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<th>Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke (Studies on complex dynamics and related topics)</th>
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<td>Author(s)</td>
<td>Sumi, Naoya</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2001, 1220: 54-62</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41284">http://hdl.handle.net/2433/41284</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
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Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke

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Abstract
We show that if \( f \) is a \( C^2 \)-local diffeomorphism with positive entropy on a \( n \)-dimensional closed manifold (\( n \geq 2 \)) then \( f \) is chaotic in the sense of Li-Yorke.

1 Introduction
We study chaotic properties of dynamical systems with positive entropy. Notions of chaos have been given by Li and Yorke [15], Devaney [5] and others. It is well known that if a continuous map of an interval has positive entropy, then the map is chaotic according to the definition of Li and Yorke (cf. [2]). For invertible maps the following holds: let \( f \) be a \( C^2 \)-diffeomorphism of a closed \( C^\infty \)-manifold. If the topological entropy of \( f \) is positive, then \( f \) is chaotic in the sense of Li-Yorke [31].

In this paper we show the following:

**Theorem A.** Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold. If the topological entropy of \( f \) is positive, then \( f \) is chaotic in the sense of Li-Yorke.

From this theorem we obtain the following corollary.

**Corollary B.** Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold. If \( f \) is not invertible, then \( f \) is chaotic in the sense of Li-Yorke.

First we shall explain here the definitions and notations used above. Let \( X \) be a compact metric space with metric \( d \) and let \( f : X \to X \) be a continuous map. A subset \( S \) of \( X \) is a scrambled set of \( f \) if there is a positive number \( \tau > 0 \) such that for any \( x, y \in S \) with \( x \neq y \),

1. \( \limsup_{n \to \infty} d(f^n(x), f^n(y)) > \tau \),
2. \( \liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0 \).

If there is an uncountable scrambled set \( S \) of \( f \), then we say that \( f \) is chaotic in the sense of Li-Yorke. Li and Yorke showed in [15] that if \( f : [0, 1] \to [0, 1] \) is a continuous map with a periodic point of period 3, then \( f \) is chaotic in this sense. Note that any scrambled set contains at most one point \( x \) which does not satisfy the following: for any periodic point \( p \in X \),

\[
\limsup_{n \to \infty} d(f^n(x), f^n(p)) > 0.
\]

For another sufficient condition for the chaos in the sense of Li-Yorke, the readers may refer to [4], [7], [8], [9], [10], [11], [19], [20], [34].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "\( \ast \)-chaos" as follows: let \( F \) be a closed subset of \( X \). A map \( f : X \to X \) is \( \ast \)-chaotic on \( F \) (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is \( \tau > 0 \) with the property that for any nonempty open subsets \( U \) and \( V \) of \( F \) with \( U \cap V = \emptyset \) and for any natural number \( N \), there is \( n \geq N \) such that \( d(f^n(x), f^n(y)) > \tau \) for some \( x \in U \), \( y \in V \), and
2. for any nonempty open subsets $U, V$ of $F$ and any $\varepsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for some $x \in U, y \in V$.

Such a set $F$ is called a $*$-chaotic set. If $S$ is a scrambled set, then the closure of $S, \tilde{S}$, is a $*$-chaotic set. In [10] Kato showed that the converse is true. This is stated precisely as follows:

**Lemma 1** ([10], Theorem 2.4) Let $X$ be a compact metric space and let $F$ be a closed subset of $X$. If $f : X \to X$ is continuous and is $*$-chaotic on $F$, then there is an $F_\sigma$-set $S \subset F$ such that $S$ is a scrambled set of $f$ and $\tilde{S} = F$. If $F$ is perfect (i.e. $F$ has no isolated points), we can choose $S$ as a countable union of Cantor sets.

By this lemma, to show the existence of uncountable scrambled sets it suffices to show the existence of perfect $*$-chaotic sets.

To obtain Theorem A we consider the inverse limit system of $f$. Let $M$ be a closed $C^\infty$-manifold and let $d$ be the distance for $M$ induced by a Riemannian metric $\| \cdot \|$ on $TM$. Let $M^Z$ denote the product topological space $M^Z = \{(x_i) : x_i \in M, i \in \mathbb{Z}\}$. Then $M^Z$ is compact. We define a compatible metric $\tilde{d}$ for $M^Z$ by

$$\tilde{d}((x_i), (y_i)) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2|i|} \quad ((x_i), (y_i) \in M^Z).$$

For $f : M \to M$ a continuous surjection, we let

$$M_f = \{(x_i) : x_i \in M \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$ 

Then $M_f$ is a closed subset of $M^Z$. The space $M_f$ is called the inverse limit space constructed by $f$. A homeomorphism $\tilde{f} : M_f \to M_f$, which is defined by

$$\tilde{f}(x_i) = (f(x_i)) \text{ for all } (x_i) \in M_f,$$

is called the shift map determined by $f$. We denote as $P^0 : M_f \to M$ the projection defined by $(x_i) \mapsto x_0$. Then $P^0 \circ \tilde{f} = f \circ P^0$ holds. Remark that $f$ is chaotic in the sense of Li-Yorke if and only if so is $\tilde{f}$.

