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Local diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke

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Abstract

We show that if $f$ is a $C^2$-local diffeomorphism with positive entropy on a $n$-dimensional closed manifold ($n \geq 2$) then $f$ is chaotic in the sense of Li-Yorke.

1 Introduction

We study chaotic properties of dynamical systems with positive entropy. Notions of chaos have been given by Li and Yorke [15], Devaney [5] and others. It is well known that if a continuous map of an interval has positive entropy, then the map is chaotic according to the definition of Li and Yorke (cf. [2]). For invertible maps the following holds: let $f$ be a $C^2$-diffeomorphism of a closed $C^\infty$-manifold. If the topological entropy of $f$ is positive, then $f$ is chaotic in the sense of Li-Yorke [31].

In this paper we show the following:

Theorem A. Let $f$ be a $C^2$-local diffeomorphism of a closed $C^\infty$-manifold. If the topological entropy of $f$ is positive, then $f$ is chaotic in the sense of Li-Yorke.

From this theorem we obtain the following corollary.

Corollary B. Let $f$ be a $C^2$-local diffeomorphism of a closed $C^\infty$-manifold. If $f$ is not invertible, then $f$ is chaotic in the sense of Li-Yorke.

First we shall explain here the definitions and notations used above. Let $X$ be a compact metric space with metric $d$ and let $f: X \to X$ be a continuous map. A subset $S$ of $X$ is a scrambled set of $f$ if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

1. $\limsup_{n \to \infty} d(f^n(x), f^n(y)) > \tau$,  
2. $\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0$.

If there is an uncountable scrambled set $S$ of $f$, then we say that $f$ is chaotic in the sense of Li-Yorke. Li and Yorke showed in [15] that if $f: [0,1] \to [0,1]$ is a continuous map with a periodic point of period $3$ then $f$ is chaotic in this sense. Note that any scrambled set contains at most one point $x$ which does not satisfy the following: for any periodic point $p \in X$,

$$\limsup_{n \to \infty} d(f^n(x), f^n(p)) > 0.$$  

For another sufficient condition for the chaos in the sense of Li-Yorke, the readers may refer to [4], [7], [8], [9], [10], [11], [19], [20], [34].

Concerning the chaos in the sense of Li-Yorke, Kato introduced the notion of "*-chaos" as follows: let $F$ be a closed subset of $X$. A map $f: X \to X$ is *-chaotic on $F$ (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ with the property that for any nonempty open subsets $U$ and $V$ of $F$ with $U \cap V = \emptyset$ and for any natural number $N$, there is $n \geq N$ such that $d(f^n(x), f^n(y)) > \tau$ for some $x \in U$, $y \in V$, and
2. for any nonempty open subsets $U, V$ of $F$ and any $\varepsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for some $x \in U, y \in V$.

Such a set $F$ is called a *-chaotic set. If $S$ is a scrambled set, then the closure of $S$, $\overline{S}$, is a *-chaotic set. In [10] Kato showed that the converse is true. This is stated precisely as follows:

**Lemma 1 ([10], Theorem 2.4)** Let $X$ be a compact metric space and let $F$ be a closed subset of $X$. If $f : X \to X$ is continuous and is *-chaotic on $F$, then there is an $F_\varepsilon$-set $S \subset F$ such that $S$ is a scrambled set of $f$ and $\overline{S} = F$. If $F$ is perfect (i.e. $F$ has no isolated points), we can choose $S$ as a countable union of Cantor sets.

By this lemma, to show the existence of uncountable scrambled sets it suffices to show the existence of perfect *-chaotic sets.

To obtain Theorem A we consider the inverse limit system of $f$. Let $M$ be a closed $C^\infty$-manifold and let $d$ be the distance for $M$ induced by a Riemannian metric $\| \cdot \|$ on $TM$. Let $M^Z$ denote the product topological space $M^Z = \{(x_i) : x_i \in M, i \in \mathbb{Z}\}$. Then $M^Z$ is compact. We define a compatible metric $\tilde{d}$ for $M^Z$ by

$$\tilde{d}((x_i), (y_i)) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}} \quad ((x_i), (y_i) \in M^Z).$$

For $f : M \to M$ a continuous surjection, we let

$$M_f = \{(x_i) : x_i \in M \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$ 

Then $M_f$ is a closed subset of $M^Z$. The space $M_f$ is called the inverse limit space constructed by $f$. A homeomorphism $\tilde{f} : M_f \to M_f$, which is defined by

$$\tilde{f}((x_i)) = (f(x_i)) \text{ for all } (x_i) \in M_f,$$

is called the shift map determined by $f$. We denote $P^0 : M_f \to M$ the projection defined by $(x_i) \mapsto x_0$. Then $P^0 \circ \tilde{f} = f \circ P^0$ holds. Remark that $f$ is chaotic in the sense of Li-Yorke if and only if so is $\tilde{f}$.

