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The ideal transforms of semigroups

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By a *semigroup* we mean a submonoid of a torsion-free abelian (additive) group in this paper. Let S be a semigroup with the quotient group $q(S)$, that is, $q(S)=\{\ s-s' \mid s,\ s'\in S\ \}$. Any semigroup T between S and $q(S)$ is called an *oversemigroup* of S.

Moreover, let $\mathbf{Z}$ be the set of all integers and let $\mathbf{Z_n}=\{\ a\in \mathbf{Z}\mid a\geq n\ \}$ and $X\cdot \mathbf{Z_m}=\{\ aX\mid a\in \mathbf{Z_m}\ \}$. And $S[X]=S+\mathbf{Z_0}X=\{\ s+nX\mid s\in S,\ n\in \mathbf{Z_0}\ \}$ is called a *polynomial semigroup* over S (cf. [KOM]).

Let I be a subset of S. I is called an *ideal* of S if $I+S=I$, that is, $a+s\in I$ for each $a\in I$ and each $s\in S$. For any $a\in S$, put $(a)=a+S=\{\ a+s\mid s\in S\ \}$. Then (a) is an ideal of S and it is called a *principal ideal* of S. For $a_1,\ a_2,\ \cdots,\ a_n\in S$, we set $I=(a_1,\ a_2,\ \cdots,\ a_n)=\cup_{i=1}^{n}(a_i)=\cup_{i=1}^{n}(a_i+S)$. The $(a_1,\ a_2,\ \cdots,\ a_n)$ is an ideal of S and it is called an *ideal generated* by $a_1,\ a_2,\ \cdots,\ a_n$ and $\{\ a_1,\ a_2,\ \cdots,\ a_n\ \}$ is called a *basis* of I.

An element u of S is called a *unit* if $u+v=0$ for some $v\in S$. Let $U(S)$ be the set of units in S. Note that $U(S)$ be a subgroup of $q(S)$.

If we put $M=S-U(S)$, then M is an ideal of S. Moreover, if I is an ideal of S such that $M\subset I$, then $M=I$ or $I=S$. M is called the *maximal ideal* of S. A proper ideal P of S is called a *prime ideal* of S if $a+b\in P$ with $a,\ b\in S$ implies either $a\in P$ or $b\in P$. We note that the maximal ideal of S is a prime ideal, ϕ is a prime ideal of S and S has the only one maximal ideal.

We give semigroup versions of some results in [F].

Let S be a semigroup. Also, let $\mathrm{Spec}(S)$ be the set of all prime ideals of S. For an ideal I of S, we put $V(I)=\{\ P\in \mathrm{Spec}(S)\mid P\supset I\ \}$ and $D(I)=\{\ P\in \mathrm{Spec}(S)\ \mid P\not\supset I\ \}=\mathrm{Spec}(S)\ -V(I)$. In particular, put $D((a))=D(a)$ for $a\in S$.

Lemma 1. *Let $\{\ I_\lambda\ \}_{\lambda\in\Lambda}$ is a family of ideals of S and let I and J are ideals of S. Then we have the following statements.*

(1) $\cap \{ I_\lambda \mid \lambda \in \Lambda \}$ *is an ideal of* S.
(2) $\cup \{ I_\lambda \mid \lambda \in \Lambda \}$ *is an ideal of* S.
(3) $I+J = \{ a+b \mid a \in I,\ b \in J \}$ *is an ideal of* S *such that* $I+J \subset I \cap J$.
(4) *If* $P = I \cap J$ *is a prime ideal of* S, *then* $I = P$ *or* $J = P$.
(5) *If* P *and* Q *are two prime ideals of* S, *then* $P \cup Q$ *is also a prime ideal of* S.

Lemma 2. *Let* S *be a semigroup. Then the following statements hold.*
(1) $V(\phi) = \mathrm{Spec}(S)$, $V(S) = \phi$.
(2) *If* $I \subset J$, *then* $V(I) \supset V(J)$.
(3) $V(I_1) \cap V(I_2) = V(I_1) \cup V(I_2)$.
(4) $V(\cup \{I_\lambda \mid \lambda \in \Lambda\}) = \cap \{V(I_\lambda) \mid \lambda \in \Lambda\}$.

