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**概要**
この論文では、非常に興味深い数列についての研究が行われています。作成者はTanaka, Taka-akiで、2001年7月に数理解析研究所講究録1219号で発表されました。論文のページ数は122-130ページです。URLはhttp://hdl.handle.net/2433/41265です。
Transcendence of the values of certain lacunary series

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1 Introduction.

Let \( f(z) = \sum_{k=0}^{\infty} z^{e_k} \) be a power series in the complex variable \( z \) with a strictly increasing sequence \( \{e_k\}_{k \geq 0} \) of exponents. From the Hadamard's gap theorem, if \( \lim \inf_{k \to \infty} e_{k+1}/e_k > 1 \), then \( f(z) \) has the unit circle \( |z| = 1 \) as a natural boundary. The transcendence of the value \( f(\alpha) \) of such a series at a nonzero algebraic number \( \alpha \) inside the unit circle has been investigated by various authors. In 1844, Liouville proved the transcendency of \( \sum_{k=0}^{\infty} 2^{-k!} \), the first example of a transcendental number. For the case of \( \lim \sup_{k \to \infty} e_{k+1}/e_k = \infty \), there were some results on the transcendence of \( f(\alpha) \), which are included in the result of Cijssouw and Tijdeman [1]. On the other hand, only special sequences \( \{e_k\}_{k \geq 0} \) have been treated in the remaining case of \( \lim_{k \to \infty} e_{k+1}/e_k < \infty \). Let \( d \) be an integer greater than 1. In 1929, Mahler [3] proved that, if \( e_k = d^k \), \( f(\alpha) \) is transcendental. Mahler's method was generalized by Loxton and van der Poorten [2], who proved the transcendence of \( f(\alpha) \) when \( \{e_{k+1}/e_k\}_{k \geq 0} \) is a sequence of integers greater than 1. However, for the case that \( \lim_{k \to \infty} e_{k+1}/e_k = d \) and \( \{e_{k+1}/e_k\}_{k \geq 0} \) is not necessarily a sequence of integers, for example \( e_k = kd^k \), no transcendence result had been known. In this paper we prove the transcendence of \( f(\alpha) \) under these conditions.

**Theorem 1.** Let \( \{r_k\}_{k \geq 0} \) be a sequence of positive integers such that \( \lim_{k \to \infty} r_{k+1}/r_k = d \), where \( d \) is an integer greater than 1. Suppose that there exists a positive number \( M \) such that \( r_{k+1} \geq dr_k - M \) for all \( k \geq 0 \). Let

\[
f(z) = \sum_{k=0}^{\infty} z^{r_k}
\]

and let \( \alpha \) be an algebraic number with \( 0 < |\alpha| < 1 \). Then the number \( f(\alpha) \) is transcendental.
EXAMPLE. Let \( \alpha \) be an algebraic number with \( 0 < |\alpha| < 1 \) and \( d \) an integer greater than 1. Then the numbers

\[
(1) \quad \sum_{k=0}^{\infty} \alpha^{kd^k}, \quad \sum_{k=0}^{\infty} \alpha^{2kd^k+(-d)^k}, \quad \sum_{k=0}^{\infty} \alpha^{[\omega d^k+\eta]}, \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha^{k\binom{2k}{n}}
\]

are transcendental, where \( \omega > 0, \eta \geq 0, [x] \) denotes the largest integer not exceeding a real number \( x \), and \( \binom{m}{n} \) is the binomial coefficient.

Applying Mahler's method, we proved in [5] the transcendence of the number \( \sum_{k=0}^{\infty} \alpha^{a_k} \) generated by a linear recurrence \( \{a_k\}_{k \geq 0} \) of nonnegative integers with \( a_k = g \rho^k + o(\rho^k) \), where \( g > 0 \) and \( \rho > 1 \), under some additional conditions. However, the transcendence of the first two numbers in (1) cannot be deduced from our result in [5] although the sequences of their exponents are linear recurrences.

