Title
Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke (Studies on complex dynamics and related topics)

Author(s)
Sumi, Naoya

Citation
数理解析研究所講究録 数学リポジトリ
Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke

Naoya Sumi (鶴見 直哉)

Abstract

We show that if $f$ is a $C^2$-local diffeomorphism with positive entropy on a $n$-dimensional closed manifold ($n \geq 2$) then $f$ is chaotic in the sense of Li-Yorke.

1 Introduction

We study chaotic properties of dynamical systems with positive entropy. Notions of chaos have been given by Li and Yorke [15], Devaney [5] and others. It is well known that if a continuous map of an interval has positive entropy, then the map is chaotic according to the definition of Li and Yorke (cf. [2]). For invertible maps the following holds: let $f$ be a $C^2$-diffeomorphism of a closed $C^\infty$-manifold. If the topological entropy of $f$ is positive, then $f$ is chaotic in the sense of Li-Yorke [31].

In this paper we show the following:

Theorem A. Let $f$ be a $C^2$-local diffeomorphism of a closed $C^\infty$-manifold. If the topological entropy of $f$ is positive, then $f$ is chaotic in the sense of Li-Yorke.

From this theorem we obtain the following corollary.

Corollary B. Let $f$ be a $C^2$-local diffeomorphism of a closed $C^\infty$-manifold. If $f$ is not invertible, then $f$ is chaotic in the sense of Li-Yorke.

First we shall explain here the definitions and notations used above. Let $X$ be a compact metric space with metric $d$ and let $f : X \to X$ be a continuous map. A subset $S$ of $X$ is a scrambled set of $f$ if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

1. $\limsup_{n \to \infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0$.

If there is an uncountable scrambled set $S$ of $f$, then we say that $f$ is chaotic in the sense of Li-Yorke. Li and Yorke showed in [15] that if $f : [0, 1] \to [0, 1]$ is a continuous map with a periodic point of period 3, then $f$ is chaotic in this sense. Note that any scrambled set contains at most one point $x$ which does not satisfy the following: for any periodic point $p \in X$,

$$\limsup_{n \to \infty} d(f^n(x), f^n(p)) > 0.$$  

For another sufficient condition for the chaos in the sense of Li-Yorke, the readers may refer to [4], [7], [8], [9], [10], [11], [19], [20], [34].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "*-chaos" as follows: let $F$ be a closed subset of $X$. A map $f : X \to X$ is *-chaotic on $F$ (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ with the property that for any nonempty open subsets $U$ and $V$ of $F$ with $U \cap V = \emptyset$ and for any natural number $N$, there is $n \geq N$ such that $d(f^n(x), f^n(y)) > \tau$ for some $x \in U$, $y \in V$, and
2. for any nonempty open subsets \( U, V \) of \( F \) and any \( \varepsilon > 0 \) there is a natural number \( n \geq 0 \) such that \( d(f^n(x), f^n(y)) < \varepsilon \) for some \( x \in U, y \in V. \)

Such a set \( F \) is called a \( \ast \)-chaotic set. If \( S \) is a scrambled set, then the closure of \( S, \bar{S}, \) is a \( \ast \)-chaotic set. In [10] Kato showed that the converse is true. This is stated precisely as follows:

**Lemma 1** ([10], Theorem 2.4) Let \( X \) be a compact metric space and let \( F \) be a closed subset of \( X. \) If \( f : X \rightarrow X \) is continuous and is \( \ast \)-chaotic on \( F, \) then there is an \( F_0 \)-set \( S \subset F \) such that \( S \) is a scrambled set of \( f \) and \( \bar{S} = F. \) If \( F \) is perfect (i.e. \( F \) has no isolated points), we can choose \( S \) as a countable union of Cantor sets.

By this lemma, to show the existence of uncountable scrambled sets it suffices to show the existence of perfect \( \ast \)-chaotic sets.