We make Spec(S) into a topological space; the topology is called the *Zariski topology.* The closed sets are defined by the $V(I) = \{ P \in \mathrm{Spec}(S) \mid P \supset I\}$.

Then $D(I)$ is an open set of Spec(S) and the $D(f) = \{ P \in \mathrm{Spec}(S) \mid f \in P\}$ is an open basis of Spec(S). For this topology, we give the following statement.

Proposition 3. Spec(S) *is a Kolmogoroff space* (T_0*-space*) *and a quasi-compact space.*

Definition 1. We call the *ideal transform of* S *with respect to an ideal* I *of* S the following oversemigroup of S:

$$T_S(I) := \{ z \in q(S) \mid (S :_S z + S) \supset nI \text{ for some } n \geq 1 \}$$

Also, we call the *Kaplansky ideal transform of* S *with respect to an ideal* I *of* S the following oversemigroup of S:

$$\Omega_S(I) := \{ z \in q(S) \mid \mathrm{rad}(S :_S z + S) \supset I \}.$$

where $\mathrm{rad}(J) = \{a \in S \mid na \in J \text{ for some positive integer } n\}$.

Note that $\Omega_S(I)$ is an oversemigroup of $T_S(I)$ and note that if I is finitely generated, then $\Omega_S(I) = T_S(I)$. For a principal ideal I, we have that $I + T_S(I) = T_S(I)$.

Proposition 4. *Let I be a princiapal ideal of S and P be a prime ideal of S. Then the following results are hold.*

(1) $I + T_S(I) = T_S(I)$, $I + \Omega_S(I) = \Omega_S(I)$.

(2) $P \in V(I)$ *if and only if* $P + T_S(I) = T_S(I)$ *if and only if* $P + \Omega_S(I) = \Omega_S(I)$.

Definition 2 ([K],[KB],[KM] and [MK]). A semigroup S is a *valuation semigroup* if $\alpha \in q(S)$ then $\alpha \in S$ or $-\alpha \in S$.

Also, we say that S is a *seminormal semigroup* if 2α, $3\alpha \in S$ for $\alpha \in q(S)$, we have $\alpha \in S$.

It is clear that valuation semigroups are seminormal.

Definition 3. A non-empty subset N of a semigroup S is called an *additive system* of S if a, $b \in N$ implies $a + b \in N$ and $0 \in N$.

Put $S_N = \{ s - t \mid s \in S, t \in N \}$. Then S_N is an oversemigroup of S and is called the *quotient semigroup* of S. If P is a prime ideal of S, then $T = S - P$ is an additive system of S and the quotient semigroup S_T is denoted by S_P.

Definition 4 ([OK]). Let T be an oversemigroup of S. Then T is said to be *flat* over S if for any prime ideal P of S, either $P + T = T$ or $T \subset S_P$. Put $\mathrm{Flat}(T) = \{P \in \mathrm{Spec}(S) \mid P + T = T \text{ or } T \subset S_P\}$.

Example 1. Let $S = (\mathbf{Z}_1 + \mathbf{Z}_1 X) \cup \{0\}$. Then $U(S) = \{0\}$ and $M = \mathbf{Z}_1 + \mathbf{Z}_1 X = S - U(S)$ is the maximal ideal of S. Also, Krull dim $S = 1$ and S is not valuation semigroup.

Also, let $T = (\mathbf{Z}_1 + \mathbf{Z}_1 X) \cup \mathbf{Z}_0$. Then T is not a valuation semigroup and $U(T) = \{0\}$. Put $N = \mathbf{Z}_1 + \mathbf{Z}_1 X$. Then $N \notin \mathrm{Flat(T)}$.

Theorem 5 ([OK]). *Let T be an oversemigroup of S. Then the following statements are equivalent.*

(1) *T is flat over S.*

(2) *$T = S_{N \cap S}$ for the maximal ideal N of T.*

(3) *For any two ideals I, J of S, $(I \cap J) + T = (I + T) \cap (J + T)$.*

Definition 5. Let S be a semigroup and let T be an oversemigroup of S. Then T is said to be *LCM-stable* over S if $((a + S) \cap (b + S)) + T = (a + T) \cap (b + T)$ for each a, $b \in S$.

A flat oversemigroup T over S is LCM-stable over S.