Theorem 1 can be deduced from Theorem 2 below. We prepare the notation for stating the theorem. For any algebraic number \( \alpha \), we denote by \( \overline{|\alpha|} \) the maximum of the absolute values of the conjugates of \( \alpha \) and by \( \text{den}(\alpha) \) the smallest positive integer such that \( \text{den}(\alpha) \cdot \alpha \) is an algebraic integer. It is easily seen that \( \overline{|\alpha + \beta|} \leq \overline{|\alpha|} + \overline{|\beta|} \) and \( \overline{|\alpha \beta|} \leq \overline{|\alpha|} \overline{|\beta|} \) for any algebraic numbers \( \alpha \) and \( \beta \). Furthermore, for any algebraic number \( \alpha \), we define

\[
\|\alpha\| = \max\{\overline{|\alpha|}, \text{den}(\alpha)\}.
\]

Then for any \( \alpha \neq 0 \) we have the inequalities

\[
(2) \quad \log |\alpha| \geq -2[Q(\alpha) : Q]\log \|\alpha\|
\]

and

\[
(3) \quad \log \|\alpha^{-1}\| \leq 2[Q(\alpha) : Q]\log \|\alpha\|
\]

(cf. [4, Lemma 2.10.2]).

Let \( K \) be an algebraic number field. We denote by \( K[[z]] \) the ring of formal power series in the variable \( z \) with coefficients in \( K \). Let

\[
f_k(z) = \sum_{l=0}^{\infty} \sigma^{(k)}_l z^l \in K[[z]] \quad (k \geq 0)
\]

and let \( \alpha \in K \) with \( 0 < |\alpha| < 1 \). In what follows, \( c_1, c_2, \ldots \) denote positive constants independent of \( k \) and depending only on \( f_k(z) \) \( (k \geq 0) \) and \( \alpha \), and if they may depend also on parameters \( x \) as well as \( y \), they will be denoted by \( c_1(x), c_2(x, y), \ldots \). Let \( \{r_k\}_{k \geq 0} \) be a sequence of positive integers with the following properties:
(I) $r_k \rightarrow \infty$ as $k$ tends to infinity;

(II) $f_k(\alpha^{r_k}) = a_k f_0(\alpha) + b_k$ ($k \geq 1$), where $a_k, b_k \in K$ and

$$\log ||a_k||, \log ||b_k|| \leq c_1 r_k;$$

(III) for any $\epsilon > 0$ and for any $l \geq 0$, there exists a constant $c_2(\epsilon, l) > 0$ such that

$$\log ||\sigma^{(k)}_l|| \leq \epsilon r_k (1 + l)$$

for all $k \geq c_2(\epsilon, l);$ 

(IV) for any $\epsilon > 0$ there exists a constant $c_3(\epsilon) > 0$ such that

$$\log |\sigma^{(k)}_l| \leq \epsilon r_k (1 + l)$$

for all $k \geq c_3(\epsilon)$ and for any $l \geq 0$.

Let $s_0, s_1, \ldots$ be variables and put $F(z; s) = \sum_{l=0}^{\infty} s_l z^l$. Then $F(z; \sigma^{(k)}) = f_k(z)$ ($k \geq 0$). We assume that

(V) if $P_0(z; s), \ldots, P_p(z; s)$ are polynomials in $z$ and $\{s_l\}_{l \geq 0}$ with degrees at most $p$ in $z$ and coefficients in $K$ and if we put

$$E(z; s) = \sum_{j=0}^{p} P_j(z; s) F(z; s)^j = \sum_{l=0}^{\infty} R_l(s) z^l,$$

then there exists a positive integer $I(p)$, independent of $k$ and depending only on $F(z; s)$ and $p$, with the following property. If $k$ is sufficiently large and $P_0(z; \sigma^{(k)}), \ldots, P_p(z; \sigma^{(k)})$ are not all zero, then there is an $l$ such that $l \leq I(p)$ and $R_l(\sigma^{(k)}) \neq 0$.