To obtain Theorem A we consider the inverse limit system of \( f. \) Let \( M \) be a closed \( C^\infty \)-manifold and let \( d \) be the distance for \( M \) induced by a Riemannian metric \( \| \cdot \| \) on \( TM. \) Let \( M^Z \) denote the product topological space \( M^Z = \{(x_i) : x_i \in M, i \in \mathbb{Z}\}. \) Then \( M^Z \) is compact. We define a compatible metric \( \tilde{d} \) for \( M^Z \) by

\[
\tilde{d}((x_i), (y_i)) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}} \quad ((x_i), (y_i) \in M^Z).
\]

For \( f : M \rightarrow M \) a continuous surjection, we let

\[
M_f = \{(x_i) : x_i \in M \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.
\]

Then \( M_f \) is a closed subset of \( M^Z. \) The space \( M_f \) is called the **inverse limit space** constructed by \( f. \) A homeomorphism \( \tilde{f} : M_f \rightarrow M_f, \) which is defined by

\[
\tilde{f}(x_i) = (f(x_i)) \quad \text{for all } (x_i) \in M_f,
\]

is called the **shift map** determined by \( f. \) We denote as \( P^0 : M_f \rightarrow M \) the projection defined by \( (x_i) \mapsto x_0. \) Then \( P^0 \circ f = f \circ P^0 \) holds. Remark that \( f \) is chaotic in the sense of Li-Yorke if and only if so is \( \tilde{f}. \)

We can show that the topological entropy, \( h(f), \) of \( f \) coincides with that of \( \tilde{f}. \) Indeed, for an \( f \)-invariant probability measure \( \nu, \) we can find an \( \tilde{f} \)-invariant probability measure \( \mu \) such that \( \nu(A) = P^0_\mu(A)(= \mu((P^0)^{-1}A)) \) for any Borel set \( A \subset M \) ([18] Lemma IV 8.3). Let us denote as \( h_{\nu}(f) \) and \( h_{\mu}(\tilde{f}) \) the metric entropy of \( (M, f, \nu) \) and \( (M_f, \tilde{f}, \mu) \) respectively. Then we have \( h_{\nu}(f) = h_{P^0 \mu}(f) = h_{\mu}(\tilde{f}) \) ([25] Lemma 5.2). Therefore, the conclusion is obtained by the variational principle ([32] Theorem 8.6).

We say that a differentiable map \( f : M \rightarrow M \) is a **local diffeomorphism** if for \( x \in M \) there is an open neighborhood \( U_x \) of \( x \) in \( M \) such that \( f(U_x) \) is open in \( M \) and \( f|_{U_x} : U_x \rightarrow f(U_x) \) is a homeomorphism. Since \( M \) is connected, then the cardinal number of \( f^{-1}(x) \) is constant. This constant is called the **covering degree** of \( f. \) If the covering degree of \( f \) is greater than one, \( (M_f, M, C, P^0) \) is a fiber bundle where \( C \) denotes the Cantor set (see [1] Theorem 6.5.1).

Let \( \mu \) be a Borel probability measure on \( M_f \) and let \( B \) be the Borel \( \sigma \)-algebra on \( M_f \) completed with respect to \( \mu. \) For \( \xi \) a measurable partition of \( M_f \) and \( \tilde{x} \in M_f \) we denote as \( \xi(\tilde{x}) \) the element of the partition \( \xi \) which contains the point \( \tilde{x}. \) Then there exists a family \( \{\mu^\xi_{\tilde{x}} | \tilde{x} \in M_f\} \) of Borel probability measures satisfying the following conditions:

1. for \( \tilde{x}, \tilde{y} \in M_f \) if \( \xi(\tilde{x}) = \xi(\tilde{y}) \) then \( \mu^\xi_{\tilde{x}} = \mu^\xi_{\tilde{y}}; \)
2. \( \mu^\xi_{\tilde{x}}(\xi(\tilde{x})) = 1 \) for \( \mu \)-almost all \( \tilde{x} \in M_f, \)
3. for \( A \subset B \) a function \( \tilde{x} \mapsto \mu^\xi_{\tilde{x}}(A) \) is measurable and \( \mu(A) = \int_{M_f} \mu^\xi_{\tilde{x}}(A) d\mu(\tilde{x}). \)

The family \( \{\mu^\xi_{\tilde{x}} | \tilde{x} \in M_f\} \) is called a **canonical system of conditional measures** for \( \mu \) and \( \xi \) (see [26] for more details).

To prove Theorem A it suffices to show the following theorem.
Theorem C Let $f$ be a $C^2$-local diffeomorphism of a closed $C^\infty$-manifold $M$ and let $\mu$ be an $f$-invariant ergodic Borel probability measure on $M_f$.

If the metric entropy of $\mu$ is positive, then there exists a measurable partition $\eta$ of $M_f$ such that $\supp(\mu^\eta)$ is a perfect $*$-chaotic set for $\mu$-almost all $\tilde{x} \in M_f$.

Here the support $\supp(\nu)$ of a finite measure $\nu$ is the smallest closed set $C$ with $\nu(C) = \nu(M_f)$. Equivalently, $\supp(\nu)$ is the set of all $\tilde{x} \in M_f$ with the property that $\nu(U) > 0$ for any open $U$ containing $\tilde{x}$.