Theorem 6. *Assume that S be a Noetherian semigroup. Let T be an oversemigroup of S. Then T is flat over S if and only if T is LCM-stable over S.*

Corollaly 7. *If S is a valuation semigroup and a proper principal ideal $I = (a)$ of S, then $P \in D(I)$ if and only if $T_S(I) = \Omega_S(I) \subset S_P$.*

Proposition 8. *The following satetements are hold.*
(1) $S_a = \Omega_S((a))$ *for a non-unit* $a \in S$.
(2) *If I and J are ideals of S such that $I \subset J$, then $\Omega_S(I) \supset \Omega_S(J)$.*
(3) $\Omega_S(I) = \cap\{S_P \mid P \in D(I)\} = \cap \ \{\Omega_S(I + S_P) \mid P \in \mathrm{Spec}(S)\}$.
(4) *If I is a proper ideal of S, then* $\Omega_S(I) = \cap\{\Omega_S(a+S) \mid a \in I\} = \cap\{S_a \mid a \in S_a\}$.

Definition 6. $x \in G$ is called an *almost integral element* of S if there exisits an element $a \in S$ such that $a + nx \in S$ for each positive integer n. Also, S is a *completely integrally closed* if x is almost integral over S then $x \in S$.

Theorem 9 ([K]). *Let S be a valuation semigroup such that $S \neq q(S)$. Then* Krull dim $S = 1$ *if and only if S is a completely integrally closed semigroup.*

Theorem 10 ([KHF]). (1) $\mathrm{Spec}(\mathbf{Z_0}[X]) = \{\ (X),\ (1),\ (1,\ X),\ \phi\ \}$.
(2) The primary ideals of $\mathbf{Z_0}[X]$ are the following:
(i) All the ideals that contains both elements of $\mathbf{Z_0}$ and $\mathbf{Z_0}X$.
(ii) $\mathbf{Z_k} + \mathbf{Z_0}X = (k)$ with $k \in \mathbf{Z_0}$.
(iii) $\mathbf{Z_0} + \mathbf{Z_k}X = (kX)$ with $k \in \mathbf{Z_0}$.

Example 2. Let $S = \mathbf{Z_0} \cup (\mathbf{Z} + \mathbf{Z_1}X)$. Put $P = \mathbf{Z} + \mathbf{Z_1}\mathbf{X}$ and $M = (1) = 1 + S = P \cup \mathbf{Z_1}$. Then $\mathrm{Spec}(S) = \{\ \phi,\ P,\ M\ \}$. Since $\phi \subset P \subset M$, we have that Krull dim $S = 2$. It is clear that S is a valuation semigroup. Since P is not finitely generated, S is not Noetherian semigroup.

Theorem 11. *Let the notation be as in Example 2 and let I be an ideal of S. Then the following statements hold.*

(1) *Let* $I = (f)$ *be a principal ideal of* S. *If* $f = 0$, *then* $T_S(I) = \Omega_S(I) = S$. *Also, if* $f \in M - P$, *then* $T_S(I) = \Omega_S(I) = S_f = \mathbf{Z}[X] = \mathbf{Z} + \mathbf{Z}_0 X$. *Next, if* $f \in P$, *then* $T_S(I) = \Omega_S(I) = S_f = q(S)$.

(2) *If* I *is not a finitely generated ideal of* S, *then* $I = \mathbf{Z} + \mathbf{Z}_n X$ $(n \geq 1)$ *and* $\Omega_S(I) = q(S)$.

(3) *Let* $I \neq S$. *Then* $\mathrm{Spec}(\Omega_S(I)) \cong D(I)$ *if and only if* $I + \Omega_S(I) = \Omega_S(I)$.

(4) S *is not a completely integrally closed and each oversemigroup of* S *is a flat semigroup over* S, *and so* $\mathrm{Flat}(T) = \mathrm{Spec}(S)$ *for each oversemigroup* T *of* S.

Theorem 12. *Let* S *be a semigroup and* I *an ideal of* S. *Then the following statements are equivalent.*

(1) $D(I)$ *is an affine open subspace of* $\mathrm{Spec}(S)$.

(2) $\Omega_S(I)$ *is flat over* S *and, for each* $P \in \mathrm{Spec}(S)$ *with* $P \supset I$, $P + \Omega_S(I) = \Omega_S(I)$.

(3) $I + \Omega_S(I) = \Omega_S(I)$.

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