**Theorem 2.** *If the properties (I) – (V) are satisfied, then the number $f_0(\alpha)$ is transcendental.*

**Remark.** If the constant $c_2(\epsilon, l)$ in the property (III) does not depend on $l$, then the property (IV) is satisfied by the property (III). This is the very case that Loxton and van der Poorten [2] dealt with.
2 Proof of the theorems.

Proof of Theorem 1. We may assume that \( r_0 = 1 \), replacing \( r_0, r_1, r_2, \ldots \) by \( 1, r_0, r_1, \ldots \) if necessary. Define

\[
f_k(z) = \sum_{h=0}^{\infty} \alpha^{r_{h+k} - r_k d^h} z^{d^h} \quad (k \geq 0).
\]

Then

\[
\sigma_l^{(k)}(z) = \begin{cases} \alpha^{r_{h+k} - r_k d^h} & (l = d^h) \\ 0 & \text{(otherwise)} \end{cases}
\]

and \( f_0(\alpha) = \sum_{h=0}^{\infty} \alpha^{r_h} = f(\alpha) \), which is transcendental by Theorem 2 if the properties (I) – (V) are satisfied.

The sequence \( \{r_k\}_{k \geq 0} \) obviously has the property (I). Let \( K = Q(\alpha) \). Then \( f_k(z) \in K[[z]] \) \((k \geq 0)\) and

\[
f_k(\alpha) = \sum_{h=0}^{\infty} \alpha^{r_{h+k}} = f_0(\alpha) - \sum_{h=0}^{k-1} \alpha^{r_h}.
\]

Since \( r_{k+1} > r_k \) for all sufficiently large \( k \) by the assumption, there is a constant \( C \geq 1 \) such that \( \max_{0 \leq h \leq k-1} r_h \leq Cr_k \) for all \( k \geq 1 \). Hence

\[
\log \left\| - \sum_{h=0}^{k-1} \alpha^{r_h} \right\| \leq \log k + \left( \max_{0 \leq h \leq k-1} r_h \right) \log \| \alpha \| \leq c_1 r_k,
\]

and the property (II) is satisfied.

Using (3), we have

\[
\log \left\| \alpha^{r_{h+k} - r_k d^h} \right\| \leq 2[K : Q]\|r_{h+k} - r_k d^h\| \log \| \alpha \|.
\]

By (4), (5), and \( \|0\| = 1 \), in order to prove that the property (III) is satisfied, it suffices to show that for any \( \varepsilon > 0 \) and for any \( h \geq 0 \), there exists a constant \( c_2(\varepsilon, h) > 0 \) such that

\[
| r_{h+k} - r_k d^h | \leq \varepsilon r_k d^h
\]

for all \( k \geq c_2(\varepsilon, h) \). If \( h = 0 \), this inequality holds for all \( k \geq 0 \). Since \( \lim_{k \to \infty} r_{k+1}/r_k = d \), for any \( \varepsilon > 0 \) and for any \( h \geq 1 \), there exists a constant \( c_2(\varepsilon, h) > 0 \) such that

\[
1 - \frac{\varepsilon}{(1 + \varepsilon) h} < \frac{r_{k+1}}{d r_k} < 1 + \frac{\varepsilon}{(1 + \varepsilon) h}
\]
for all $k \geq c_2(\epsilon, h)$. Then

$$\frac{|r_{h+k} - r_k d^h|}{r_k d^h} = \frac{r_{k+h} \cdots r_{k+1}}{dr_{k+h-1} \cdots dr_k} - 1 \leq \sum_{m=1}^{h} h^m \left( \frac{\epsilon}{(1 + \epsilon)h} \right)^m \leq \frac{\epsilon}{1 + \epsilon} = \epsilon.$$

Next we prove that the property (IV) is satisfied. Since

$$r_{h+k} - r_k d^h = (r_{k+h} - dr_{k+h-1}) + d(r_{k+h-1} - dr_{k+h-2}) + \cdots + d^{h-1}(r_{k+1} - dr_k) \geq -M(1 + d + \cdots + d^{h-1})$$

by the assumption in the theorem,

$$\log |\sigma_{d^h}^{(k)}| = (r_{h+k} - r_k d^h) \log |\alpha| \leq \frac{-M(d^h - 1)}{d - 1} \log |\alpha| < -M(1 + d^h) \log |\alpha|.$$ 

Then for any $\epsilon > 0$ there exists a constant $c_3(\epsilon) > 0$ such that $\epsilon r_k \geq -M \log |\alpha|$ for all $k \geq c_3(\epsilon)$, and the property (IV) is fulfilled.