Let us see how Theorem A follows from Theorem C. We know that $h(\tilde{f}) = \sup\{h_\nu(\tilde{f}) : \nu(\mathcal{M}_e(\tilde{f}))\}$ where $\mathcal{M}_e(\tilde{f})$ is the set of all $\tilde{f}$-invariant ergodic probability measures (cf. [27]). Thus, if $h(\tilde{f}) = h(f) > 0$, then we can choose $\mu \in \mathcal{M}_e(\tilde{f})$ with $h_\mu(\tilde{f}) > 0$. Therefore, by Theorem C and Lemma 1, $f$ is chaotic in the sense of Li-Yorke.

2 Key Lemmas

In this section we prepare some lemmas which need to prove Theorem C. Let $f$ be a $C^2$-local diffeomorphism of a closed $C^\infty$-manifold $M$ and $\mu$ be an $f$-invariant ergodic Borel probability measure on $M_f$ with $h_\mu(f) > 0$. As in the previous section we denote as $B$ the Borel $\sigma$-algebra on $M_f$ completed with respect to $\mu$. For $\mu$-almost all $\tilde{x} = (x_i) \in M_f$, there exist a splitting of the tangent space $T_{x_0}M = \oplus_{i=1}^\epsilon E_i(\tilde{x})$ and real numbers $\lambda_1(x_0) < \cdots < \lambda_\epsilon(x_0)$ such that

(a) the maps $\tilde{x} \mapsto E_i(\tilde{x})$, $\lambda_i(x_0)$ and $s(x_0)$ are measurable, moreover $E_i(f(\tilde{x})) = D_{x_0}f(E_i(\tilde{x}))$ and $\lambda_i(x_0), s(x_0)$ are $f$-invariant ($i = 1, \cdots, s(x_0)$),

(b) $\lim_{n \to \pm \infty} \frac{1}{n} \log |(D_{x_0}f^{|n|})^{\pm 1}(v)| = \lambda_i(x_0)$ ($0 \neq v \in E_i(\tilde{x})$, $i = 1, \cdots, s(x_0)$) and

(c) $\lim_{n \to \pm \infty} \frac{1}{n} \log |\det(D_{x_0}f^{|n|})^{\pm 1}| = \sum_{i=1}^{s(x_0)} \lambda_i(x_0) \dim E_i(\tilde{x})$.

([21], [33], [29], [30]). The numbers $\lambda_1(x_0), \cdots, \lambda_\epsilon(x_0)$ are called Lyapunov exponents of $f$ at $x_0$. Since $\mu$ is ergodic, we can put $s = s(x_0)$, $\lambda_i = \lambda_i(x_0)$ and $m_i = \dim E_i(\tilde{x})$ ($i = 1, \cdots, s$) for $\mu$-almost all $\tilde{x} = (x_i) \in M_f$.

A well-known theorem of Margulis and Ruelle [28] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. $h_{\mu}(f) \leq \sum_{\lambda_i > 0} \lambda_i m_i$. Since $\tilde{f}$ has positive entropy, we have $0 < h_{\mu}(\tilde{f}) = h_{\mu}(f) \leq \max(\lambda_1) = \lambda_s$. Fix $0 < \lambda < \min(\lambda_i : \lambda_i > 0)$. From [24], [29] and [30] there are measurable functions $\tilde{\beta} > \tilde{\alpha} > 0$ and $\tilde{\gamma} > 1$ with the following properties: For $\tilde{x} = (x_i) \in M_f$ we put

$$W_{loc}^u(\tilde{x}) = \{y = (y_i) \in M_f : d(x_0, y_0) \leq \tilde{\alpha} (\tilde{x}), d(x_i, y_i) \leq \tilde{\beta}(\tilde{x}) e^{-\lambda_i} (i \geq 1)\}.$$ 

Then

(a) the map $P^0$ restricted to $W_{loc}^u(\tilde{x})$ is injective,

(b) $P^0(W_{loc}^u(\tilde{x}))$ is a $C^2$-submanifold of the ball $\{y \in M : d(x_0, y) \leq \tilde{\alpha}(\tilde{x})\}$,

(c) $T_{x_0}P^0(W_{loc}^u(\tilde{x})) = \oplus_{\lambda_i > 0} E_i(\tilde{x})(\neq \{0\})$ for $\mu$-almost all $\tilde{x} \in M_f$,

(d) $d(y_i, z_i) \leq \tilde{\gamma}(\tilde{x}) d(y_0, x_0) e^{-\lambda i}$ for $(y_n), (z_n) \in W_{loc}^u(\tilde{x})$.