We can show that the topological entropy, $h(f)$, of $f$ coincides with that of $\tilde{f}$. Indeed, for an $f$-invariant probability measure $\nu$, we can find an $\tilde{f}$-invariant probability measure $\mu$ such that $\nu(A) = P^0_\mu(A) = \mu((P^0)^{-1}A))$ for any Borel set $A \subset M$ ([18] Lemma IV 8.3). Let us denote as $h_{\nu}(f)$ and $h_{\mu}(\tilde{f})$ the metric entropy of $(M, f, \nu)$ and $(M_f, \tilde{f}, \mu)$ respectively. Then we have $h_{\nu}(f) = h_{P^0\mu}(f) = h_{\mu}(\tilde{f})$ ([25] Lemma 5.2). Therefore, the conclusion is obtained by the variational principle ([32] Theorem 8.6).

We say that a differentiable map $f : M \to M$ is a local diffeomorphism if for $x \in M$ there is an open neighborhood $U_x$ of $x$ in $M$ such that $f(U_x)$ is open in $M$ and $f|U_x : U_x \to f(U_x)$ is a diffeomorphism. Since $M$ is connected, then the cardinal number of $f^{-1}(x)$ is constant. This constant is called the covering degree of $f$. If the covering degree of $f$ is greater than one, $(M_f, M, C, P^0)$ is a fiber bundle where $C$ denotes the Cantor set (see [1] Theorem 6.5.1).

Let $\mu$ be a Borel probability measure on $M_f$ and let $B$ be the Borel $\sigma$-algebra on $M_f$ completed with respect to $\mu$. For $\xi$ a measurable partition of $M_f$ and $\tilde{x} \in M_f$ we denote as $\xi(\tilde{x})$ the element of the partition $\xi$ which contains the point $\tilde{x}$. Then there exists a family $\{\mu^t_{\tilde{x}} | \tilde{x} \in M_f\}$ of Borel probability measures satisfying the following conditions:

1. for $\tilde{x}, \tilde{y} \in M_f$ if $\xi(\tilde{x}) = \xi(\tilde{y})$ then $\mu^t_{\tilde{x}} = \mu^t_{\tilde{y}}$,

2. $\mu^t_{\tilde{x}}(\xi(\tilde{x})) = 1$ for $\mu$-almost all $\tilde{x} \in M_f$,

3. for $A \in B$ a function $\tilde{x} \mapsto \mu^t_{\tilde{x}}(A)$ is measurable and $\mu(A) = \int_{M_f} \mu^t_{\tilde{x}}(A) d\mu(\tilde{x})$.

The family $\{\mu^t_{\tilde{x}} | \tilde{x} \in M_f\}$ is called a canonical system of conditional measures for $\mu$ and $\xi$ (see [26] for more details).

To prove Theorem A it suffices to show the following theorem.

To prove Theorem A it suffices to show the following theorem.
Theorem C Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and let \( \mu \) be an \( f \)-invariant ergodic Borel probability measure on \( M_f \).

If the metric entropy of \( \mu \) is positive, then there exists a measurable partition \( \eta \) of \( M_f \) such that \( \text{supp}(\mu_0^\eta) \) is a perfect \( \ast \)-chaotic set for \( \mu \)-almost all \( \tilde{x} \in M_f \).

Here the support \( \text{supp}(\nu) \) of a finite measure \( \nu \) is the smallest closed set \( C \) with \( \nu(C) = \nu(M_f) \). Equivalently, \( \text{supp}(\nu) \) is the set of all \( \tilde{x} \in M_f \) with the property that \( \nu(U) > 0 \) for any open \( U \) containing \( \tilde{x} \).

Let us see how Theorem A follows from Theorem C. We know that \( h(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}_f(f)\} \) where \( \mathcal{M}_f(f) \) is the set of all \( f \)-invariant ergodic probability measures (cf. [27]). Thus, if \( h(f) = h(f) > 0 \), then we can choose \( \mu \in \mathcal{M}_f(f) \) with \( h_\mu(f) > 0 \). Therefore, by Theorem C and Lemma 1, \( f \) is chaotic in the sense of Li-Yorke.