We can show that the topological entropy, $h(f)$, of $f$ coincides with that of $\tilde{f}$. Indeed, for an $f$-invariant probability measure $\nu$, we can find an $\tilde{f}$-invariant probability measure $\mu$ such that $\nu(A) = P^0_\nu(A) = \mu((P^0)^{-1}(A))$ for any Borel set $A \subset M$ ([18] Lemma IV.8.3). Let us denote as $h_\nu(f)$ and $h_\mu(\tilde{f})$ the metric entropy of $(M, f, \nu)$ and $(M_f, \tilde{f}, \mu)$ respectively. Then we have $h_\nu(f) = h_{P^0\nu}(f) = h_\mu(\tilde{f})$ ([25] Lemma 5.2). Therefore, the conclusion is obtained by the variational principle ([32] Theorem 8.6).

We say that a differentiable map $f : M \to M$ is a local diffeomorphism if for $x \in M$ there is an open neighborhood $U_x$ of $x$ in $M$ such that $f(U_x)$ is open in $M$ and $f|_{U_x} : U_x \to f(U_x)$ is a diffeomorphism. Since $M$ is connected, then the cardinal number of $f^{-1}(x)$ is constant. This constant is called the covering degree of $f$. If the covering degree of $f$ is greater than one, $(M_f, M, C, P^0)$ is a fiber bundle where $C$ denotes the Cantor set (see [1] Theorem 6.5.1). Let $\mu$ be a Borel probability measure on $M_f$ and let $B$ be the Borel $\sigma$-algebra on $M_f$ completed with respect to $\mu$. For $\xi$ a measurable partition of $M_f$ and $\tilde{z} \in M_f$ we denote as $\xi(\tilde{z})$ the element of the partition $\xi$ which contains the point $\tilde{z}$. Then there exists a family $\{\mu^z_\xi | \tilde{z} \in M_f\}$ of Borel probability measures satisfying the following conditions:

1. for $\tilde{z}, \tilde{y} \in M_f$ if $\xi(\tilde{z}) = \xi(\tilde{y})$ then $\mu^z_\xi = \mu^y_\xi$,
2. $\mu^z_\xi(\xi(\tilde{z})) = 1$ for $\mu$-almost all $\tilde{z} \in M_f$,
3. for $A \in B$ a function $\tilde{z} \mapsto \mu^z_\xi(A)$ is measurable and $\mu(A) = \int_{M_f} \mu^z_\xi(A) d\mu(\tilde{z})$.

The family $\{\mu^z_\xi | \tilde{z} \in M_f\}$ is called a canonical system of conditional measures for $\mu$ and $\xi$ (see [26] for more details).

To prove Theorem A it suffices to show the following theorem.
Theorem C Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and let \( \mu \) be an \( f \)-invariant ergodic Borel probability measure on \( M_f \).

If the metric entropy of \( \mu \) is positive, then there exists a measurable partition \( \eta \) of \( M_f \) such that \( \text{supp}(\mu^2_{\xi}) \) is a perfect \( * \)-chaotic set for \( \mu \)-almost all \( \hat{x} \in M_f \).

Here the support \( \text{supp}(\nu) \) of a finite measure \( \nu \) is the smallest closed set \( C \) with \( \nu(C) = \nu(M_f) \). Equivalently, \( \text{supp}(\nu) \) is the set of all \( \hat{x} \in M_f \) with the property that \( \nu(U) > 0 \) for any open \( U \) containing \( \hat{x} \).

Let us see how Theorem A follows from Theorem C. We know that \( h(\tilde{f}) = \sup\{ h_{\mu}(\tilde{f}) : \mu \in \mathcal{M}_e(\tilde{f}) \} \) where \( \mathcal{M}_e(\tilde{f}) \) is the set of all \( \tilde{f} \)-invariant ergodic probability measures (cf. [27]). Thus, if \( h(\tilde{f}) = h(f) > 0 \), then we can choose \( \mu \in \mathcal{M}_e(\tilde{f}) \) with \( h_{\mu}(\tilde{f}) > 0 \). Therefore, by Theorem C and Lemma 1, \( f \) is chaotic in the sense of Li-Yorke.