For the case when $f$ is invertible we may refer to [6], [22] and [23].

Let $\xi$ and $\eta$ be measurable partitions of $M_f$. Put $f^n \xi = \{f^nC : C \in \xi\}$ for $n \in \mathbb{Z}$ and then $(f^n \xi)(\tilde{x}) = \tilde{f}(\xi(f^{-n}(\tilde{x})))$ for $\tilde{x} \in M_f$. $\eta \leq \xi$ means that for $\mu$-almost all $\tilde{x} \in M_f$ one has $\xi(\tilde{x}) \subset \eta(\tilde{x})$.

Lemma 2 Let $f$ and $\mu$ be as above. Then there exists a measurable partition $\xi$ of $M_f$ such
(a) $\xi \leq \tilde{f}^{-1}\xi$,

(b) for $\mu$-almost all $\tilde{x} \in M_f$, $\xi(\tilde{x}) \subset \tilde{W}^u_{loc}(\tilde{x})$ and $\xi(\tilde{x})$ contains a neighborhood of $\tilde{x}$ open in $\tilde{W}^u_{loc}(\tilde{x})$,

(c) $\sqrt[\infty]{\lim_{n \to \infty} \tilde{f}^{-n}\xi}$ is the partition into points.

This lemma is similar to [13] Proposition 3.1, [16] Proposition 5.2 and [17] Lemma 2.2. So we omit the proof.

Let $\mathcal{C}$ denote the family of all nonempty closed subsets of $M_f$ and define a metric $d_H$ by

$$d_H(A, B) = \max\{\sup_{b \in B} d(A, b), \sup_{a \in A} d(a, B)\} \quad (A, B \subset C)$$

where $d(A, b) = \inf\{d(a, b) : a \in A\}$. Then it is known that $(\mathcal{C}, d_H)$ is a compact metric space (cf.[12]). If $\xi$ is a measurable partition, then $\tilde{x} \mapsto \xi(\tilde{x}) \in \mathcal{C}$ is measurable. Indeed, this follows from [3] Theorems III.2, III.9, III.22 and the fact that $\{(\tilde{x}, \xi(\tilde{x})) : \tilde{x} \in M_f\}$ is a Borel subset of $M_f \times M_f$. For $A \subset M_f$ we put $\text{diam}(A) = \sup\{d(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in A\}$. Then we have $\text{diam}(A) = \text{diam}(A)$. Since $\tilde{x} \mapsto \xi(\tilde{x}) \subset C$ is measurable, $\tilde{x} \mapsto \text{diam}(\xi(\tilde{x}))$ is also a measurable function. By Lemma 2 (c) we have that for $\mu$-almost all $\tilde{x} \in M_f$

$$\text{diam}(\tilde{f}^{-n}\xi(\tilde{x})) \to 0 \quad (1)$$

as $n \to \infty$.

Let $\xi$ and $\eta$ be measurable partitions of $M_f$ and let $\{\mu_{\tilde{x}}\mid \tilde{x} \in M_f\}$ be a canonical system of conditional measures for $\mu$ and $\xi$. The mean conditional entropy of $\eta$ with respect to $\xi$ is defined by

$$H_\mu(\eta|\xi) = \int -\log \mu_{\tilde{x}}^\xi(\eta(\tilde{x})) d\mu(\tilde{x})$$

(see [27] for details).

Lemma 3 Let $f$ and $\mu$ be as above and let $\xi$ be as in Lemma 2. Then,

$$h_\mu(\tilde{f}) = H_\mu(\tilde{f}^{-1}\xi|\xi).$$

For the case when $f$ is invertible this lemma is proved by Ledrappier and Young [14]. We recall that if the covering degree of $f$ is greater than one, then $(M_f, M, C, P^\infty)$ is a fiber bundle where $C$ denotes the Cantor set. In view of this fact, the above lemma can be proved by almost the same arguments as the proof of [14] Corollary 5.3 and [16] Corollary 7.1 with some slight modifications. Here we omit the proof.

