Transcendence of the values of certain lacunary series

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1 Introduction.

Let $f(z) = \sum_{k=0}^{\infty} z^{e_k}$ be a power series in the complex variable $z$ with a strictly increasing sequence $\{e_k\}_{k \geq 0}$ of exponents. From the Hadamard's gap theorem, if $\lim \inf_{k \to \infty} e_{k+1}/e_k > 1$, then $f(z)$ has the unit circle $|z| = 1$ as a natural boundary. The transcendence of the value $f(\alpha)$ of such a series at a nonzero algebraic number $\alpha$ inside the unit circle has been investigated by various authors. In 1844, Liouville proved the transcendence of $\sum_{k=0}^{\infty} 2^{-k!}$, the first example of a transcendental number. For the case of $\lim \sup_{k \to \infty} e_{k+1}/e_k = \infty$, there were some results on the transcendence of $f(\alpha)$, which are included in the result of Cijsouw and Tijdeman [1]. On the other hand, only special sequences $\{e_k\}_{k \geq 0}$ have been treated in the remaining case of $\lim \sup_{k \to \infty} e_{k+1}/e_k < \infty$. Let $d$ be an integer greater than 1. In 1929, Mahler [3] proved that, if $e_k = d^k$, $f(\alpha)$ is transcendental. Mahler's method was generalized by Loxton and van der Poorten [2], who proved the transcendence of $f(\alpha)$ when $\{e_{k+1}/e_k\}_{k \geq 0}$ is a sequence of integers greater than 1. However, for the case that $\lim_{k \to \infty} e_{k+1}/e_k = d$ and $\{e_{k+1}/e_k\}_{k \geq 0}$ is not necessarily a sequence of integers, for example $e_k = kd^k$, no transcendence result had been known. In this paper we prove the transcendence of $f(\alpha)$ under these conditions.

Theorem 1. Let $\{r_k\}_{k \geq 0}$ be a sequence of positive integers such that $\lim_{k \to \infty} r_{k+1}/r_k = d$, where $d$ is an integer greater than 1. Suppose that there exists a positive number $M$ such that $r_{k+1} \geq dr_k - M$ for all $k \geq 0$. Let

$$f(z) = \sum_{k=0}^{\infty} z^{r_k}$$

and let $\alpha$ be an algebraic number with $0 < |\alpha| < 1$. Then the number $f(\alpha)$ is transcendental.
Example. Let \( \alpha \) be an algebraic number with \( 0 < |\alpha| < 1 \) and \( d \) an integer greater than 1. Then the numbers
\[
\sum_{k=0}^{\infty} \alpha^{kd^{k}}, \quad \sum_{k=0}^{\infty} \alpha^{2kd^{k}+(-d)^{k}}, \quad \sum_{k=0}^{\infty} \alpha^{[\omega d^{k}+\eta]}, \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha^{k\binom{2k}{k}}
\]
are transcendental, where \( \omega > 0, \eta \geq 0 \), \([x]\) denotes the largest integer not exceeding a real number \( x \), and \( \binom{m}{n} \) is the binomial coefficient.

Applying Mahler's method, we proved in [5] the transcendence of the number \( \sum_{k=0}^{\infty} \alpha^{a_{k}} \) generated by a linear recurrence \( \{a_{k}\}_{k\geq 0} \) of nonnegative integers with \( a_{k} = g\rho^{k} + o(\rho^{k}) \), where \( g > 0 \) and \( \rho > 1 \), under some additional conditions. However, the transcendence of the first two numbers in (1) cannot be deduced from our result in [5] although the sequences of their exponents are linear recurrences.

