<table>
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<th>Title</th>
<th>A Localization of a Semigroup Ring, II (Algebraic Semigroups, Formal Languages and Computation)</th>
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<tr>
<td>Author(s)</td>
<td>Matsuda, Ryuki</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1222: 11-17</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41308">http://hdl.handle.net/2433/41308</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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A Localization of a Semigroup Ring, II

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This is a continuation of our [M4]. Thus a submonoid $S$ of a torsion-free abelian (additive) group is called a g-monoid. For a g-monoid $S$, the quotient group of $S$ is denoted by $q(S)$, and for a commutative ring $R$, the total quotient ring of $R$ is denoted by $q(R)$. Throughout the paper $S$ denotes a g-monoid which is not $\{0\}$.

Let $F(S)$ be the set of fractional ideals of the g-monoid $S$. A mapping \( I \mapsto I^* \) of $F(S)$ to $F(S)$ is called a star-operation on $S$ if the following conditions hold for every element $a \in q(S)$ and $I, J \in F(S)$:

\[
(a)^* = (a); (a + I)^* = a + I^*; I \subset I^*;
\]

If $I \subset J$, then $I^* \subset J^*$; $(I^*)^* = I^*$.

Let $*$ be a star-operation on $S$. If, for all finitely generated fractional ideals $J_1, J_2$ and $I$, $(I + J_1)^* \subset (I + J_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. star-operation on $S$.

Let $F'(S)$ be the set of non-empty subsets of $q(S)$ such that $S + I \subset I$. A mapping \( I \mapsto I^* \) of $F'(S)$ to $F'(S)$ is called a semistar-operation on $S$ if the following conditions hold for every element $a \in q(S)$ and $I, J \in F'(S)$:

\[
(a + I)^* = a + I^*; I \subset I^*;
\]

If $I \subset J$, then $I^* \subset J^*$; $(I^*)^* = I^*$.

Let $*$ be a semistar-operation on $S$. If, for all finitely generated fractional ideals $J_1, J_2$ and $I$, $(I + J_1)^* \subset (I + J_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. semistar-operation on $S$.

Let $R$ be a commutative ring. A non-zerodivisor of $R$ is called a regular element of $R$. If an ideal $I$ of $R$ contains at least one regular element, then $I$ is called a regular ideal of $R$. If every regular ideal is generated by regular elements, then $R$ is called a Marot ring. If, for every regular element $f$ of the polynomial ring $R[X]$, the ideal of $R$ generated by the coefficients of $f$ is a regular ideal of $R$, then $R$ is said to have property (A).

Let $I$ be an $R$-submodule of $q(R)$ such that $rI \subset R$ for some regular $r \in R$. Then $I$ is called a fractional ideal of $R$. Let $F(R)$ be the set of non-zero fractional ideals of $R$. A mapping $I \mapsto I^*$ of $F(R)$ to $F(R)$ is called a
star-operation on $R$ if the following conditions hold for every regular element $a \in q(R)$ and $I, J \in F(R)$:

1. $(a)^* = (a)$;  
2. $(aI)^* = aI^*$;  
3. $I \subset I^*$;  
4. If $I \subset J$, then $I^* \subset J^*$;  
5. $(I^*)^* = I^*$.

Let $*$ be a star-operation on $R$. If, for all finitely generated non-zero fractional ideals $J_1, J_2, I$ with $I$ regular, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. star-operation on $R$.

Let $F'(R)$ be the set of non-zero $R$-submodules of $q(R)$. A mapping $I \mapsto I^*$ of $F'(R)$ to $F'(R)$ is called a semistar-operation on $R$ if the following conditions hold for every regular element $a \in q(R)$ and $I, J \in F'(R)$:

1. $(aI)^* = aI^*$;  
2. $I \subset I^*$;  
3. If $I \subset J$, then $I^* \subset J^*$;  
4. $(I^*)^* = I^*$.

Let $*$ be a semistar-operation on $R$. If, for all finitely generated non-zero fractional ideals $J_1, J_2, I$ with $I$ regular, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. semistar-operation on $R$.

Let $f = \sum a_i X^{s_i} \in R[X; S]$, where $s_i \neq s_j$ for $i \neq j$, and $a_i \neq 0$ for each $i$. Then the ideal $(s_1, \cdots, s_n)$ of $S$ is denoted by $e(f)$, and the ideal $(a_1, \cdots, a_n)$ of $R$ is denoted by $c(f)$.

**Proposition 1.** Let $*$ be a star-operation on a domain $D$. The following conditions are equivalent:

1. $*$ is e.a.b.
2. If $f/g = f'/g'$, where $f, g, f', g' \in D[X]$ with $g, g'$ non-zero, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$.