By Lemma 2(a) we have that $\xi \geq \tilde{f}_n\xi \geq \tilde{f}^2\xi \geq \cdots$. Let us introduce a measurable partition defined by $\eta = \bigwedge_{n=0}^\infty \tilde{f}^n\xi$. Then we have $\tilde{f}\eta = \eta$. For simplicity put

$$\mu_{\tilde{x}} = \mu_{\tilde{x}}^\xi \quad \text{and} \quad \mu_{\tilde{x}}^n = \mu_{\tilde{x}}^{\tilde{f}^n\xi} \quad (n \in \mathbb{Z}).$$

By Doob's theorem it follows that for a $\mu$-integrable function $\psi : M_f \to \mathbb{R}$

$$\int \psi d\mu_{\tilde{x}} = \lim_{n \to \infty} \int \psi d\mu_{\tilde{x}}^n \quad (2)$$

for $\mu$-almost all $\tilde{x}$. Since $\tilde{f}\eta = \eta$ and $\tilde{f}_n\mu = \mu$, by the uniqueness of a canonical system of conditional measures (cf.[26]) we have that for $\mu$-almost all $\tilde{x}$

$$\tilde{f}_n\mu_{\tilde{x}} = \mu_{\tilde{x}}^n \quad \text{and} \quad \tilde{f}_n\mu_{\tilde{x}}^n = \mu_{\tilde{x}}^{\tilde{f}^n\xi} \quad (n \in \mathbb{Z}).$$

(3)

Here $(\tilde{f}, \nu)(A) = \nu(\tilde{f}^{-1}A)$ for a Borel probability measure $\nu$ on $M_f$ and $A \in B$.

Let $\mathcal{C}(M_f)$ be the Banach space of continuous real-valued functions of $M_f$ with the sup norm $|\cdot|_\infty$, and let $\mathcal{M}(M_f)$ be a set of all Borel probability measures on $M_f$ with the weak
topology. Since $C(M_f)$ is separable, there exists a countable set \{\varphi_1, \varphi_2, \cdots\} which is dense in $C(M_f)$. For $\nu, \nu' \in \mathcal{M}(M_f)$ define

$$
\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\nu - \int \varphi_n d\nu'|}{2^n |\varphi_n|_{\infty}}.
$$

Then $\rho$ is a compatible metric for $\mathcal{M}(M_f)$ and $(\mathcal{M}(M_f), \rho)$ is compact (cf.[18]). Since (2) holds for \{\varphi_i\}, we have

$$
\mu_{\tilde{\varphi}} = \lim_{n \to \infty} \mu_{\varphi}^n
$$

for $\mu$-almost all $\tilde{x}$.

**Lemma 4** Let $f$, $\mu$ and $\{\mu_\varphi | \varphi \in M_f\}$ be as above. Then for $\epsilon > 0$ there exists a closed set $F_\epsilon$ with $\mu(F_\epsilon) \geq 1 - \epsilon$ satisfying the map

$$
F_\epsilon \ni \tilde{x} \mapsto \int \varphi_n d\mu_{\varphi}^\epsilon
$$

is measurable for $n \geq 1$ and thus $\tilde{x} \mapsto \nu_{\varphi}^\epsilon \in \mathcal{M}(M_f)$ is measurable.

**Proof.** Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\epsilon > 0$. Since $\tilde{x} \mapsto \int \varphi_n d\mu_{\varphi}^\epsilon$ is measurable for $i \geq 1$, by Lusin's theorem there exists a closed set $F_i$ $(i \geq 1)$ with $\mu(F_i) \geq 1 - \epsilon/2^i$ satisfying

$$
F_i \ni \tilde{x} \mapsto \int \varphi_n d\mu_{\varphi}^\epsilon : \text{continuous}.
$$

Then $F_\epsilon = \bigcap_{i=1}^{\infty} F_i$ has the desired property.

\[\Box\]

For $\nu \in \mathcal{M}(M_f)$ and $E \in B$ let $\nu|_E$ denote the restriction of $\nu$ to $E$, i.e. $\nu|_E(A) = \nu(A \cap E)$ for $A \in B$. Clearly $\nu|_E$ is a finite measure. We denote as $B(\tilde{x}, r)$ and $U(\tilde{x}, r)$ the closed and open balls in $M_f$ with center $\tilde{x} \in M_f$ and radius $r > 0$ respectively. Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\nu \in \mathcal{M}(M_f)$. For $\tilde{x} \in \text{supp}(\nu)$ and $\epsilon > 0$ we can find $i$ such that

$$
\int_{U(\tilde{x}, \epsilon)} \varphi_i d\nu > \int \varphi_i d\nu - \epsilon.
$$