Theorem 1 can be deduced from Theorem 2 below. We prepare the notation for stating the theorem. For any algebraic number \( \alpha \), we denote by \( \bar{|\alpha|} \) the maximum of the absolute values of the conjugates of \( \alpha \) and by den(\( \alpha \)) the smallest positive integer such that den(\( \alpha \))\( \cdot \alpha \) is an algebraic integer. It is easily seen that \( \lceil \alpha + \beta \rceil \leq \lceil \alpha \rceil + \lceil \beta \rceil \) and \( \lfloor \alpha \beta \rfloor \leq \lceil \alpha \rceil \lceil \beta \rceil \) for any algebraic numbers \( \alpha \) and \( \beta \). Furthermore, for any algebraic number \( \alpha \), we define
\[
\|\alpha\| = \max\{\bar{|\alpha|}, \text{den}(\alpha)\}.
\]
Then for any \( \alpha \neq 0 \) we have the inequalities
\[
\log |\alpha| \geq -2[Q(\alpha) : Q] \log \|\alpha\|
\]
and
\[
\log \|\alpha^{-1}\| \leq 2[Q(\alpha) : Q] \log \|\alpha\|
\]
(cf. [4, Lemma 2.10.2]).

Let \( K \) be an algebraic number field. We denote by \( K[[z]] \) the ring of formal power series in the variable \( z \) with coefficients in \( K \). Let
\[
f_{k}(z) = \sum_{l=0}^{\infty} \sigma_{l}^{(k)} z^{l} \in K[[z]] \quad (k \geq 0)
\]
and let \( \alpha \in K \) with \( 0 < |\alpha| < 1 \). In what follows, \( c_{1}, c_{2}, \ldots \) denote positive constants independent of \( k \) and depending only on \( f_{k}(z) \) (\( k \geq 0 \)) and \( \alpha \), and if they may depend also on parameters \( x \) as well as \( y \), they will be denoted by \( c_{1}(x), c_{2}(x, y), \ldots \). Let \( \{r_{k}\}_{k\geq 0} \) be a sequence of positive integers with the following properties:
(I) \( r_k \to \infty \) as \( k \) tends to infinity;

(II) \( f_k(\alpha^r_k) = a_k f_0(\alpha) + b_k \) \((k \geq 1)\), where \( a_k, b_k \in K \) and
\[
\log ||a_k||, \log ||b_k|| \leq c_1 r_k;
\]

(III) for any \( \varepsilon > 0 \) and for any \( l \geq 0 \), there exists a constant \( c_2(\varepsilon, l) > 0 \) such that
\[
\log \|\sigma_l^{(k)}\| \leq \varepsilon r_k (1 + l)
\]
for all \( k \geq c_2(\varepsilon, l) \);

(IV) for any \( \varepsilon > 0 \) there exists a constant \( c_3(\varepsilon) > 0 \) such that
\[
\log |\sigma_l^{(k)}| \leq \varepsilon r_k (1 + l)
\]
for all \( k \geq c_3(\varepsilon) \) and for any \( l \geq 0 \).

Let \( s_0, s_1, \ldots \) be variables and put \( F(z; s) = \sum_{l=0}^{\infty} s_l z^l \). Then \( F(z; \sigma^{(k)}) = f_k(z) \) \((k \geq 0)\). We assume that

(V) if \( P_0(z; s), \ldots, P_p(z; s) \) are polynomials in \( z \) and \( \{s_l\}_{l \geq 0} \) with degrees at most \( p \) in \( z \) and coefficients in \( K \) and if we put
\[
E(z; s) = \sum_{j=0}^{p} P_j(z; s)F(z; s)^j = \sum_{l=0}^{\infty} R_l(s)z^l,
\]
then there exists a positive integer \( I(p) \), independent of \( k \) and depending only on \( F(z; s) \) and \( p \), with the following property. If \( k \) is sufficiently large and \( P_0(z; \sigma^{(k)}), \ldots, P_p(z; \sigma^{(k)}) \) are not all zero, then there is an \( l \) such that \( l \leq I(p) \) and \( R_l(\sigma^{(k)}) \neq 0 \).

**Theorem 2.** If the properties (I) – (V) are satisfied, then the number \( f_0(\alpha) \) is transcendental.

**Remark.** If the constant \( c_2(\varepsilon, l) \) in the property (III) does not depend on \( l \), then the property (IV) is satisfied by the property (III). This is the very case that Loxton and van der Poorten [2] dealt with.
2 Proof of the theorems.