2 Key Lemmas

In this section we prepare some lemmas which need to prove Theorem C. Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and \( \mu \) be an \( f \)-invariant ergodic Borel probability measure on \( M_f \) with \( h_{\mu}(f) > 0 \). As in the previous section we denote as \( \mathcal{B} \) the Borel \( \sigma \)-algebra on \( M_f \) completed with respect to \( \mu \). For \( \mu \)-almost all \( \hat{x} = (x_i) \in M_f \), there exist a splitting of the tangent space \( T_{x_0}M = \oplus_{i=1}^\epsilon E_i(\hat{x}) \) and real numbers \( \lambda_1(x_0) < \cdots < \lambda_{\epsilon}(x_0) \) such that

(a) the maps \( \hat{x} \mapsto E_i(\hat{x}) \), \( \lambda_1(x_0) \) and \( s(x_0) \) are measurable, moreover \( E_i(f(\hat{x})) = D_{x_0}f(E_i(\hat{x})) \)
and \( \lambda_i(x_0), s(x_0) \) are \( f \)-invariant (\( i = 1, \cdots, s(x_0) \)) and

(b) \( \lim_{n \to \pm \infty} \frac{1}{n} \log \| (D_{x_0}f^{|n|})^{\pm 1}(v) \| = \lambda_i(x_0) \) (\( 0 \neq v \in E_i(\hat{x}), i = 1, \cdots, s(x_0) \)) and

(c) \( \lim_{n \to \pm \infty} \frac{1}{n} \log \| \text{det}(D_{x_0}f^{|n|})^{\pm 1} \| = \sum_{i=1}^{s(x_0)} \lambda_i(x_0) \dim E_i(\hat{x}) \)

([21], [33], [29], [30]). The numbers \( \lambda_1(x_0), \cdots, \lambda_{\epsilon}(x_0) \) are called Lyapunov exponents of \( f \) at \( x_0 \). Since \( \mu \) is ergodic, we can put \( s = s(x_0) \), \( \lambda_i = \lambda_i(x_0) \) and \( m_i = \dim E_i(\hat{x}) \) (\( i = 1, \cdots, s \)) for \( \mu \)-almost all \( \hat{x} = (x_i) \in M_f \).

A well-known theorem of Margulis and Ruelle [28] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. \( h_{\mu}(f) \leq \sum_{i > 0} \lambda_i m_i \). Since \( \bar{f} \) has positive entropy, we have \( 0 < h_{\mu}(f) = h_{\mu}(\tilde{f}) \leq \max(\lambda_i) = \lambda_\epsilon \). Fix \( 0 < \lambda < \min(\lambda_i : \lambda_i > 0) \). From [24], [29] and [30] there are measurable functions \( \tilde{\beta} > \tilde{\alpha} > 0 \) and \( \tilde{\gamma} > 1 \) with the following properties: For \( \hat{x} = (x_i) \in M_f \) we put

\[
\tilde{W}_\text{loc}^u(\hat{x}) = \{ \tilde{y} = (y_i) \in M_f : d(x_0, y_0) \leq \tilde{\alpha}(\hat{x}), d(x_i, y_i) \leq \tilde{\beta}(\hat{x}) e^{-\lambda i} (i \geq 1) \}.
\]

Then

(a) the map \( P^0 \) restricted to \( \tilde{W}_\text{loc}^u(\hat{x}) \) is injective,

(b) \( P^0(\tilde{W}_\text{loc}^u(\hat{x})) \) is a \( C^2 \)-submanifold of the ball \( \{ y \in M : d(x_0, y) \leq \tilde{\alpha}(\hat{x}) \} \),

(c) \( T_{x_0}P^0(\tilde{W}_\text{loc}^u(\hat{x})) = \oplus_{i > 0} E_i(\hat{x})(\neq \{0\}) \) for \( \mu \)-almost all \( \hat{x} \in M_f \),

(d) \( d(y_i, z_i) \leq \tilde{\gamma}(\hat{x}) d(y_0, z_0) e^{-\lambda i} \) for \( (y_n), (z_n) \in \tilde{W}_\text{loc}^u(\hat{x}) \).

For the case when \( f \) is invertible we may refer to [6], [22] and [23].

Let \( \xi \) and \( \eta \) be measurable partitions of \( M_f \). Put \( f^n \xi = \{ f^n C : C \in \xi \} \) for \( n \in \mathbb{Z} \) and then \( (f^n \xi)(\hat{x}) = f^n (\xi(f^{-n}(\hat{x}))) \) for \( \hat{x} \in M_f \). \( \eta \leq \xi \) means that for \( \mu \)-almost all \( \hat{x} \in M_f \) one has \( \xi(\hat{x}) \subset \eta(\hat{x}) \).

