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Some remarks on generalized inverse $*$-semigroups II [1]

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Abstract

By using a concept of representations of generalized inverse $*$-semigroups [2], we introduce a new partial product on a generalized inverse $*$-semigroup. The purpose of this paper is to give a characterization of prehomomorphisms of generalized inverse $*$-semigroups.

1 Introduction

A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies

$$\text{(i) } (x^*)^* = x; \qquad \text{(ii) } (xy)^* = y^*x^*; \qquad \text{(iii) } xx^*x = x.$$

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if $e^* = e$. For a subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular $*$-semigroup. Define a relation $\leq$ on S as follows:

$$a \leq b \iff a = eb = bf \quad \text{for some } \ e, f \in P(S).$$

A regular $*$-semigroup S is called a *generalized inverse $*$-semigroup* if $E(S)$ satisfies the identity $xyzw = xzyw$. In this case, $E(S)$ forms a band.

Result 1.1. [1] *Let a and b be elements of S. Then the following conditions are equivalent:*

(1) $a \leq b$,

(2) $aa^* = ba^*$ *and* $a^*a = b^*a$,

(3) $aa^* = ab^*$ *and* $a^*a = a^*b$,

(4) $a = aa^*b = ba^*a$.

Moreover, if S is a generalized inverse $*$-semigroup, the conditions above are equivalent to the following:

[1]This paper is an abstract and the details will be published elsewhere.

(5) $a = eb = bf$ *for some* $e, f \in E(S)$.

Result 1.2. [1] *The relation* $\leq$ *on a regular* $*$*-semigroup, defined above, is a partial order on* S *which preserves the unary operation. Moreover, if* S *is a generalized inverse* $*$*-semigroup,* $\leq$ *is compatible.*

We call the partial order $\leq$, defined above, *the natural order* on S.

Let S and T be regular $*$-semigroups. A mapping $\phi : S \to T$ is called a *prehomomorphism*, if it satisfies

(i) $(ab)\phi \leq (a\phi)(b\phi)$,

(ii) $(a\phi)^* = a^*\phi$,

for all $a, b \in S$.

Result 1.3. [1] *Let* ϕ *be a prehomomorphism of a reguar* $*$*-semigroup* S *to a regular* $*$*-semigroup* T*. Then we have the following:*

(1) ϕ *maps an idempotent of* S *to an idempotent of* T*, and so it maps a projection of* S *to a projection of* T*,*

(2) ϕ *is isotone, that is,* $a \leq b$ *implies* $a\phi \leq b\phi$*,*

As a generalization of the Preston-Vagner representations, we obtain a representation of a generalized inverse $*$-semigroup [2]. A non-empty set X with an equivalence relation σ is called a *transitive* ι*-set*, and denoted by $(X;\sigma)$. Let $(X;\sigma)$ be a transitive ι-set. A subset A of X is called an ι*-single subset* of $(X;\sigma)$ if there exists at most one element of A for each equivalence class induced by σ, that is, $x\sigma y\,(x, y \in A)$ implies $x = y$. Denot the set of all ι-single subsets of $(X;\sigma)$ by $\mathbf{T}$. A mapping α in $\mathcal{I}_X$, the symmetric inverse semigroup on X, is called a *partial one-to-one* ι*-mapping* on $(X;\sigma)$ if $d(\alpha), r(\alpha)$ are both ι-single subsets of $(X;\sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of α, respectively. Denote the set of all partial one-to-one ι-mappings of $(X;\sigma)$ by $\mathcal{GI}_{(X;\sigma)}$.

For any ι-single subsets A and B of $(X;\sigma)$, define $\theta_{A,B}$ by

$$\theta_{A,B} = \{(a,b) \in A \times B : (a,b) \in \sigma\} = (A \times B) \cap \sigma.$$

Since a subset of an ι-single subset is also an ι-single subset, $\theta_{A,B} \in \mathcal{GI}_{(X;\sigma)}$. For any $\alpha, \beta \in \mathcal{GI}_{(X;\sigma)}$, define $\theta_{\alpha,\beta}$ by $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$, and let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{LI}_{(X;\sigma)}\}$, an indexed set of one-to-one partial functions. Now, define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{GI}_{(X;\sigma)}$ as follows:

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

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[1]It is called a $\vee$-*prehomomorphism* in [4]

where the multiplication of the right side of the first equality is that of $\mathcal{I}_X$. Denote $(\mathcal{GI}_{(X;\sigma)}, \circ, *)$ by $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$ or simply by $\mathcal{GI}_{(X;\sigma)}$. In this paper, we use $\mathcal{GI}_{(X;\sigma)}$ rather than $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$.