Since the inequality holds for $\nu'$ sufficiently close to $\nu$, we can easily prove that

$$
\mathcal{M}(M_f) \ni \nu \mapsto \text{supp}(\nu) \in \mathcal{C}
$$

is lower semi-continuous and so the map is measurable ([3] Corollary III.3). Since $\nu \mapsto \text{diam}(\text{supp}(\nu))$ is lower semi-continuous,

$$
\mathcal{P}(M_f) = \{\nu \in \mathcal{M}(M_f) : \nu \text{ is a point measure}\}
= \{\nu \in \mathcal{M}(M_f) : \text{diam}(\text{supp}(\nu)) = 0\}
$$

is a closed set of $\mathcal{M}(M_f)$. Since $(\tilde{\varphi})^n(\tilde{x}) \subset \eta(\tilde{x})$, we have

$$
\text{supp}(\mu_{\tilde{\varphi}}^n) \subset \text{supp}(\mu_{\tilde{\varphi}}) \quad (n \in \mathbb{Z})
$$

for $\mu$-almost all $\tilde{x} \in M_f$.

**Lemma 5** Let $f$, $\mu$ and $\{\mu_{\tilde{\varphi}} | \tilde{x} \in M_f\}$ be as above. Then for $\mu$-almost all $\tilde{x} \in M$, $\text{supp}(\mu_{\tilde{\varphi}})$ has no isolated points.
Proof. Let $\xi$ and $\mu_{\tilde{x}}^n$ be as above. Then it is easily checked that for $n \in \mathbb{Z}$

$$P_n = \{\tilde{x} \in M_f : \mu_{\tilde{x}}^n \in \mathcal{P}(M_f)\} \supset \{\tilde{x} \in M_f : \mu_{\tilde{x}}|_{f^n(\xi)}(\tilde{x}) \text{ is a point measure}\}.$$ 

If this lemma is false, then there exists a measurable set with positive measure such that for any $\tilde{x}$ belonging to the set, $\text{supp}(\mu_{\tilde{x}})$ has an isolated point. Since $\text{diam}((f^{-k}\xi)(\tilde{x})) \to 0$ \((k \to \infty)\) by (1), we have $\mu(P_{-k}) > 0$ for $k$ large enough. Put $P = \bigcap_{j \geq 1} \bigcup_{n \geq j} f^n P_{-k}$ and then $\mu(P) = 1$ because $\mu$ is ergodic.

By (3) we have

$$\tilde{f}^n(P_{-k}) = \{\tilde{f}^n(\tilde{x}) \in M_f : \mu_{\tilde{x}}^{-k} \in \mathcal{P}(M)\} = \{\tilde{x} \in M_f : \tilde{f}^n \mu_{\tilde{x}}^{-k} \in \mathcal{P}(M)\} = P_{n-k} \quad (n \in \mathbb{Z}),$$

and so $P = \bigcap_{j \geq 1} \bigcup_{n \geq j} P_{n-k}$. Thus, for $\tilde{x} \in P$ there exists an increasing sequence \({n_i}\)\(_\geq 0\) such that $\tilde{x} \in P_{n_i}$ for $i \geq 0$. Since $\mu_{\tilde{x}} = \lim_{i \to \infty} \mu_{\tilde{x}}^{n_i}$ (by (4)) and $\mu_{\tilde{x}}^{n_i} \in \mathcal{P}(M_f)$ for $i$, we have $\mu_{\tilde{x}} \in \mathcal{P}(M_f)$ for $\tilde{x} \in P$.

Since $\xi \geq \eta$ and $\mu_{\tilde{x}}$ is a point measure for $\mu$-almost all $\tilde{x} \in M_f$, so is $\mu_{\tilde{x}}^\xi$. Thus

$$\mu_{\tilde{x}}^\xi((\tilde{f}^{-1}\xi)(\tilde{x})) = 1 \text{ for } \mu\text{-almost all } \tilde{x}.$$ Therefore

$$h_\mu(\tilde{f}) = H_\mu((\tilde{f}^{-1}\xi)\xi) = \int -\log \mu_{\tilde{x}}^\xi((\tilde{f}^{-1}\xi)(\tilde{x})) d\mu(\tilde{x}) = 0$$

by Lemma 3. This is a contradiction.

\[\square\]

3 Proof of Theorem C

In this section we will prove Theorem C. Let $f$, $\mu$, $\eta$ and $\{\mu_{\tilde{x}}|\tilde{x} \in M_f\}$ be as in §2. By Lemma 5, supp($\mu_{\tilde{x}}$) is perfect for $\mu$-almost all $\tilde{x} \in M_f$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 1 If $\mu_{\tilde{x}}$ is not a point measure for $\mu$-almost all $\tilde{x} \in M_f$, then supp($\mu_{\tilde{x}}$) is a $\ast$-chaotic set for $\mu$-almost all $\tilde{x} \in M_f$.