Lemma 2 Let \( f \) and \( \mu \) be as above. Then there exists a measurable partition \( \xi \) of \( M_f \) such
(a) $\xi \leq \tilde{f}^{-1}\xi$,

(b) for $\mu$-almost all $\tilde{x} \in M_f$, $\xi(\tilde{x}) \subset \tilde{W}^u_{\mathrm{loc}}(\tilde{x})$ and $\xi(\tilde{x})$ contains a neighborhood of $\tilde{x}$ open in $\tilde{W}^u_{\mathrm{loc}}(\tilde{x})$,

(c) $\bigvee_{n=0}^{\infty} \tilde{f}^{-n}\xi$ is the partition into points.

This lemma is similar to [13] Proposition 3.1, [16] Proposition 5.2 and [17] Lemma 2.2. So we omit the proof.

Let $C$ denote the family of all nonempty closed subsets of $M_f$ and define a metric $d_H$ by

$$d_H(A, B) = \max\{\sup_{b \in B} d(a, b), \sup_{a \in A} d(a, B)\} \quad (A, B \subset C)$$

where $d(a, b) = \inf\{d(a, b) : a \in A\}$. Then it is known that $(C, d_H)$ is a compact metric space (cf.[12]). If $\xi$ is a measurable partition, then $\tilde{x} \mapsto \xi(\tilde{x}) \in C$ is measurable. Indeed, this follows from [3] Theorems III.2, III.9, III.22 and the fact that $\{(\tilde{x}, \xi(\tilde{x})) : \tilde{x} \in M_f\}$ is a Borel subset of $M_f \times M_f$. For $A \subset M_f$ we put $\diam(A) = \sup\{d(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in A\}$. Then we have $\diam(A) = \diam(A)$. Since $\tilde{x} \mapsto \xi(\tilde{x}) \in C$ is measurable, $\tilde{x} \mapsto \diam(\xi(\tilde{x}))$ is also a measurable function. By Lemma 2 (c) we have that for $\mu$-almost all $\tilde{x} \in M_f$

$$\diam(\tilde{f}^{-n}\xi(\tilde{x})) \to 0 \quad (1)$$

as $n \to \infty$.

Let $\xi$ and $\eta$ be measurable partitions of $M_f$ and let $\{\mu_{\tilde{x}}^\xi : \tilde{x} \in M_f\}$ be a canonical system of conditional measures for $\mu$ and $\xi$. The mean conditional entropy of $\eta$ with respect to $\xi$ is defined by

$$H_\mu(\eta|\xi) = \int -\log \mu_{\tilde{x}}^\xi(\eta(\tilde{x})) d\mu(\tilde{x})$$

(see [27] for details).

Lemma 3 Let $f$ and $\mu$ be as above and let $\xi$ be as in Lemma 2. Then,

$$h_\mu(\tilde{f}) = H_\mu(\tilde{f}^{-1}\xi|\xi).$$

For the case when $f$ is invertible this lemma is proved by Ledrappier and Young [14]. We recall that if the covering degree of $f$ is greater than one, then $(M_f, M, C, P^\mu)$ is a fiber bundle where $C$ denotes the Cantor set. In view of this fact, the above lemma can be proved by almost the same arguments as the proof of [14] Corollary 5.3 and [16] Corollary 7.1 with some slight modifications. Here we omit the proof.

By Lemma 2(a) we have that $\xi \geq \tilde{f}\xi \geq \tilde{f}^2\xi \geq \cdots$. Let us introduce a measurable partition defined by $\eta = \bigwedge_{i=0}^{\infty} \overline{f}^i\xi$. Then we have $\tilde{f}\eta = \eta$. For simplicity put

$$\mu_{\tilde{x}} = \mu_{\tilde{x}}^\eta \quad \text{and} \quad \mu_{\tilde{x}}^n = \mu_{\tilde{x}}^{\tilde{f}^n\xi} \quad (n \in \mathbb{Z}).$$

By Doob's theorem it follows that for a $\mu$-integrable function $\psi : M_f \to \mathbb{R}$

$$\int \psi d\mu_{\tilde{x}} = \lim_{n \to \infty} \int \psi d\mu_{\tilde{x}}^n \quad (2)$$

for $\mu$-almost all $\tilde{x}$. Since $\tilde{f}\eta = \eta$ and $\tilde{f}\mu = \mu$, by the uniqueness of a canonical system of conditional measures (cf.[26]) we have that for $\mu$-almost all $\tilde{x}$

$$\tilde{f}\mu_{\tilde{x}} = \mu_{\tilde{x}} \quad \text{and} \quad \tilde{f}\mu_{\tilde{x}}^n = \mu_{\tilde{x}}^{\tilde{f}^n\xi} \quad (n \in \mathbb{Z}).$$

(3)

Here $(\tilde{f}, \nu)(A) = \nu(\tilde{f}^{-1}A)$ for a Borel probability measure $\nu$ on $M_f$ and $A \in B$.