**Proof.** Assume that, if $f/g = f'/g'$, where $f, g, f', g' \in D[X]$ with $g, g'$ non-zero, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$. Let $I, J_1, J_2$ be finitely generated non-zero fractional ideals of $D$, and assume that $(IJ_1)^* \subset (IJ_2)^*$. We may assume that $I, J_1, J_2$ are ideals of $D$. Let $I = (a_0, \cdots, a_n), J_1 = (b_0, \cdots, b_m)$ and $J_2 = (c_0, \cdots, c_l)$. Put $f = \sum a_i X^i, g = \sum b_i X^{i(n+1)}$ and $h = \sum c_i X^{i(n+1)}$. Then $c(fg) = IJ_1, c(fh) = IJ_2$. Since, $(fg)/(fh) = g/h$ and $c(fg)^* \subset c(fh)^*$, we have $c(g)^* \subset c(h)^*$. That is, $J_1^* \subset J_2^*$. Hence $*$ is e.a.b.

In the following, let $D$ be a domain, and let $A$ be a Marot ring with property
Theorem 1. Let $*$ be a star-operation on $A$. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f / g = f' / g'$, where $f, g, f', g' \in A[X; S]$ with $g, g'$ regular, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$.

(2) Let $*$ be a star-operation on $S$. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f / g = f' / g'$, where $f, g, f', g' \in D[X; S]$ with $g, g'$ non-zero, and if $e(f)^* \subset e(g)^*$, then $e(f')^* \subset e(g')^*$.

(3) Let $*$ be a semistar-operation on $A$. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f / g = f' / g'$, where $f, g, f', g' \in A[X; S]$ with $g, g'$ regular, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$.

(4) Let $*$ be a semistar-operation on $S$. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f / g = f' / g'$, where $f, g, f', g' \in D[X; S]$ with $g, g'$ non-zero, and if $e(f)^* \subset e(g)^*$, then $e(f')^* \subset e(g')^*$.

The proof of Theorem 1 is similar to that of Proposition 1.

Let $R$ be a commutative ring. If every finitely generated regular ideal of $R$ is principal, $R$ is called an r-Bezout ring. If every finitely generated regular ideal of $R$ is invertible, $R$ is called a Prüfer ring. A multiplicative system of $R$ consisting of regular elements is called a regular multiplicative system of $R$, and a quotient ring of $R$ with respect to a regular multiplicative system is called a regular quotient ring of $R$.

Let $*$ be a star-operation (resp. semistar-operation) on a $g$-monoid $S$. If the set $\{I^* \mid I \text{ is a finitely generated fractional ideal of } S\}$ is a group under the sum $(I_1^*, I_2^*) \mapsto (I_1^* + I_2^*)^*$, then $S$ is called a Prüfer $*$-multiplication semigroup. Assume that $*$ is an e.a.b. star-operation (resp. semistar-operation) on $S$, let $D$ be a domain. Then the ring $S_D^* = \{f / g \mid f, g \in D[X; S] - \{0\}$ with
Theorem 2. Let * be an e.a.b. star-operation on $A$, and let $T = \{g \mid g$ is a regular element of $A[X; S]$ with $c(g)^* = A\}$. Then the following conditions are equivalent:

1. $A[X; S]_T$ is a Prüfer ring.
2. $A$ is a Prüfer *-multiplication ring.
3. $A[X; S]_T$ is an r-Bezout ring.
4. Each regular prime ideal of $A[X; S]_T$ is the extension of a prime ideal of $A$.
5. $A_S^* = A[X; S]_T$ is a regular quotient ring of $A[X; S]$.
6. Each prime ideal of $A[X; S]_T$ is the contraction of a prime ideal of $A_S^*$.
7. Each regular prime ideal of $A[X; S]_T$ is the contraction of a prime ideal of $A_S^*$.
8. Each valuation overring of $A_S^*$ is of the form $A[X; S][P_{A[X; S]}]$, where $P$ is a prime ideal of $A$ such that $A_{[P]}$ is a valuation overring of $A$.
9. $A_S^*$ is a flat $A[X; S]$-module.

Moreover, there exists a Prüfer Marot ring $A$ with property (A) which satisfies the following conditions: Let * be any e.a.b. *-operation on $A$. Then there exists a prime ideal of $A[X; Z_0]_T$ which is not the extension of a prime ideal of $A$, where $Z_0$ is the g-monoid of non-negative integers.

For the proof of equivalence of $(0) \sim (9)$ we confer [M3, Propositions 3.1 and 3.9 and Theorem 3.7]. Let $k$ be a field, let $X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be indeter-
minimates, and let $D_0$ be a Prüfer domain. Let $R = k[[X_1, X_2, \cdots, Y_1, Y_2, \cdots]]_1/(X_iX_j, Y_iY_j \mid i \neq j)$, and let $A = R \oplus D_0$, where $k[[X_1, X_2, \cdots, Y_1, Y_2, \cdots]]_1 = \bigcup_{n=1}^{\infty} k[[X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n]]$, and $(X_iX_j, Y_iY_j \mid i \neq j)$ is the ideal of $k[[X_1, X_2, \cdots, Y_1, Y_2, \cdots]]_1$ generated by the subset $\{X_iX_j, Y_iY_j \mid i \neq j\}$. Then $A$ is such a ring (cf. [M1, Theorem (1.3)]).