2 Key Lemmas

In this section we prepare some lemmas which need to prove Theorem C. Let \( f \) be a \( C^2 \)-local diffeomorphism of a closed \( C^\infty \)-manifold \( M \) and \( \mu \) be an \( f \)-invariant ergodic Borel probability measure on \( M_f \) with \( h_\mu(f) > 0 \). As in the previous section we denote as \( B \) the Borel \( \sigma \)-algebra on \( M_f \) completed with respect to \( \mu \). For \( \mu \)-almost all \( \tilde{x} = (x_i) \in M_f \), there exist a splitting of the tangent space \( T_{x_i}M = \oplus_{i=1}^s \mathcal{E}_i(\tilde{x}) \) and real numbers \( \lambda_i(x_0) < \lambda_2(x_0) < \cdots < \lambda_{\dim E_\mu}(x_0) \) such that

(a) the maps \( \tilde{x} \mapsto E_i(\tilde{x}) \), \( \lambda_i(x_0) \) and \( s(x_0) \) are measurable, moreover \( E_i(\tilde{f}(\tilde{x})) = D_{x_0}f(E_i(\tilde{x})) \) and \( \lambda_i(x_0), s(x_0) \) are \( f \)-invariant \((i = 1, \cdots, s(x_0))\),

(b) \( \lim_{n \to \pm \infty} \frac{1}{n} \log ||D_{x_0}f^{[n]}||^{\pm 1}(v)|| = \lambda_i(x_0) \quad (0 \neq v \in E_i(\tilde{x}), \ i = 1, \cdots, s(x_0)) \) and

(c) \( \lim_{n \to \pm \infty} \frac{1}{n} \log |\det(D_{x_0}f^{[n]})^{\pm 1}| = \sum_{i=1}^{s(x_0)} \lambda_i(x_0) \dim E_i(\tilde{x}) \)

([21], [33], [29], [30]). The numbers \( \lambda_1(x_0), \cdots, \lambda_{s(x_0)}(x_0) \) are called Lyapunov exponents of \( f \) at \( x_0 \). Since \( \mu \) is ergodic, we can put \( s = s(x_0), \lambda_i = \lambda_i(x_0) \) and \( m_i = \dim E_i(\tilde{x}) \) \((i = 1, \cdots, s)\) for \( \mu \)-almost all \( \tilde{x} = (x_i) \in M_f \).

A well-known theorem of Margulis and Ruelle [28] says that entropy is always bounded above by the sum of positive Lyapunov exponents; i.e. \( h_{\mu}^{\infty}(f) \leq \sum_{\lambda_i > 0} \lambda_i m_i \). Since \( \tilde{f} \) has positive entropy, we have \( 0 < h_{\mu}(f) = h_{\mu}(f) \leq \max(\lambda_i) = \lambda_\ast \). Fix \( 0 < \lambda < \min(\lambda_i : \lambda_i > 0) \). From [24], [29] and [30] there are measurable functions \( \tilde{\beta} > \tilde{\alpha} > 0 \) and \( \tilde{\gamma} > 1 \) with the following properties: For \( \tilde{x} = (x_i) \in M_f \) we put

\[ \tilde{W}_{loc}^u(\tilde{x}) = \{ \tilde{y} = (y_i) : d(x_0, y_0) \leq \tilde{\alpha}(\tilde{x}), d(x_i, y_i) \leq \tilde{\beta}(\tilde{x}) e^{-\lambda i} \ (i \geq 1) \}. \]

Then

(a) the map \( P^0 \) restricted to \( \tilde{W}_{loc}^u(\tilde{x}) \) is injective,

(b) \( P^0(\tilde{W}_{loc}^u(\tilde{x})) = \mathcal{C}^2 \)-submanifold of the ball \{ \( y \in M : d(x_0, y) \leq \tilde{\alpha}(\tilde{x}) \} \),

(c) \( T_{x_i}P^0(\tilde{W}_{loc}^u(\tilde{x})) = \oplus_{\lambda_i > 0} E_i(\tilde{x})(\neq 0) \) for \( \mu \)-almost all \( \tilde{x} \in M_f \),

(d) \( d(y_i, z_i) \leq \tilde{\gamma}(\tilde{x}) d(y_0, z_0) e^{-\lambda i} \) for \( (y_n), (z_n) \in \tilde{W}_{loc}^u(\tilde{x}) \).

For the case when \( f \) is invertible we may refer to [6], [22] and [23].

Let \( \xi \) and \( \eta \) be measurable partitions of \( M_f \). Put \( f^n \xi = \{f^n C : C \in \xi \} \) for \( n \in \mathbb{Z} \) and then \( \{(f^n \xi)(\tilde{x}) = f^n(\xi(f^{-n}(\tilde{x}))) \) for \( \tilde{x} \in M_f \). \( \eta \leq \xi \) means that for \( \mu \)-almost all \( \tilde{x} \in M_f \) one has \( \xi(\tilde{x}) \subset \eta(\tilde{x}) \).