Let $C(M_f)$ be the Banach space of continuous real-valued functions of $M_f$ with the sup norm $| \cdot |_\infty$, and let $\mathcal{M}(M_f)$ be a set of all Borel probability measures on $M_f$ with the weak
topology. Since $C(M_f)$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \cdots\}$ which is dense in $C(M_f)$. For $\nu, \nu' \in M(M_f)$ define

$$
\rho(\nu, \nu') = \sum_{n=1}^{\infty} \frac{|\int \varphi_n d\nu - \int \varphi_n d\nu'|}{2^n|\varphi_n|_{\infty}}.
$$

Then $\rho$ is a compatible metric for $M(M_f)$ and $(M(M_f), \rho)$ is compact (cf.[18]). Since (2) holds for $\{\varphi_i\}$, we have

$$
\mu_{\tilde{x}} = \lim_{n \to \infty} \mu_{\tilde{x}}^n
$$

for $\mu$-almost all $\tilde{x}$. For $\nu \in M(M_f)$ and a measurable partition $\xi$, by the definition of conditional measures $\{\nu_{\tilde{x}}^\xi\}$, the map

$$
M_f \ni \tilde{x} \mapsto \int \varphi_n d\nu_{\tilde{x}}^\xi
$$

is measurable for $n \geq 1$ and thus $\tilde{x} \mapsto \nu_{\tilde{x}}^\xi \in M(M_f)$ is measurable.

**Lemma 4** Let $f, \mu$ and $\{\mu_{\tilde{x}}|\tilde{x} \in M_f\}$ be as above. Then for $\epsilon > 0$ there exists a closed set $F_\epsilon$ with $\mu(F_\epsilon) \geq 1 - \epsilon$ satisfying the map

$$
F_\epsilon \ni \tilde{x} \mapsto \mu_{\tilde{x}} \in M(M_f)
$$

is continuous.

**Proof.** Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\epsilon > 0$. Since $\tilde{x} \mapsto \int \varphi_n d\nu_{\tilde{x}}^\xi$ is measurable for $i \geq 1$, by Lusin's theorem there exists a closed set $F_i$ ($i \geq 1$) with $\mu(F_i) \geq 1 - \epsilon/2^i$ satisfying

$$
F_i \ni \tilde{x} \mapsto \int \varphi_i d\mu_{\tilde{x}}^\xi : \text{continuous}.
$$

Then $F_\epsilon = \bigcap_{i=1}^{\infty} F_i$ has the desired property.

For $\nu \in M(M_f)$ and $E \in B$ let $\nu|_E$ denote the restriction of $\nu$ to $E$, i.e. $\nu|_E(A) = \nu(A \cap E)$ for $A \in B$. Clearly $\nu|_E$ is a finite measure. We denote as $B(\tilde{x}, r)$ and $U(\tilde{x}, r)$ the closed and open balls in $M_f$ with center $\tilde{x} \in M_f$ and radius $r > 0$ respectively. Let $\{\varphi_1, \varphi_2, \cdots\}$ be as above and let $\nu \in M(M_f)$. For $\tilde{x} \in \text{supp}(\nu)$ and $\epsilon > 0$ we can find $i$ such that

$$
\int_{U(\tilde{x}, \epsilon)} \varphi_i d\nu > \int \varphi_i d\nu - \epsilon.
$$

Since the inequality holds for $\nu'$ sufficiently close to $\nu$, we can easily prove that

$$
M(M_f) \ni \nu \mapsto \text{supp}(\nu) \in \mathcal{C}
$$

is lower semi-continuous and so the map is measurable ([3] Corollary III.3). Since $\nu \mapsto \text{diam}(\text{supp}(\nu))$ is lower semi-continuous,

$$
\mathcal{P}(M_f) = \{ \nu \in M(M_f) : \nu \text{ is a point measure} \}
$$

$$
= \{ \nu \in M(M_f) : \text{diam}(\text{supp}(\nu)) = 0 \}
$$

is a closed set of $M(M_f)$. Since $(\tilde{f}^n \xi)(\tilde{x}) \subset \eta(\tilde{x})$, we have

$$
\text{supp}(\mu_{\tilde{x}}^\xi) \subset \text{supp}(\mu_{\tilde{x}}) \quad (n \in \mathbb{Z})
$$

for $\mu$-almost all $\tilde{x} \in M_f$.