Proof of Theorem 1. We may assume that $r_0 = 1$, replacing $r_0, r_1, r_2, \ldots$ by 1, $r_0, r_1, \ldots$ if necessary. Define

$$f_k(z) = \sum_{h=0}^{\infty} \alpha^{r_k k - r_k d^h} z^{d^h} \quad (k \geq 0).$$

Then

$$\sigma_l^{(k)} = \begin{cases} \alpha^{r_k k - r_k d^h} & (l = d^h) \\ 0 & \text{otherwise} \end{cases}$$

and $f_0(\alpha) = \sum_{h=0}^{\infty} \alpha^{r_k} = f(\alpha)$, which is transcendental by Theorem 2 if the properties (I) – (V) are satisfied.

The sequence $\{r_k\}_{k \geq 0}$ obviously has the property (I). Let $K = Q(\alpha)$. Then $f_k(z) \in K[[z]]$ $(k \geq 0)$ and

$$f_k(\alpha^{r_k}) = \sum_{h=0}^{\infty} \alpha^{r_k k} = f(\alpha) - \sum_{h=0}^{k-1} \alpha^{r_k}.$$  

Since $r_{k+1} > r_k$ for all sufficiently large $k$ by the assumption, there is a constant $C \geq 1$ such that $\max_{0 \leq h \leq k-1} r_h \leq Cr_k$ for all $k \geq 1$. Hence

$$\log \left\| - \sum_{h=0}^{k-1} \alpha^{r_h} \right\| \leq \log k + \left( \max_{0 \leq h \leq k-1} r_h \right) \log \|\alpha\| \leq c_1 r_k,$$

and the property (II) is satisfied.

Using (3), we have

$$\log \left\| \alpha^{r_k k - r_k d^h} \right\| \leq 2[K : Q]|r_k k - r_k d^h| \log \|\alpha\|. \quad (5)$$

By (4), (5), and $\|0\| = 1$, in order to prove that the property (III) is satisfied, it suffices to show that for any $\varepsilon > 0$ and for any $h \geq 0$, there exists a constant $c_2(\varepsilon, h) > 0$ such that

$$|r_{k+1} - r_k d^h| \leq \varepsilon r_k d^h$$

for all $k \geq c_2(\varepsilon, h)$. If $h = 0$, this inequality holds for all $k \geq 0$. Since $\lim_{k \to \infty} r_{k+1}/r_k = d$, for any $\varepsilon > 0$ and for any $h \geq 1$, there exists a constant $c_2(\varepsilon, h) > 0$ such that

$$1 - \frac{\varepsilon}{(1 + \varepsilon)h} < \frac{r_{k+1}}{dr_k} < 1 + \frac{\varepsilon}{(1 + \varepsilon)h}.$$
for all $k \geq c_2(\epsilon, h)$. Then

$$|r_{h+k} - r_{k}d^{h}| \frac{1}{r_{k}d^{h}} = \left| \frac{r_{k+h}}{dr_{k+h-1}} \cdots \frac{r_{k+1}}{dr_{k}} - 1 \right| \leq \sum_{m=1}^{h} h^{m} \left( \frac{\epsilon}{(1+\epsilon)h} \right)^{m} \leq \frac{\frac{\epsilon}{1+\epsilon}}{1-\frac{\epsilon}{1+\epsilon}} = \epsilon.$$

Next we prove that the property (IV) is satisfied. Since

$$r_{h+k} - r_{k}d^{h} = (r_{k+h} - dr_{k+h-1}) + d(r_{k+h-1} - dr_{k+h-2}) + \cdots + d^{h-1}(r_{k+1} - dr_{k}) \geq -M(1 + d + \cdots + d^{h-1})$$

by the assumption in the theorem,

$$\log |\sigma_{d^{h}}^{(k)}| = (r_{h+k} - r_{k}d^{h}) \log |\alpha| \leq \frac{-M(d^{h} - 1)}{d-1} \log |\alpha| < -M(1 + d^{h}) \log |\alpha|.$$ 

Then for any $\epsilon > 0$ there exists a constant $c_3(\epsilon) > 0$ such that $\epsilon r_{k} \geq -M \log |\alpha|$ for all $k \geq c_3(\epsilon)$, and the property (IV) is fulfilled.