Lemma 2 Let \( f \) and \( \mu \) be as above. Then there exists a measurable partition \( \xi \) of \( M_f \) such
(a) $\xi \leq \tilde{f}^{-1}\xi$,

(b) for $\mu$-almost all $\tilde{x} \in M_f$, $\xi(\tilde{x}) \subset \tilde{W}_{loc}^{u}(\tilde{x})$ and $\xi(\tilde{x})$ contains a neighborhood of $\tilde{x}$ open in $\tilde{W}_{loc}^{u}(\tilde{x})$,

(c) $\bigvee_{n=0}^{\infty} \tilde{f}^{-n}\xi$ is the partition into points.

This lemma is similar to [13] Proposition 3.1, [16] Proposition 5.2 and [17] Lemma 2.2. So we omit the proof.

Let $C$ denote the family of all nonempty closed subsets of $M_f$ and define a metric $d_H$ by

$$d_H(A, B) = \max\{\sup_{b \in B} d(A, b), \sup_{a \in A} d(a, B)\} \quad (A, B \subset C)$$

where $d(a, b) = \inf\{d(a, b) : a \in A\}$. Then it is known that $(C, d_H)$ is a compact metric space (cf.[12]). If $\xi$ is a measurable partition, then $\tilde{x} \mapsto \xi(\tilde{x}) \in C$ is measurable. Indeed, this follows from [3] Theorems III.2, III.9, III.22 and the fact that $\{(\tilde{x}, \xi(\tilde{x})) : \tilde{x} \in M_f\}$ is a Borel subset of $M_f \times M_f$. For $A \subset M_f$ we put $\mathrm{diam}(A) = \sup\{d(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in A\}$. Then we have $\mathrm{diam}(A) = \mathrm{diam}(\tilde{A})$. Since $\tilde{x} \mapsto \xi(\tilde{x}) \in C$ is measurable, $\tilde{x} \mapsto \mathrm{diam}(\xi(\tilde{x}))$ is also a measurable function. By Lemma 2 (c) we have that for $\mu$-almost all $\tilde{x} \in M_f$

$$\mathrm{diam}((\tilde{f}^{-n}\xi)(\tilde{x})) \to 0 \quad (1)$$

as $n \to \infty$.

Let $\xi$ and $\eta$ be measurable partitions of $M_f$ and let $\{\mu^\xi_{\tilde{x}} : \tilde{x} \in M_f\}$ be a canonical system of conditional measures for $\mu$ and $\xi$. The mean conditional entropy of $\eta$ with respect to $\xi$ is defined by

$$H_\mu(\eta|\xi) = \int -\log \mu^\xi_{\tilde{x}}(\eta(\tilde{x}))d\mu(\tilde{x})$$

(see [27] for details).

**Lemma 3** Let $f$ and $\mu$ be as above and let $\xi$ be as in Lemma 2. Then,

$$h_\mu(\tilde{f}) = H_\mu(\tilde{f}^{-1}\xi|\xi).$$

For the case when $f$ is invertible this lemma is proved by Ledrappier and Young [14]. We recall that if the covering degree of $f$ is greater than one, then $(M_f, M, C, P)$ is a fiber bundle where $C$ denotes the Cantor set. In view of this fact, the above lemma can be proved by almost the same arguments as the proof of [14] Corollary 5.3 and [16] Corollary 7.1 with some slight modifications. Here we omit the proof.

By Lemma 2(a) we have that $\xi \geq \tilde{f}\xi \geq \tilde{f}^2\xi \geq \cdots$. Let us introduce a measurable partition defined by $\eta = \bigwedge_{i=0}^{\infty} \tilde{f}^i\xi$. Then we have $\tilde{f}\eta = \eta$. For simplicity put

$$\mu_{\tilde{x}} = \mu^\eta_{\tilde{x}} \quad \text{and} \quad \mu_{\tilde{x}}^n = \mu^{\tilde{f}^n\eta}_{\tilde{x}} \quad (n \in \mathbb{Z}).$$

By Doob’s theorem it follows that for a $\mu$-integrable function $\psi : M_f \to \mathbb{R}$

$$\int \psi d\mu_{\tilde{x}} = \lim_{n \to \infty} \int \psi d\mu_{\tilde{x}}^n \quad (2)$$

for $\mu$-almost all $\tilde{x}$. Since $\tilde{f}\eta = \eta$ and $\tilde{f}_*\mu = \mu$, by the uniqueness of a canonical system of conditional measures (cf.[26]) we have that for $\mu$-almost all $\tilde{x}$

$$\tilde{f}_*\mu_{\tilde{x}} = \mu_{\tilde{x}} \quad \text{and} \quad \tilde{f}_*\mu_{\tilde{x}}^n = \mu_{\tilde{x}}^{n+1} \quad (n \in \mathbb{Z}). \quad (3)$$

Here $(\tilde{f}_*\nu)(A) = \nu(\tilde{f}^{-1}A)$ for a Borel probability measure $\nu$ on $M_f$ and $A \in B$.

