Sigma_{n=1}^{\infty} p(n) q^{-n}$ for any $q \in \overline{Q}$ with $|q| > 1$.

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