Finally we show that the property (V) is satisfied by the same way as in the proof of Theorem 2.10.1 in [4]. Choose a positive integer $\lambda(p)$, depending on $p$, such that

$$\max_{0 \leq j \leq p} \deg_z P_j(z; s) < d^{\lambda(p)}.$$

Suppose that $P_0(z; \sigma^{(k)}), \ldots, P_p(z; \sigma^{(k)})$ are not all zero and put

$$p' = p'(k) = \max\{j \mid P_j(z; \sigma^{(k)}) \neq 0\}, \quad a = a(k) = \deg_z P_{p'}(z; \sigma^{(k)}).$$

Then

$$E(z; \sigma^{(k)}) = \sum_{j=0}^{p'} P_j(z; \sigma^{(k)}) f_k(z)^j = \sum_{l=0}^{\infty} R_l(\sigma^{(k)}) z^l.$$ 

We prove that $R_l(\sigma^{(k)}) \neq 0$ for some $l$. This can be done by choosing

$$l = a + \sum_{m=1}^{p'} d^{\lambda(p)+m}$$

and considering the $d$-adic expansion of the positive integer $l$ in place of the $\{d_1, d_2, \ldots\}$-adic expansion in the proof of Theorem 2.10.1 in [4]. Since $a(k) < d^{\lambda(p)}$ and $p'(k) \leq p$ for any $k$, we can take $I(p) = d^{\lambda(p)+p+1}$ and the property (V) is fulfilled. Then by Theorem 2, $f(\alpha)$ is transcendental, and the proof of the theorem is completed.
We prove Theorem 2 by the method of Loxton and van der Poorten [2] and Nishioka [4].

Proof of Theorem 2. We assume on the contrary that \( f_0(\alpha) \) is algebraic. We may suppose \( f_0(\alpha) \in K \).

Proposition 1 (Loxton and van der Poorten [2], see also Nishioka [4, Proposition 2.9.2]). Let \( m \) be a nonnegative integer. There exists an infinite subset \( \Lambda(m) \) of the set \( N \) of positive integers such that for any polynomial \( P(s_0, \ldots, s_m) \in K[s_0, \ldots, s_m] \) the following two properties are equivalent:

(i) \( P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0 \) for infinitely many \( k \in \Lambda(m) \).

(ii) \( P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0 \) for all \( k \in \Lambda(m) \).

Let \( m \) be a nonnegative integer and put

\[
V(m) = \{ P(s_0, \ldots, s_m) \in K[s_0, \ldots, s_m] \mid P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0 \text{ for all } k \in \Lambda(m) \}.
\]

Then \( V(m) \) is a prime ideal of \( K[s_0, \ldots, s_m] \) by Proposition 1.

Proposition 2 (Loxton and van der Poorten [2], see also Nishioka [4, Proposition 2.9.3]). For any positive integer \( p \), there exist \( p + 1 \) polynomials \( P_0(z; s_0, \ldots, s_{p^2}), \ldots, P_p(z; s_0, \ldots, s_{p^2}) \in K[z, s_0, \ldots, s_{p^2}] \) with degrees at most \( p \) in \( z \) such that the function

\[
E_p(z; s) = \sum_{j=0}^{p} P_j(z; s_0, \ldots, s_{p^2}) F(z; s)^j = \sum_{l=0}^{\infty} R_l(s) z^l
\]

has the following two properties:

(i) \( R_l(s) = R_l(s_0, \ldots, s_{p^2}) \in V(p^2) \) for all \( l \) with \( l \leq p^2 \);

(ii) there exists a positive integer \( I(p) \), independent of \( k \) and depending only on \( F(z; s) \) and \( p \), such that \( \text{ord}_{z=0} E_p(z; \sigma^{(k)}) \leq I(p) \) for all sufficiently large \( k \in \Lambda(p^2) \).