Let $C(M_f)$ be the Banach space of continuous real-valued functions of $M_f$ with the sup norm $| \cdot |_{\infty}$, and let $\mathcal{M}(M_f)$ be a set of all Borel probability measures on $M_f$ with the weak
topology. Since $C(M_f)$ is separable, there exists a countable set $\{\varphi_1, \varphi_2, \ldots\}$ which is dense in $C(M_f)$. For $\nu, \nu' \in \mathcal{M}(M_f)$ define

$$\rho(\nu, \nu') = \sum_{n=1}^{\infty} \left| \frac{\int \varphi_n d\nu - \int \varphi_n d\nu'}{2^n|\varphi_n|_\infty} \right|.$$

Then $\rho$ is a compatible metric for $\mathcal{M}(M_f)$ and $(\mathcal{M}(M_f), \rho)$ is compact (cf.[18]). Since (2) holds for $\{\varphi_i\}$, we have

$$\mu_\bar{x} = \lim_{n \to \infty} \mu_{\bar{x}}^n$$

(4)

for $\mu$-almost all $\bar{x}$. For $\nu \in \mathcal{M}(M_f)$ and a measurable partition $\xi$, by the definition of conditional measures $\nu_{\bar{x}}^\xi$, the map

$$M_f \ni \bar{x} \mapsto \int \varphi_n d\nu_{\bar{x}}^\xi$$

is measurable for $n \geq 1$ and thus $\bar{x} \mapsto \nu_{\bar{x}}^\xi \in \mathcal{M}(M_f)$ is measurable.

**Lemma 4** Let $f$, $\mu$ and $\{\mu_\bar{x}\}_{\bar{x} \in M_f}$ be as above. Then for $\epsilon > 0$ there exists a closed set $F_\epsilon$ with $\mu(F_\epsilon) \geq 1 - \epsilon/2^i$ satisfying the map

$$F_\epsilon \ni \bar{x} \mapsto \mu_\bar{x} \in \mathcal{M}(M_f)$$

is continuous.

**Proof.** Let $\{\varphi_1, \varphi_2, \ldots\}$ be as above and let $\epsilon > 0$. Since $\bar{x} \mapsto \int \varphi_n d\nu_{\bar{x}}^\xi$ is measurable for $i \geq 1$, by Lusin's theorem there exists a closed set $F_i$ $(i \geq 1)$ with $\mu(F_i) \geq 1 - \epsilon/2^i$ satisfying

$$F_i \ni \bar{x} \mapsto \int \varphi_n d\mu_\bar{x} : \text{continuous}.$$

Then $F_\epsilon = \bigcap_{i=1}^{\infty} F_i$ has the desired property.

\[ \square \]

For $\nu \in \mathcal{M}(M_f)$ and $E \in \mathcal{B}$ let $\nu|_E$ denote the restriction of $\nu$ to $E$, i.e. $\nu|_E(A) = \nu(A \cap E)$ for $A \in \mathcal{B}$. Clearly $\nu|_E$ is a finite measure. We denote as $B(\bar{x}, r)$ and $U(\bar{x}, r)$ the closed and open balls in $M_f$ with center $\bar{x} \in M_f$ and radius $r > 0$ respectively. Let $\{\varphi_1, \varphi_2, \ldots\}$ be as above and let $\nu \in \mathcal{M}(M_f)$. For $\bar{x} \in \text{supp}(\nu)$ and $\epsilon > 0$ we can find $i$ such that

$$\int_{U(\bar{x}, \epsilon)} \varphi_i d\nu > \int \varphi_i d\nu - \epsilon.$$

Since the inequality holds for $\nu'$ sufficiently close to $\nu$, we can easily prove that

$$\mathcal{M}(M_f) \ni \nu \mapsto \text{supp}(\nu) \in \mathcal{C}$$

is lower semi-continuous and so the map is measurable ([3] Corollary III.3). Since $\nu \mapsto \text{diam}(\text{supp}(\nu))$ is lower semi-continuous,

$$\mathcal{P}(M_f) = \{\nu \in \mathcal{M}(M_f) : \nu \text{ is a point measure}\}$$

$$= \{\nu \in \mathcal{M}(M_f) : \text{diam}(\text{supp}(\nu)) = 0\}$$

is a closed set of $\mathcal{M}(M_f)$. Since $(\tilde{f}^n \xi)(\bar{x}) \subset \eta(\bar{x})$, we have

$$\text{supp}(\mu_{\bar{x}}^\xi) \subset \text{supp}(\mu_{\bar{x}}) \quad (n \in \mathbb{Z})$$

for $\mu$-almost all $\bar{x} \in M_f$.