**Lemma 5** Let $f, \mu$ and $\{\mu_{\tilde{x}}|\tilde{x} \in M_f\}$ be as above. Then for $\mu$-almost all $\tilde{x} \in M$, $\text{supp}(\mu_{\tilde{x}})$ has no isolated points.
Proof. Let \( \xi \) and \( \mu_{\overline{x}}^{n} \) be as above. Then it is easily checked that for \( n \in \mathbb{Z} \)

\[
P_{n} = \{ \tilde{x} \in M_{f} : \mu_{\overline{x}}^{n} \in \mathcal{P}(M_{f}) \} \supset \{ \tilde{x} \in M_{f} : \mu_{\tilde{x}}|_{\tilde{f}^{-1}\xi}(\tilde{x}) \text{ is a point measure} \}.
\]

If this lemma is false, then there exists a measurable set with positive measure such that for any \( \tilde{x} \) belonging to the set, \( \text{supp}(\mu_{\tilde{x}}) \) has an isolated point. Since \( \text{diam}(\tilde{f}^{-k}\xi)(\tilde{x})) \to 0 \) \((k \to \infty)\) by (1), we have \( \mu(P_{-k}) > 0 \) for \( k \) large enough. Put \( P = \bigcap_{j \geq 1} \bigcup_{n \geq j} P_{n-k} \) and then \( \mu(P) = 1 \) because \( \mu \) is ergodic.

By (3) we have

\[
\tilde{f}^{n}(P_{-k}) = \{ \tilde{x} \in M_{f} : \mu_{\overline{x}}^{-k} \in \mathcal{P}(M) \}
= \{ \tilde{x} \in M_{f} : \tilde{f}_{*}^{n} \mu_{\overline{x}}^{-k} \in \mathcal{P}(M) \}
= \{ \tilde{x} \in M_{f} : \mu_{\overline{x}}^{-k} \in \mathcal{P}(M) \}
= P_{n-k} \quad (n \in \mathbb{Z}),
\]

and so \( P = \bigcap_{j \geq 1} \bigcup_{n \geq j} P_{n-k} \). Thus, for \( \tilde{x} \in P \) there exists an increasing sequence \( \{ n_{i} \}_{i \geq 0} \) such that \( \tilde{x} \in P_{n_{i}} \) for \( i \geq 0 \). Since \( \mu_{\tilde{x}} = \lim_{n \to \infty} \mu_{\overline{x}}^{n} \) (by (4)) and \( \mu_{\tilde{x}}^{0} \in \mathcal{P}(M_{f}) \) for \( i \), we have \( \mu_{\tilde{x}} \in \mathcal{P}(M_{f}) \) for \( \tilde{x} \in P \).

Since \( \xi \geq \eta \) and \( \mu_{\tilde{x}} \) is a point measure for \( \mu \)-almost all \( \tilde{x} \in M_{f} \), so is \( \mu_{\tilde{x}}^{\xi} \). Thus \( \mu_{\tilde{x}}^{\xi}((\tilde{f}^{-1}\xi)(\tilde{x})) = 1 \) for \( \mu \)-almost all \( \tilde{x} \). Therefore

\[
h_{\mu}(\tilde{f}) = H_{\mu}(\tilde{f}^{-1}\xi|\xi) = \int -\log \mu_{\tilde{x}}^{\xi}((\tilde{f}^{-1}\xi)(\tilde{x})) d\mu_{\overline{x}}(\tilde{x}) = 0
\]

by Lemma 3. This is a contradiction. \( \square \)

3 Proof of Theorem C

In this section we will prove Theorem C. Let \( f, \mu, \eta \) and \( \{ \mu_{\tilde{x}}| \tilde{x} \in M_{f} \} \) be as in §2. By Lemma 5, \( \text{supp}(\mu_{\tilde{x}}) \) is perfect for \( \mu \)-almost all \( \tilde{x} \in M_{f} \). Therefore, to obtain the conclusion it suffices to show the following.

Proposition 1 If \( \mu_{\tilde{x}} \) is not a point measure for \( \mu \)-almost all \( \tilde{x} \in M_{f} \), then \( \text{supp}(\mu_{\tilde{x}}) \) is a \( * \)-chaotic set for \( \mu \)-almost all \( \tilde{x} \in M_{f} \).

Proof. The proof of this proposition is similar to that of [31] Proposition 2. However, for completeness we give the proof.