Proposition 3. For any positive integer \( p \) and any positive number \( \epsilon \), if \( k \geq c_4(\epsilon, p) \), then

\[
\log \| E_p(\alpha^k; \sigma^{(k)}) \| \leq \epsilon r_k c_5(p) + c_6 r_k p.
\]
Proof. By the property (III), \( \| \sigma_{l}^{(k)} \| \leq e^{r_{k}(1+l)} \) for all \( k \geq c_{2}(\epsilon, l) \). Let 
\[ P_{j}(z; s_{0}, \ldots, s_{p^{2}}) = \sum_{l=0}^{p} Q_{jl}(s_{0}, \ldots, s_{p^{2}}) z^{l}. \]
Since \( Q_{jl}(s_{0}, \ldots, s_{p^{2}}) \in K[s_{0}, \ldots, s_{p^{2}}] \), we have
\[ \| Q_{jl}(\sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)}) \| \leq c_{7}(p)e^{e_{k}c_{8}(p)} \]
for all \( k \geq \max_{0 \leq l \leq p^{2}} c_{2}(\epsilon, l) \). Since
\[ E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) = \sum_{j=0}^{p} P_{j}(\alpha^{r_{k}}; \sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)}) F(\alpha^{r_{k}}; \sigma^{(k)})^{j} \]
\[ = \sum_{j=0}^{p} P_{j}(\alpha^{r_{k}}; \sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)}) F(\alpha^{r_{k}}; \sigma^{(k)})^{j} \]
\[ = \sum_{j=0}^{p} P_{j}(\alpha^{r_{k}}; \sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)}) (a_{k}f_{0}(\alpha) + b_{k})^{j} \]
noting that \( \| \alpha^{r_{k}} \| \leq c_{9}^{r_{k}} \), we obtain
\[ \| E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) \| \leq c_{10}(p)e^{e_{k}c_{11}(p)c_{9}^{p}} \left( e^{2c_{1}r_{k}}(\| f_{0}(\alpha) \| + 1) \right)^{p} \]
for \( k \geq \max_{0 \leq l \leq p^{2}} c_{2}(\epsilon, l) \), which implies the proposition.

Proposition 4. For any positive integer \( p \) and any positive number \( \epsilon \), there exist infinitely many \( k \in \Lambda(p^{2}) \) such that \( E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) \neq 0 \) and
\[ \log | E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) | \leq -c_{7}r_{k}p^{2} + \epsilon r_{k}c_{8}(p). \]

Proof. In what follows, we always assume that \( k \in \Lambda(p^{2}) \). By the property (i) of Proposition 2,
\[ E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) = \sum_{l>p^{2}} R_{l}(\sigma^{(k)}) \alpha^{r_{k}l}. \]
Let
\[ n_{k} = \min \{ l \mid R_{l}(\sigma^{(k)}) \neq 0 \} \quad (k \geq 0). \]
By the property (ii) of Proposition 2, there is an \( l \) such that \( l \leq I(p) \) and \( R_{l}(\sigma^{(k)}) \neq 0 \) for all sufficiently large \( k \). Hence there exists an integer \( N \) such that \( n_{k} = N \) for infinitely many \( k \). If \( n_{k} = N \),
\[ | E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) - R_{N}(\sigma^{(k)}) \alpha^{r_{k}N} | \leq \sum_{l=N+1}^{\infty} | R_{l}(\sigma^{(k)}) \alpha^{r_{k}l} |. \]
\[ P_j(z; s_0, \ldots, s_{p^2}) = \sum_{l=0}^{p} Q_{jl}(s_0, \ldots, s_{p^2}) z^l, \quad F(z; s)^j = \sum_{l=0}^{\infty} G_{jl}(s) z^l. \]