**Lemma 5** Let $f$, $\mu$ and $\{\mu_\bar{x}|\bar{x} \in M_f\}$ be as above. Then for $\mu$-almost all $\bar{x} \in M$, $\text{supp}(\mu_{\bar{x}})$ has no isolated points.
Proof. Let $\xi$ and $\mu^0_\overline{x}$ be as above. Then it is easily checked that for $n \in \mathbb{Z}$

$$P_n = \{\tilde{x} \in M_f : \mu^0_\overline{x} \in \mathcal{P}(M_f)\} \supset \{\tilde{x} \in M_f : \mu_\overline{x}|_{(\overline{f}^{-1}\xi)(\tilde{x})} \text{ is a point measure}\}.$$ 

If this lemma is false, then there exists a measurable set with positive measure such that for any $\tilde{x}$ belonging to the set, $\text{supp}(\mu_\overline{x})$ has an isolated point. Since $\text{diam}((\overline{f}^{-k}\xi)(\tilde{x})) \to 0$ ($k \to \infty$) by (1), we have $\mu(P_{-k}) > 0$ for $k$ large enough. Put $P = \cap_{i \geq 1} \bigcup_{n \geq i} f^n P_{-k}$ and then $\mu(P) = 1$ because $\mu$ is ergodic.

By (3) we have

$$\tilde{f}^n(P_{-k}) = \{\tilde{f}^n(\tilde{x}) \in M_f : \mu^{-k}_{\tilde{x}} \in \mathcal{P}(M)\}$$

$$= \{\tilde{x} \in M_f : \tilde{f}^n \mu^{-k}_{\tilde{x}} \in \mathcal{P}(M)\}$$

$$= \{\tilde{x} \in M_f : \mu^{-k}_{\tilde{x}} \in \mathcal{P}(M)\}$$

$$= P_{n-k} \quad (n \in \mathbb{Z}),$$

and so $P = \cap_{i \geq 1} \bigcup_{n \geq i} P_{n-k}$. Thus, for $\tilde{x} \in P$ there exists an increasing sequence $\{n_i\}_{i \geq 0}$ such that $\tilde{x} \in P_{n_i}$ for $i \geq 0$. Since $\mu^{-k}_\overline{x} = \lim_{i \to \infty} \mu^{-n_i}_\overline{x}$ (by (4)) and $\mu^{-n_i}_\overline{x} \in \mathcal{P}(M_f)$ for $i$, we have $\mu_\overline{x} \in \mathcal{P}(M_f)$ for $\tilde{x} \in P$.

Since $\xi \geq \eta$ and $\mu_\overline{x}$ is a point measure for $\mu$-almost all $\tilde{x} \in M_f$, so is $\mu^0_\overline{x}$. Thus $\mu^0_\overline{x}(\overline{x}) = 1$ for $\mu$-almost all $\tilde{x}$. Therefore

$$h_\mu(\tilde{f}) = H_\mu(\tilde{f}^{-1}\xi|\xi) = \int -\log \mu(\tilde{f}^{-1}\xi|\xi)d\mu(\tilde{x}) = 0$$

by Lemma 3. This is a contradiction.

3 Proof of Theorem C

In this section we will prove Theorem C. Let $f$, $\mu$, $\eta$ and $\{\mu_\overline{x}|\overline{x} \in M_f\}$ be as in §2. By Lemma 5, $\text{supp}(\mu_\overline{x})$ is perfect for $\mu$-almost all $\tilde{x} \in M_f$. Therefore, to obtain the conclusion it suffices to show the following.

Proposition 1 If $\mu_\overline{x}$ is not a point measure for $\mu$-almost all $\tilde{x} \in M_f$, then $\text{supp}(\mu_\overline{x})$ is a $*$-chaotic set for $\mu$-almost all $\tilde{x} \in M_f$.

Proof. The proof of this proposition is similar to that of [31] Proposition 2. However, for completeness we give the proof.