Finally we show that the property (V) is satisfied by the same way as in the proof of Theorem 2.10.1 in [4]. Choose a positive integer $\lambda(p)$, depending on $p$, such that

$$\max_{0 \leq j \leq p} \deg_{z} P_{j}(z; s) < d^\lambda(p).$$

Suppose that $P_0(z; \sigma^{(k)}), \ldots, P_p(z; \sigma^{(k)})$ are not all zero and put

$$p' = p'(k) = \max\{ j \mid P_j(z; \sigma^{(k)}) \neq 0 \}, \quad a = a(k) = \deg_{z} P_{p'}(z; \sigma^{(k)}).$$

Then

$$E(z; \sigma^{(k)}) = \sum_{j=0}^{p'} P_{j}(z; \sigma^{(k)}) f_{k}(z)^{j} = \sum_{l=0}^{\infty} R_{l}(\sigma^{(k)}) z^{l}.$$ 

We prove that $R_{l}(\sigma^{(k)}) \neq 0$ for some $l$. This can be done by choosing

$$l = a + \sum_{m=1}^{p'} d^\lambda(p) + m$$

and considering the $d$-adic expansion of the positive integer $l$ in place of the $\{d_1, d_2, \ldots\}$-adic expansion in the proof of Theorem 2.10.1 in [4]. Since $a(k) < d^\lambda(p)$ and $p'(k) \leq p$ for any $k$, we can take $I(p) = d^\lambda(p) + p + 1$ and the property (V) is fulfilled. Then by Theorem 2, $f(\alpha)$ is transcendental, and the proof of the theorem is completed.
We prove Theorem 2 by the method of Loxton and van der Poorten [2] and Nishioka [4].

**Proof of Theorem 2.** We assume on the contrary that $f_0(\alpha)$ is algebraic. We may suppose $f_0(\alpha) \in K$.

**Proposition 1** (Loxton and van der Poorten [2], see also Nishioka [4, Proposition 2.9.2]). Let $m$ be a nonnegative integer. There exists an infinite subset $\Lambda(m)$ of the set $\mathbb{N}$ of positive integers such that for any polynomial $P(s_0, \ldots, s_m) \in K[s_0, \ldots, s_m]$ the following two properties are equivalent:

(i) $P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0$ for infinitely many $k \in \Lambda(m)$.

(ii) $P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0$ for all $k \in \Lambda(m)$.

Let $m$ be a nonnegative integer and put

$V(m) = \{P(s_0, \ldots, s_m) \in K[s_0, \ldots, s_m] \mid P(\sigma_0^{(k)}, \ldots, \sigma_m^{(k)}) = 0 \text{ for all } k \in \Lambda(m)\}$.

Then $V(m)$ is a prime ideal of $K[s_0, \ldots, s_m]$ by Proposition 1.

**Proposition 2** (Loxton and van der Poorten [2], see also Nishioka [4, Proposition 2.9.3]). For any positive integer $p$, there exist $p + 1$ polynomials $P_0(z; s_0, \ldots, s_{p^2}), \ldots, P_p(z; s_0, \ldots, s_{p^2}) \in K[z, s_0, \ldots, s_{p^2}]$ with degrees at most $p$ in $z$ such that the function

$$E_p(z; s) = \sum_{j=0}^{p} P_j(z; s_0, \ldots, s_{p^2})F(z; s)^j = \sum_{l=0}^{\infty} R_l(s)z^l$$

has the following two properties:

(i) $R_l(s) = R_l(s_0, \ldots, s_{p^2}) \in V(p^2)$ for all $l$ with $l \leq p^2$;

(ii) there exists a positive integer $I(p)$, independent of $k$ and depending only on $F(z; s)$ and $p$, such that $\text{ord}_{z=0} E_p(z; \sigma^{(k)}) \leq I(p)$ for all sufficiently large $k \in \Lambda(p^2)$.