Proof. The proof of this proposition is similar to that of [31] Proposition 2. However, for completeness we give the proof.

Fix $0 < \varepsilon < 1$ and let $F_\varepsilon$ be as in Lemma 4. By assumption we can take and fix $\tilde{x}_0 \in \text{supp}(\mu|F_\varepsilon)$ such that $\mu_{\tilde{x}_0}$ is not a point measure. Choose two distinct points $\tilde{y}_1, \tilde{y}_2 \in \text{supp}(\mu_{\tilde{x}_0})$ and put $\tau = d(\tilde{y}_1, \tilde{y}_2)/2 > 0$. For $0 < \tau < \tau/2$ we can take $\delta = \delta(\tau) > 0$ with

$$\mu_{\tilde{x}_0}(U(\tilde{y}_i, \tau)) > \delta \quad (i = 1, 2).$$

Since $U(\tilde{y}_i, \tau)$ are open, there exists a large integer $m' = m'(\tau) > 0$ such that if $\rho(\nu, \mu_{\tilde{x}_0}) < 1/m'$ ($\nu \in \mathcal{M}(M_f)$), then

$$\nu(U(\tilde{y}_i, \tau)) > \delta = \delta(\tau) \quad (i = 1, 2).$$

By Lemma 4 we can find $\varepsilon' = \varepsilon'(r) > 0$ such that for $\tilde{x} \in U(\tilde{x}_0, \varepsilon') \cap F_\varepsilon$

$$\rho(\mu_{\tilde{x}}, \mu_{\tilde{x}_0}) < 1/2m' = 1/2m'(r).$$

Remark that

$$d(U(\tilde{y}_1, \tau), U(\tilde{y}_2, \tau)) = \inf\{d(\tilde{x}, \tilde{y}) : d(\tilde{x}, \tilde{y}_1) < \tau, d(\tilde{y}, \tilde{y}_2) < \tau\} > \tau.$$
Let $\xi$ be as in Lemma 2 and put

$$B_m(n) = \left\{ \tilde{x} \in M | \frac{\rho(\mu_{\overline{x}}^{\ell_k/2}, \mu_{\tilde{x}})}{m} < 1/m, \right. $$

$$\left. \text{diam}(\tilde{f}^{-k}[\ell_k/2] \xi(\tilde{f}^{-k}\tilde{x})) < 1/m \right\}$$

for $n, m \geq 1$. Then $B_m(n) \subset B_m(n + 1)$ and $\mu(\bigcup_{n=0}^{\infty} B_m(n)) = 1$ by (1) and (4), and so there exists an increasing sequence $\{n_m\}$ such that $\mu(B_m(n_m)) \geq 1 - 1/2^{m+1}$ ($m \geq 1$).

Since $\mu(\bigcap_{k=m}^{\infty} B_k(n_k)) \geq 1 - 1/2^{m+1}$ for $m \geq 1$, we can find $D_m \in \mathcal{B}$ with $\mu(D_m) \geq 1 - 2^{-m/2}$ satisfying

$$\mu_{\tilde{x}}(\bigcap_{k=m}^{\infty} B_k(n_k)) \geq 1 - 2^{-m/2} \quad (\tilde{x} \in D_m). \quad (7)$$

For $0 < r < \tau/2$ we put

$$K_r = \bigcap_{k=m}^{\infty} \bigcup_{n=0}^{\infty} \left( \bigcap_{n=m}^{\infty} \tilde{f}^{-\ell_k}(U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon} \cap D_{m}) \right).$$

Since $\mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon} \cap D_m) \geq \mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon}) - 2^{-m/2} > 0$ for $m$ sufficiently large, we have $\mu(K_r) = 1$ ($0 < r < \tau/2$) by the ergodicity of $\mu$. Therefore, to obtain the conclusion it suffices to show that $\text{supp}(\mu_{\tilde{x}})$ is a $\ast$-chaotic set for $\tilde{x} \in K = \bigcap_{n \geq 1} K_{1/n}$.