Fix $0 < \epsilon < 1$ and let $F_\epsilon$ be as in Lemma 4. By assumption we can take and fix $\tilde{x}_0 \in \text{supp}(\mu|F_\epsilon)$ such that $\mu_{\tilde{x}_0}$ is not a point measure. Choose two distinct points $\tilde{y}_1, \tilde{y}_2 \in \text{supp}(\mu_{\tilde{x}_0})$ and put $\tau = d(\tilde{y}_1, \tilde{y}_2)/2(>0)$. For $0 < r < \tau/2$ we can take $\delta = \delta(r) > 0$ with

$$\mu_{\tilde{x}_0}(U(\tilde{y}_i, r)) > \delta \quad (i = 1, 2).$$

Since $U(\tilde{y}_i, r)$ are open, there exists a large integer $m' = m'(r) > 0$ such that if $\rho(\nu, \mu_{\tilde{x}_0}) < 1/m'$ ($\nu \in \mathcal{M}(M_f)$), then

$$\nu(U(\tilde{y}_i, r)) > \delta = \delta(r) \quad (i = 1, 2).$$

By Lemma 4 we can find $\epsilon' = \epsilon'(r) > 0$ such that for $\tilde{x} \in U(\tilde{x}_0, \epsilon') \cap F_\epsilon$

$$\rho(\mu_\overline{x}, \mu_{\tilde{x}_0}) < 1/2m' = 1/2m'(r).$$

Remark that

$$d(U(\tilde{y}_1, r), U(\tilde{y}_2, r)) = \inf \{d(\tilde{x}, \tilde{y}) : d(\tilde{x}, \tilde{y}_1) < r, \ d(\tilde{y}, \tilde{y}_2) < r\} > \tau.$$
Let $\xi$ be as in Lemma 2 and put

$$B_m(n) = \left\{ \tilde{x} \in M_f \right\} \rho(\mu_{\tilde{x}}^{[k/2]}, \mu_{\tilde{z}}) < 1/m,\quad \text{diam}(f^{k+[k/2]}(\tilde{z})) < 1/m \quad (k \geq n)$$

for $n, m \geq 1$. Then $B_m(n) \subset B_m(n+1)$ and $\mu(\bigcup_{n=0}^{\infty} B_m(n)) = 1$ by (1) and (4), and so there exists an increasing sequence $\{n_1\}$ such that $\mu(B_m(n_1)) \geq 1 - 1/2^{m+1}$ ($m \geq 1$). Since $\mu(\bigcap_{k=m}^{\infty} B_k(n_k)) \geq 1 - 1/2^m$ for $m \geq 1$, we can find $D_m \in B$ with $\mu(D_m) \geq 1 - 2^{-m/2}$ satisfying

$$\mu_{\tilde{x}}(\bigcap_{k=m}^{\infty} B_k(n_k)) \geq 1 - 2^{-m/2} \quad (\tilde{x} \in D_m). \quad (7)$$

For $0 < r < \tau/2$ we put

$$K_r = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \left( \bigcap_{n=0}^{\infty} \bigcup_{\ell=n}^{\infty} f^{-\ell}(U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon} \cap D_{m}) \right).$$

Since $\mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon} \cap D_m) \geq \mu(U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon}) - 2^{-m/2} > 0$ for $m$ sufficiently large, we have $\mu(K_r) = 1$ ($0 < r < \tau/2$) by the ergodicity of $\mu$. Therefore, to obtain the conclusion it suffices to show that supp($\mu_{\tilde{x}}$) is a *-chaotic set for $\tilde{x} \in K = \bigcap_{n \geq 1} K_{1/n}$.

To do this fix $\tilde{x} \in K_r$ ($r = 1/n, n \geq 1$) and suppose that nonempty open sets $U_1$ and $U_2$ satisfy

$$U_1 \cap U_2 \neq \emptyset, \quad U_j \cap \text{supp}(\mu_{\tilde{x}}) \neq \emptyset \quad (j = 1, 2).$$

Choose $m_0 > 0$ with

$$0 < 2^{-m_0/2} < \min\{\mu_{\tilde{x}}(U_j) : j = 1, 2\} \quad \text{and} \quad m_0 \geq 2m'.$$

Since $\tilde{x} \in K_r$, by the definition of $K_r$, there exist $m_1 > m_0$ and a sequence of positive integers $\{\ell_k\}$ with $\ell_k > n_k$ such that

$$f^{\ell_k}(\tilde{x}) \in U(\tilde{x}_0, \epsilon'(r)) \cap F_{\epsilon} \cap D_{m_1} \quad (k \geq 1). \quad (8)$$