Fix \( 0 < \epsilon < 1 \) and let \( F_{\epsilon} \) be as in Lemma 4. By assumption we can take and fix \( \tilde{x}_{0} \in \text{supp}(\mu|F_{\epsilon}) \) such that \( \mu_{\tilde{x}_{0}} \) is not a point measure. Choose two distinct points \( \tilde{y}_{1}, \tilde{y}_{2} \) in \( \text{supp}(\mu_{\tilde{x}_{0}}) \) and put \( r = d(\tilde{y}_{1}, \tilde{y}_{2})/2(>0) \). For \( 0 < r < \tau/2 \) we can take \( \delta = \delta(r) > 0 \) with

\[
\mu_{\tilde{x}_{0}}(U(\tilde{y}_{i}, r)) > \delta \quad (i = 1, 2).
\]

Since \( U(\tilde{y}_{i}, r) \) are open, there exists a large integer \( m' = m'(r) > 0 \) such that if \( \rho(\nu, \mu_{\tilde{x}_{0}}) < 1/m' \) \((\nu \in \mathcal{M}(M_{f})) \), then

\[
\nu(U(\tilde{y}_{i}, r)) > \delta = \delta(r) \quad (i = 1, 2).
\]

By Lemma 4 we can find \( \epsilon' = \epsilon'(r) > 0 \) such that for \( \tilde{x} \in U(\tilde{x}_{0}, \epsilon') \cap F_{\epsilon} \)

\[
\rho(\mu_{\tilde{x}}, \mu_{\tilde{x}_{0}}) < 1/2m' = 1/2m'(r).
\]

Remark that

\[
d(U(\tilde{y}_{1}, r), U(\tilde{y}_{2}, r)) = \inf\{d(\tilde{x}, \tilde{y}) : d(\tilde{x}, \tilde{y}_{1}) < r, d(\tilde{y}, \tilde{y}_{2}) < r \} > \tau.
\]
Let \( \xi \) be as in Lemma 2 and put
\[
B_m(n) = \left\{ \tilde{x} \in M_f \middle| \begin{array}{c}
\rho(\mu_{\tilde{x}}^{[k/2]}, \mu_{\tilde{x}}) < 1/m, \\
\operatorname{diam}(\hat{f}^{-k+[k/2]}(\xi) \cap \hat{f}^{-k}\tilde{x})) < 1/m \end{array} \right\}
\]
for \( n, m \geq 1 \). Then \( B_m(n) \subset B_m(n+1) \) and \( \mu(\bigcap_{n=0}^\infty B_m(n)) = 1 \) by (1) and (4), and so there exists an increasing sequence \( \{n_m\} \) such that \( \mu(B_m(n_m)) \geq 1 - 1/2^{m+1} \) \( (m \geq 1) \). Since \( \mu(\bigcap_{k=m}^\infty B_k(n_k)) \geq 1 - 1/2^m \) for \( m \geq 1 \), we can find \( D_m \in B \) with \( \mu(D_m) \geq 1 - 2^{-m/2} \) satisfying
\[
\mu_{\tilde{x}}(\bigcap_{k=m}^\infty B_k(n_k)) \geq 1 - 2^{-m/2} \quad (\tilde{x} \in D_m).
\]
For \( 0 < r < \tau/2 \) we put
\[
K_r = \bigcap_{k=1}^\infty \bigcup_{m=k}^\infty \left( \bigcap_{n=0}^\infty \hat{f}^{-\ell}(U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon \cap D_m) \right).
\]
Since \( \mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon \cap D_m) \geq \mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon) - 2^{-m/2} > 0 \) for \( m \) sufficiently large, we have \( \mu(K_r) = 1 \) \( (0 < r < \tau/2) \) by the ergodicity of \( \mu \). Therefore, to obtain the conclusion it suffices to show that \( \operatorname{supp}(\mu_{\tilde{x}}) \) is \( \ast \)-chaotic for \( \tilde{x} \in K = \bigcap_{n \geq 1} K_{1/n} \).