Result 1.4. [2] *For a transitive ι-set $(X;\sigma)$, we have the following:*

(1) *The $*$-groupoid $\mathcal{GI}_{(X;\sigma)}$, defined above, is a generalized inverse $*$-semigroup. Moreover, any generalized inverse $*$-semigroup can be embedded (up to $*$-isomorphism) in $\mathcal{GI}_{(X;\sigma)}$ on some transitive ι-set $(X;\sigma)$.*

(2) $E(\mathcal{GI}_{(X;\sigma)}) = \mathcal{M}$ *and* $P(\mathcal{LI}_{(X;\sigma)}) = \{1_A : A$ *is an ι-single subset of* $(X;\sigma)\}$.

(3) *If σ is the identity relation on X, then $\mathcal{GI}_{(X;\sigma)}$ is the symmetric inverse semigroup $\mathcal{I}_X$ on X.*

2 Characterization of prehomomorphisms

Let S be a generalized inverse $*$-semigroup. For any element $a \in S$, aa^* and a^*a by $d(a)$ and $r(a)$, respectively. Define a new partial product $\cdot$ on S as follows:

$$a \cdot b = \begin{cases} ab & \text{if } \ r(a) = d(a^*abb^*) \text{ and } d(b) = r(a^*abb^*) \\ \text{undefined} & \text{otherwise} \end{cases}$$

The partial product $\cdot$ is called a *restricted product* of S.

Lemma 2.1. *Let a and b be elements of a generalized inverse $*$-semigroup S.*

(1) *$a \cdot b$ is defined if and only if $a^*a = a^*abb^*a^*a$ and $bb^* = bb^*a^*abb^*$.*

(2) *If $a \cdot b$ is defined, then $d(a \cdot b) = d(a)$ and $r(a \cdot b) = r(b)$.*

The follwing lemma is a basic property of the restricted product of S.

Lemma 2.2. *Let S be a generalized inverse $*$-semigroup.*

(1) *Let x be an element of S and e a projection of S such that $e \leq x^*x$. Then $a = xe$ is the unique element in S such that $a \leq x$ and $a^*a = e$.*

(2) *Let x be an element of S and e a projection of S such that $e \leq xx^*$. Then $a = ex$ is the unique element in S such that $a \leq x$ and $aa^* = e$.*

(3) *For any elemants $x, y \in S$, $xy = a \cdot b$ where $a = xe$, $b = fy$, $e = x^*xyy^*x^*x$ and $f = yy^*x^*xyy^*$.*

Lemma 2.3. *Let $\phi : S \to T$ be a prehomomorphism of a generalized inverse $*$-semigroup S to a generalized inverse $*$-semigroup T, and a, b elements of S.*

(1) $(aa^*)\phi = (a\phi)(a\phi)^*$ *and* $(a^*a)\phi = (a\phi)^*(a\phi)$.

(2) *If* $a \cdot b$ *is defined, then* $a\phi \cdot b\phi$ *is defined and* $((a \cdot b)\phi)^*(a \cdot b)\phi) = (a\phi \cdot b\phi)^*(a\phi \cdot b\phi)$.

(3) *By the lemma above,* $ab = (ae) \cdot (fb)$ *where* $e = a^*abb^*a^*a$ *and* $f = bb^*a^*abb^*$. *If* ϕ *satisfies that* $(gh)\phi = (g\phi)(h\phi)$ *for any* $g, h \in E(S)$, *then* $(ae)\phi = (a\phi)(e\phi)$ *and* $(fb)\phi = (f\phi)(b\phi)$.

Now, we have the main theorem.

Theorem 2.4. *Let* S *and* T *be generalized inverse* $*$*-semigroups and* $\phi : S \to T$ *a mapping.*

(1) ϕ *is a prehomomorphism if and only if it preserves the restricred product and the natural order.*

(2) ϕ *is a homomorphism if and only if it is a prehomomorphism which satisfies* $(ef)\phi = (e\phi)(f\phi)$ *for all* $e, f \in E(S)$.

References

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