Thus, by (3) and (7) we have

$$\mu_{\tilde{x}}(f^{\ell_k}(B_k(n_k))) \geq \mu_{\tilde{x}}(f^{\ell_k}(\bigcap_{k=m_1}^{\infty} B_k(n_k))) \geq \mu_{\tilde{x}}(f^{\ell_k}(\bigcap_{k=m}^{\infty} B_k(n_k))) \geq 1 - 2^{-m/2} \geq 1 - 2^{-m_0/2} \geq 1 - 2^{-m_1/2} \quad (k \geq m_1),$$

and so $\mu_{\tilde{x}}(U_j \cap f^{\ell_k}(B_k(n_k))) \geq \mu_{\tilde{x}}^2(U_j) - 2^{-m_0/2} > 0$. Therefore we can choose

$$\tilde{z}_j = \tilde{z}_j(k) \in U_j \cap f^{\ell_k}(B_k(n_k)) \cap \eta(\tilde{x})$$

for $j = 1, 2$ and $k \geq m_1$.

Since $f^{\ell_k}(\tilde{z}_j) \in B_k(n_k) \cap f^{\ell_k}(\eta(\tilde{x})) \subset B_k(\ell_k) \cap \eta(f^{\ell_k}(\tilde{x}))$, we have

$$\rho(\mu_{f^{\ell_k}(\tilde{z}_j)}, \mu_{f^{\ell_k}(\tilde{z}_j)}) = \rho(\mu_{f^{\ell_k/(2)}(\tilde{z}_j)}, \mu_{f^{\ell_k/(2)}(\tilde{z}_j)}) \leq 1/k \leq 1/m_0 \leq 1/2m', \quad \text{diam}(f^{\ell_k+[\ell_k/2]}(\tilde{x}))/f^{\ell_k}(\tilde{x}) < 1/k,$$

for $j = 1, 2$ and $k \geq m_1$. By use of (6) and (8)

$$\rho(\mu_{f^{\ell_k/(2)}(\tilde{z}_j)}, \mu_{\tilde{x}}) \leq \rho(\mu_{f^{\ell_k/(2)}(\tilde{z}_j)}, \mu_{f^{\ell_k/(2)}(\tilde{z}_j)}) + \rho(\mu_{f^{\ell_k/(2)}(\tilde{z}_j)}, \mu_{\tilde{x}}) < 1/2m' + 1/2m' = 1/m'.$$

and so $\mu_{\tilde{x}}(f^{\ell_k+[\ell_k/2]}(\tilde{z}_j) \cap \tilde{z}_j \neq \emptyset$ by (5). Thus we have

$$(f^{\ell_k+[\ell_k/2]}(\tilde{z}_j) \cap \tilde{z}_j \neq \emptyset$$

and

$$\mu_{\tilde{z}_j}^{\ell_k+[\ell_k/2]}(\tilde{z}_j) \cap \tilde{z}_j \neq \emptyset.$$
for $1 \leq i, j \leq 2$ and $k \geq m_1$. Since $\tilde{z}_j \in U_j$, by (9) we may assume
\[ \tilde{z}_j \in \left( \tilde{f}^{-t_k + \lfloor t_k/2 \rfloor} \xi \right)(\tilde{z}_j) \subset U_j \]
for $k$ large enough. Therefore
\[ U_j \cap \tilde{f}^{-t_k} U(\tilde{y}_i, r) \supset \left( \tilde{f}^{-t_k + \lfloor t_k/2 \rfloor} \xi \right)(\tilde{z}_j) \cap \tilde{f}^{-t_k} U(\tilde{y}_i, r) \neq \emptyset \]
for $1 \leq i, j \leq 2$ and $k$ large enough.

Now we take $b_{i,j} = b_{i,j}(k) \in U_j \cap \tilde{f}^{-t_k} U(\tilde{y}_i, r)$ for $1 \leq i, j \leq 2$ and then
\[
\begin{align*}
& b_{i,j} \in U_j \quad (1 \leq i, j \leq 2), \\
& d(f^{t_k}(b_{1,1}), f^{t_k}(b_{2,2})) > d(U(\tilde{y}_1, r), U(\tilde{y}_2, r)) > \tau \quad \text{and} \\
& d(f^{t_k}(b_{1,1}), f^{t_k}(b_{1,2})) \leq \text{diam}(U(\tilde{y}_1, r)) = 2r = 2/n.
\end{align*}
\]
This implies that supp$(\mu_{\tilde{x}})$ is a $*$-chaotic set for $\tilde{x} \in K = \cap_{n \geq 1} K_{1/n}$. \hfill $\square$

References