To do this fix $\tilde{x} \in K$, ($r = 1/n, n \geq 1$) and suppose that nonempty open sets $U_1$ and $U_2$ satisfy

$$U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \text{supp}(\mu_{\tilde{x}}) \neq \emptyset \quad (j = 1, 2).$$

Choose $m_0 > 0$ with

$$0 < 2^{-m_0/2} < \min\{\mu_{\tilde{x}}(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2m'.$$

Since $\tilde{x} \in K_r$, by the definition of $K_r$, there exist $m_1 > m_0$ and a sequence of positive integers $\{\ell_k\}_k$ with $\ell_k > n_k$ such that

$$\tilde{f}^\ell\tilde{x}(\tilde{x}) \in U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon} \cap D_{m_1} \quad (k \geq 1). \quad (8)$$

Thus, by (3) and (7) we have

$$\mu_{\tilde{x}}(\tilde{f}^{-\ell_k}(B_k(n_k))) \geq \mu_{\tilde{x}}(\tilde{f}^{-\ell_k}(\bigcap_{k=m_1}^{\infty} B_k(n_k)))$$

$$= \mu_{\tilde{f}^{\ell_k}\tilde{x}(\tilde{x})}(\bigcap_{k=m_1}^{\infty} B_k(n_k))$$

$$\geq 1 - 2^{-m_1/2} \geq 1 - 2^{-m_0/2} \quad (k \geq m_1),$$

and so $\mu_{\tilde{x}}(U_j \cap \tilde{f}^{-\ell_k}(B_k(n_k))) \geq \mu_{\tilde{x}}(U_j) - 2^{-m_0/2} > 0$. Therefore we can choose

$$\tilde{z}_j = \tilde{z}_j(k) \in U_j \cap \tilde{f}^{-\ell_k}(B_k(n_k)) \cap \eta(\tilde{x})$$

for $j = 1, 2$ and $k \geq m_1$.

Since $\tilde{f}^\ell\tilde{z}_j \in B_k(n_k) \cap \tilde{f}^\ell(\eta(\tilde{x})) \subset B_k(\ell_k) \cap \eta(\tilde{f}^\ell(\tilde{x}))$, we have

$$\rho(\mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}, \mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}) = \rho(\mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}, \mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}) < 1/k \leq 1/m_0 \leq 1/2m',$$

$$\text{diam}(\tilde{f}^{-\ell_k+[\ell_k/2]}(\tilde{z}_j)) < 1/k$$

for $j = 1, 2$ and $k \geq m_1$. By use of (6) and (8)

$$\rho(\mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}, \mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}, \mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}) < 1/2m' + 1/2m' = 1/m',$$

and so $\mu_{\tilde{z}_j}^{-\ell_k+[\ell_k/2]}(\tilde{f}^{-\ell_k}U(\tilde{y}_i, r)) = \mu_{\tilde{f}^{[\ell_k/2]}(\tilde{z}_j)}(U(\tilde{y}_i, r)) > \delta$ by (5). Thus we have

$$\tilde{f}^{-\ell_k+[\ell_k/2]}(\tilde{z}_j) \cap \tilde{f}^{-\ell_k}U(\tilde{y}_i, r) \neq \emptyset$$
for $1 \leq i,j \leq 2$ and $k \geq m_1$. Since $\tilde{z}_j \in U_j$, by (9) we may assume

$$\tilde{z}_j \in (\tilde{f}^{-\ell_k + [\ell_k/2]} \xi)(\tilde{z}_j) \subset U_j$$

for $k$ large enough. Therefore

$$U_j \cap \tilde{f}^{-\ell_k} U(\tilde{y}_1, r) \supset (\tilde{f}^{-\ell_k + [\ell_k/2]} \xi)(\tilde{z}_j) \cap \tilde{f}^{-\ell_k} U(\tilde{y}_1, r) \neq \emptyset$$

for $1 \leq i,j \leq 2$ and $k$ large enough.

Now we take $b_{i,j} = b_{i,j}(k) \in U_j \cap \tilde{f}^{-\ell_k} U(\tilde{y}_1, r)$ for $1 \leq i,j \leq 2$ and then

$$b_{i,j} \in U_j \quad (1 \leq i,j \leq 2),$$

$$d(\tilde{f}^{\ell_k}(b_{1,1}), \tilde{f}^{\ell_k}(b_{2,2})) > d(U(\tilde{y}_1, r), U(\tilde{y}_2, r)) > \tau$$

and

$$d(\tilde{f}^{\ell_k}(b_{1,1}), \tilde{f}^{\ell_k}(b_{1,2})) \leq \text{diam}(U(\tilde{y}_1, r)) = 2r = 2/n.$$

This implies that $\text{supp}(\mu_{\tilde{x}})$ is a $*$-chaotic set for $\tilde{x} \in K = \cap_{n \geq 1} K_{1/n}$. \hfill $\square$

References