Then by the property (IV),
\[ |Q_{jl}(\sigma_0^{(k)}, \ldots, \sigma_{p^2}^{(k)})| \leq c_9(p) e^{\epsilon r_k c_{10}(p)} \]
and
\[ |G_{jl}(\sigma^{(k)})| = \left| \sum_{l_1 + \cdots + l_j = l} \sigma_{l_1}^{(k)} \cdots \sigma_{l_j}^{(k)} \right| \leq (l+1)^j e^{\epsilon r_k (j+l)} \]
for \( k \geq c_3(\epsilon) \). Therefore
\[ |R_l(\sigma^{(k)})| \leq c_{11}(p) e^{\epsilon r_k c_{12}(p)} (l+1)^p e^{\epsilon r_k (p+l)} \]
for \( k \geq c_3(\epsilon) \). On the other hand, noting that \( N \leq I(p) \), we obtain
\[ \|R_N(\sigma^{(k)})\| \leq c_{13}(p) e^{\epsilon r_k c_{14}(p)} \]
for \( k \geq c_{15}(\epsilon, p) \). By (7)
\[ \log |R_l(\sigma^{(k)})| \leq \log c_{11}(p) + \epsilon r_k c_{12}(p) + p \log(l+1) + \epsilon r_k (p+l) + r_k l \log |\alpha| \]
if \( k \geq c_{18}(\epsilon, p) \). Choose \( \epsilon \) so small that \( 1 - c_{17}\epsilon > 0 \). Then for \( k \geq c_{18}(\epsilon, p) \),
\[ \sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})| \leq c_{19} e^{\epsilon r_k (1-c_{17}\epsilon) r_k (N+1) \log |\alpha|} \]
By (2), (8), and (9), if \( k \geq c_{20}(\epsilon, p) \) and \( n_k = N \), then
\[
\log \sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})|/|R_N(\sigma^{(k)})| \leq \epsilon r_k c_{16}(p) + \log c_{19} + (1 - c_{17}\epsilon) r_k (N+1) \log |\alpha| \]
Noting that \( N \leq I(p) \), we have
\[ \epsilon(c_{16}(p) + 2[K : Q]c_{14}(p) - c_{17}(N + 1) \log |\alpha|) + \log |\alpha| < 0 \]
if $\epsilon < c_{21}(p)$. Hence we have

$$\sum_{l=N+1}^{\infty} |R_{l}(\sigma^{(k)})\alpha^{r_{k}l}|/|R_{N}(\sigma^{(k)})\alpha^{r_{k}N}| \rightarrow 0 \text{ as } k \rightarrow \infty (n_{k} = N).$$

Therefore by (6)

$$E_{p}(\alpha^{r_{k}}; \sigma^{(k)})/R_{N}(\sigma^{(k)})\alpha^{r_{k}N} \rightarrow 1 \text{ as } k \rightarrow \infty (n_{k} = N).$$

Noting that $N > p^{2}$ and using (7), we obtain the assertions of the proposition.

Now we complete the proof of the theorem by choosing $p > 2[K : Q]c_{6}/c_{7}$. By Proposition 3, 4, and (2), for infinitely many $k \in \Lambda(p^{2})$, we have

$$-c_{7}r_{k}p^{2} + \epsilon r_{k}c_{8}(p) \geq \log |E_{p}(\alpha^{r_{k}}; \sigma^{(k)})| \geq -2[K : Q] \log \|E_{p}(\alpha^{r_{k}}; \sigma^{(k)})\| \geq -2[K : Q] \log (\epsilon r_{k}c_{5}(p) + c_{6}r_{k}p).$$

Dividing both sides by $r_{k}$, we get

$$-c_{7}p^{2} + \epsilon c_{8}(p) \geq -2[K : Q]c_{5}(p) + c_{6}p.$$

Letting $\epsilon$ tend to 0, we obtain

$$-c_{7}p^{2} \geq -2[K : Q]c_{6}p,$$

which contradicts the choice of $p$, and the proof of the theorem is completed.

References