**Proposition 3.** For any positive integer $p$ and any positive number $\epsilon$, if $k \geq c_4(\epsilon, p)$, then

$$\log \|E_p(\alpha^r; \sigma^{(k)})\| \leq \epsilon r_k c_5(p) + c_6 r_k p.$$
Proof. By the property (III), $\|\sigma^{(k)}\| \leq e^{e^{r_{k}+1}}$ for all $k \geq c_{2}(\epsilon, l)$. Let

$$P_{j}(z; s_{0}, \ldots, s_{p^{2}}) = \sum_{l=0}^{p} Q_{jl}(s_{0}, \ldots, s_{p^{2}})z^{l}.$$ Since $Q_{jl}(s_{0}, \ldots, s_{p^{2}}) \in K[s_{0}, \ldots, s_{p^{2}}]$, we have

$$\|Q_{jl}(\sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)})\| \leq c_{7}(p)e^{er_{k}c_{8}(p)}$$

for all $k \geq \max_{0 \leq l \leq p^{2}} c_{2}(\epsilon, l)$. Since

$$E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) = \sum_{j=0}^{p} P_{j}(\alpha^{r_{k}} ; \sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)})F(\alpha^{r_{k}} ; \sigma^{(k)})^{j}$$

$$= \sum_{j=0}^{p} P_{j}(\alpha^{r_{k}} ; \sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)})f_{k}(\alpha)^{j}$$

$$= \sum_{j=0}^{p} \left( \sum_{l=0}^{p} Q_{jl}(\sigma_{0}^{(k)}, \ldots, \sigma_{p^{2}}^{(k)})\alpha^{r_{k}l} \right)(a_{k}f_{0}(\alpha)+b_{k})^{j},$$

noting that $\|\alpha^{r_{k}}\| \leq c_{9}^{r_{k}}$, we obtain

$$\|E_{p}(\alpha^{r_{k}}; \sigma^{(k)})\| \leq c_{10}(p)e^{er_{k}c_{11}(p)}c_{9}^{r_{k}p}(e^{2c_{1}r_{k}}(\|f_{0}(\alpha)\|+1))^{p}$$

for $k \geq \max_{0 \leq l \leq p^{2}} c_{2}(\epsilon, l)$, which implies the proposition.

**Proposition 4.** For any positive integer $p$ and any positive number $\epsilon$, there exist infinitely many $k \in \Lambda(p^{2})$ such that $E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) \neq 0$ and

$$\log|E_{p}(\alpha^{r_{k}}; \sigma^{(k)})| \leq -c_{7}r_{k}p^{2} + \epsilon r_{k}c_{8}(p).$$

Proof. In what follows, we always assume that $k \in \Lambda(p^{2})$. By the property (i) of Proposition 2,

$$E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) = \sum_{l>p^{2}} R_{l}(\sigma^{(k)})\alpha^{r_{k}l}.$$ Let

$$n_{k} = \min\{l \mid R_{l}(\sigma^{(k)}) \neq 0\} \quad (k \geq 0).$$

By the property (ii) of Proposition 2, there is an $l$ such that $l \leq I(p)$ and $R_{l}(\sigma^{(k)}) \neq 0$ for all sufficiently large $k$. Hence there exists an integer $N$ such that $n_{k} = N$ for infinitely many $k$. If $n_{k} = N$,

$$|E_{p}(\alpha^{r_{k}}; \sigma^{(k)}) - R_{N}(\sigma^{(k)})\alpha^{r_{k}N}| \leq \sum_{l=N+1}^{\infty} |R_{l}(\sigma^{(k)})\alpha^{r_{k}l}|.$$
\begin{align*}
P_j(z; s_0, \ldots, s_{p^2}) &= \sum_{i=0}^{p} Q_{ji}(s_0, \ldots, s_{p^2}) z^i, \quad F(z; s)^j = \sum_{i=0}^{\infty} G_{ji}(s) z^i.
\end{align*}