A similar result to Theorem 2 holds for semistar-operations on the ring $A$ as follows.

**Theorem 3.** Let $*$ be an e.a.b. semistar-operation on $A$, and let $W = \{g \mid g$ is a regular element of $A^*[S]$ such that $c(g)^* = A^*\}$. Then the following conditions are equivalent:

1. $A^*[X;S]_W$ is a Prüfer ring.
2. $A^*[X;S]_W$ coincides with the Kronecker function ring $A^*_S$ of $A$ with respect to $*$ and $S$.
3. $A^*[X;S]_W$ is an r-Bezout ring.
4. Each regular prime ideal of $A^*[X;S]_W$ is the extension of a prime ideal of $A^*$.
5. $A^*_S$ is a regular quotient ring of $A^*[X;S]$.
6. Each prime ideal of $A^*[X;S]_W$ is the contraction of a prime ideal of $A^*_S$.
7. Each regular prime ideal of $A^*[X;S]_W$ is the contraction of a prime ideal of $A^*_S$.
8. Each valuation overring of $A^*_S$ is of the form $A^*[X;S]_WQ$, where $Q$ is a prime ideal of $A^*$ such that $(A^*)_Q$ is a valuation overring of $A^*$.
9. $A^*_S$ is a flat $A^*[X;S]$-module.

For the proof we confer [M3, Propositions 3.2, 3.8 and 3.9].

**Theorem 4.** Let $D$ be a domain, and let $*$ be an e.a.b. star-operation on a g-monoid $S$, and let $T = \{g \mid g$ is a non-zero element of $D[X;S]$ with $e(g)^* = S\}$. The following conditions are equivalent:

1. $D[X;S]_T$ is a Prüfer ring.
2. $S$ is a Prüfer $*$-multiplication semigroup.
3. $D[X;S]_T$ coincides with the Kronecker function ring $S^*_D$ of $S$ with respect to $*$ and $S$. 

For the proof we confer [M3, Propositions 3.2, 3.8 and 3.9].
to \( \ast \) and \( D \).

(3) \( D[X; S]_T \) is a Bezout ring.

(4) Each prime ideal of \( D[X; S]_T \) is the extension of a prime ideal of \( S \).

(5) \( S^D \) is a quotient ring of \( D[X; S] \).

(6) Each prime ideal of \( D[X; S]_T \) is the contraction of a prime ideal of \( S^D \).

(7) Each valuation overring of \( S^D \) is of the form \( D[X; S]_{PD[X; S]} \), where \( P \) is a prime ideal of \( S \) such that \( S_P \) is a valuation oversemigroup of \( S \).

(8) \( S^D \) is a flat \( D[X; S] \)-module.

For the proof we confer [MS, Theorems 8 and 25].

A similar result to Theorem 4 holds for semistar-operations on \( S \).

**Theorem 5.** Let \( \ast \) be an e.a.b. semistar-operation on \( S \), and let \( W = \{ g \mid g \text{ is a non-zero element of } D[X; S^\ast] \text{ such that } e(g)^\ast = S^\ast \} \). The following conditions are equivalent:

(0) \( D[X; S^\ast]_W \) is a Prüfer ring.

(1) \( S \) is a Prüfer \( \ast \)-multiplication semigroup.

(2) \( D[X; S^\ast]_W = S^D \).

(3) \( D[X; S^\ast]_W \) is a Bezout ring.

(4) Each prime ideal of \( D[X; S^\ast]_W \) is the extension of a prime ideal of \( S^\ast \).

(5) \( S^D \) is a quotient ring of \( D[X; S^\ast] \).

(6) Each prime ideal of \( D[X; S^\ast]_W \) is the contraction of a prime ideal of \( S^D \).

(7) Each valuation overring of \( S^D \) is of the form \( D[X; S^\ast]_{QD[X; S^\ast]} \), where \( Q \) is a prime ideal of \( S^\ast \) such that \( (S^\ast)_Q \) is a valuation oversemigroup of \( S^\ast \).

(8) \( S^D \) is a flat \( D[X; S^\ast] \)-module.

For the proof we confer [M2, Proposition 4 and Theorem 23].

**Appendix**

**Theorem.** Let \( S \) be a \( g \)-monoid, and let \( T \) be an extension semigroup. If \( T \) is a Noetherian semigroup, and if \( T \) is a finitely generated \( S \)-module, then \( S \) is a Noetherian semigroup.
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