To do this fix \( \tilde{x} \in K_r \) \( (r = 1/n, n \geq 1) \) and suppose that nonempty open sets \( U_1 \) and \( U_2 \) satisfy
\[
U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \operatorname{supp}(\mu_{\tilde{x}}) \neq \emptyset \quad (j = 1, 2).
\]
Choose \( m_0 > 0 \) with
\[
0 < 2^{-m_0/2} < \min\{\mu_{\tilde{x}}(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2m'.
\]
Since \( \tilde{x} \in K_r \), by the definition of \( K_r \), there exist \( m_1 > m_0 \) and a sequence of positive integers \( \{\ell_k\}_k \) with \( \ell_k > n_k \) such that
\[
\hat{f}^{\ell_k}(\tilde{x}) \in U(\tilde{x}_0, \epsilon'(r)) \cap F_\epsilon \cap D_m \quad (k \geq 1).
\]
Thus, by (3) and (7) we have
\[
\mu_{\tilde{x}}(\hat{f}^{-\ell_k}(B_k(n_k))) \geq \mu_{\tilde{x}}(\hat{f}^{-\ell_k}(\bigcap_{k=m_1}^\infty B_k(n_k)))
= \mu_{\hat{f}^{\ell_k}(\tilde{x})}(\bigcap_{k=m_1}^\infty B_k(n_k))
\geq 1 - 2^{-m_1} \geq 1 - 2^{-m_0/2} \quad (k \geq m_1),
\]
and so \( \mu_{\tilde{x}}(U_j \cap \hat{f}^{-\ell_k}(B_k(n_k))) \geq \mu_{\tilde{x}}^\ast(U_j) - 2^{-m_0/2} > 0 \). Therefore we can choose
\[
\tilde{z}_j = \tilde{z}_j(k) \in U_j \cap \hat{f}^{-\ell_k}(B_k(n_k)) \cap \eta(\tilde{x})
\]
for \( j = 1, 2 \) and \( k \geq m_1 \).

Since \( \hat{f}^{\ell_k}(\tilde{z}_j) \in B_k(n_k) \cap \hat{f}^{\ell_k}(\eta(\tilde{x})) \subset B_k(\ell_k) \cap \eta(\hat{f}^{\ell_k}(\tilde{x})) \), we have
\[
\rho(\mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}) \leq \rho(\mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}, \mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}) < 1/k \leq 1/m_0 \leq 1/2m',
\]
\[
\operatorname{diam}(\hat{f}^{-\ell_k+[\ell_k/2]}(\xi)(\tilde{z}_j)) < 1/k
\]
for \( j = 1, 2 \) and \( k \geq m_1 \). By use of (6) and (8)
\[
\rho(\mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\tilde{x}_0}) \leq \rho(\mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}, \mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}) + \rho(\mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}, \mu_{\tilde{x}_0})
< 1/2m' + 1/2m' = 1/m',
\]
and so \( \mu_{\tilde{x}_0}^{-[\ell_k+[\ell_k/2]}(\hat{f}^{-\ell_k}U(\tilde{y}_i, r)) = \mu_{\hat{f}^{\ell_k}(\tilde{z}_j)}^{[\ell_k/2]}(U(\tilde{y}_i, r)) > \delta \) by (5). Thus we have
\[
(\hat{f}^{-\ell_k+[\ell_k/2]}(\xi)(\tilde{z}_j) \cap \hat{f}^{-\ell_k}U(\tilde{y}_i, r) \neq \emptyset)
\]
for $1 \leq i,j \leq 2$ and $k \geq m_1$. Since $\tilde{z}_j \in U_j$, by (9) we may assume
\[ \tilde{z}_j \in (\tilde{f}^{-t^k + [t^k/2]} \xi)(\tilde{z}_j) \subset U_j \]
for $k$ large enough. Therefore
\[ U_j \cap \tilde{f}^{-t^k} U(\tilde{y}_1, r) \supset (\tilde{f}^{-t^k + [t^k/2]} \xi)(\tilde{z}_j) \cap \tilde{f}^{-t^k} U(\tilde{y}_1, r) \neq \emptyset \]
for $1 \leq i,j \leq 2$ and $k$ large enough. Therefore

Now we take $b_{i,j} = b_{i,j}(k) \in U_j \cap \tilde{f}^{-t^k} U(\tilde{y}_1, r) \cap \tilde{f}^{-t^k} U(\tilde{y}_2, r) \neq \emptyset \cap \overline{f}^{-\ell_k}[\ell_k/2] \xi(\tilde{z}_j) \cap \tilde{f}^{-\ell_k} U(\tilde{y}_1, r)$ for $1 \leq i,j \leq 2$ and then
\[ b_{i,j} \in U_j \quad (1 \leq i,j \leq 2), \]
\[ d(f^{t_k}(b_{1,1}), f^{t_k}(b_{2,2})) > d(U(\tilde{y}_1, r), U(\tilde{y}_2, r)) > \tau \quad \text{and} \]
\[ d(f^{t_k}(b_{1,1}), f^{t_k}(b_{1,2})) \leq \text{diam}(U(\tilde{y}_1, r)) = 2r = 2/n. \]
This implies that support($\mu_\tilde{x}$) is a $*$-chaotic set for $\tilde{x} \in K = \cap_{n \geq 1} K_{1/n}$. \qed

References