Then by the property (IV),

\[ |Q_{ji}(\sigma_{0}^{(k)}, \ldots, \sigma_{p^2}^{(k)})| \leq c_9(p) e^{r_k c_{10}(p)} \]

and

\[ |G_{ji}(\sigma^{(k)})| = \left| \sum_{l_1 + \cdots + l_j = l} \sigma_{l_1}^{(k)} \cdots \sigma_{l_j}^{(k)} \right| \leq (l+1)^j e^{r_k (j+l)} \]

for \( k \geq c_3(\epsilon) \). Therefore

\begin{equation}
|R_l(\sigma^{(k)})| \leq c_{11}(p) e^{r_k c_{12}(p)} (l+1)^p e^{r_k (p+l)}
\end{equation}

for \( k \geq c_3(\epsilon) \). On the other hand, noting that \( N \leq I(p) \), we obtain

\begin{equation}
\|R_N(\sigma^{(k)})\| \leq c_{13}(p) e^{r_k c_{14}(p)}
\end{equation}

for \( k \geq c_{15}(\epsilon, p) \). By (7)

\[ \log |R_l(\sigma^{(k)})\alpha^{r_k l}| \leq \log c_{11}(p) + \varepsilon r_k c_{12}(p) + p \log(l+1) + \varepsilon r_k (p+l) + r_k l \log |\alpha| \]

if \( k \geq c_{18}(\epsilon, p) \). Choose \( \varepsilon \) so small that \( 1 - c_{17} \varepsilon > 0 \). Then for \( k \geq c_{18}(\epsilon, p) \),

\begin{equation}
\sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})\alpha^{r_k l}| \leq e^{r_k c_{16}(p)} c_{19} e^{\log c_{13}(p) + 2[K : Q] c_{14}(p)}
\end{equation}

By (2), (8), and (9), if \( k \geq c_{20}(\epsilon, p) \) and \( n_k = N \), then

\[ \log \sum_{l=N+1}^{\infty} |R_l(\sigma^{(k)})\alpha^{r_k l}| / |R_N(\sigma^{(k)})| \leq \varepsilon r_k c_{16}(p) + \log c_{19} + (1 - c_{17} \varepsilon) r_k (N+1) \log |\alpha| \]

if \( k \geq c_{18}(\epsilon, p) \). Noting that \( N \leq I(p) \), we have

\[ \varepsilon (c_{16}(p) + 2[K : Q] c_{14}(p) - c_{17} (N+1) \log |\alpha|) + \log |\alpha| < 0 \]
if $\epsilon < c_{21}(p)$. Hence we have
\[ \sum_{l=N+1}^{\infty} |R_{l}(\sigma^{(k)})\alpha^{r_{k}l}|/|R_{N}(\sigma^{(k)})\alpha^{r_{k}N}| \to 0 \quad \text{as} \quad k \to \infty \quad (n_{k} = N). \]
Therefore by (6)
\[ E_{p}(\alpha^{r_{k}}; \sigma^{(k)})/R_{N}(\sigma^{(k)})\alpha^{r_{k}N} \to 1 \quad \text{as} \quad k \to \infty \quad (n_{k} = N). \]
Noting that $N > p^{2}$ and using (7), we obtain the assertions of the proposition.

Now we complete the proof of the theorem by choosing $p > 2[K : Q]c_{6}/c_{7}$. By Proposition 3, 4, and (2), for infinitely many $k \in \Lambda(p^{2})$, we have
\[ -c_{7}r_{k}p^{2} + \epsilon r_{k}c_{8}(p) \geq \log |E_{p}(\alpha^{r_{k}}; \sigma^{(k)})| \geq -2[K : Q] \log \|E_{p}(\alpha^{r_{k}}; \sigma^{(k)})\| \geq -2[K : Q]((\epsilon r_{k}c_{5}(p) + c_{6}r_{k}p). \]
Dividing both sides by $r_{k}$, we get
\[ -c_{7}p^{2} + \epsilon c_{8}(p) \geq -2[K : Q]((\epsilon c_{5}(p) + c_{6}p). \]
Letting $\epsilon$ tend to 0, we obtain
\[ -c_{7}p^{2} \geq -2[K : Q]c_{6}p, \]
which contradicts the choice of $p$, and the proof of the theorem is completed.

References