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<th>Title</th>
<th>$CPN$ Languages and Codes (Algebraic Semigroups, Formal Languages and Computation)</th>
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<tbody>
<tr>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1222: 46-49</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41313">http://hdl.handle.net/2433/41313</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
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\textbf{CPN Languages and Codes}

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Let $D = (P, X, \delta, \mu_0)$ be a Petri net with a initial marking $\mu_0$ where $P$ is the set of places, $X$ is the set of transitions, $\delta$ is the transition function and $\mu_0 \in N_+^P$ is a positive marking, i.e. $\pi_p(\mu_0) > 0$ for any $p \in P$. Notice that $\pi_p(\mu_0)$ is meant the number of tokens at $p$ of the marking $\mu_0$. A language $C$ is called a CPN language over $X$ generated by $D$ and denoted by $C = \mathcal{L}(D)$ if $C = \{u \in X^+ | \exists p \in P, \pi_p(\delta(\mu_0, u)) = 0, \forall q \in P, \pi_q(\delta(\mu_0, u)) \geq 0, \quad \forall q' \in P, \pi_{q'}(\delta(\mu_0, u')) > 0 \text{ for } u' \in P_r(u) \setminus \{u\} \text{ where } P_r(u) \text{ is the set of all prefixes of } u\}$. Then it is obvious that $C = \mathcal{L}(D)$ is a prefix code over $X$. If $C$ is a maximal prefix code over $X$, then $C$ is called an mCPN language over $X$.

**Theorem 1** Let $A, B \subseteq X^+$ be finite maximal prefix codes over $X$. If $AB$ is an mCPN language over $X$, then $A, B$ are full uniform codes over $X$.

**Remark 1** In the above theorem, the condition for $A$ and $B$ to be finite is necessary. For instance, let $X = \{a, b\}$ and let $A = B = b^*a$. Then $AB = b^*ab^*a$ is an mCPN language over $X$ but neither $A$ nor $B$ is a full uniform code over $X$.

Now we consider some constructions of mCPN languages.

**Definition 1** Let $A, B \subseteq X^+$. Then by $A \oplus B$ we denote the language $(\cup_{b \in X}\{(P_r(A) \setminus A) \circ Bab^{-1}\}b) \cup (\cup_{a \in X}\{(P_r(B) \setminus B) \circ Aa^{-1}\}a)$ where $\circ$ is meant the shuffle operation.

**Proposition 1** Let $X = Y \cup Z$ where $Y, Z \neq \emptyset, Y \cap Z = \emptyset$. If $A \subseteq Y^+$ is an mCPN language over $Y$ and $B \subseteq Z^+$ is an mCPN language over $Z$, then $A \oplus B$ is an mCPN language over $X$. 
Example 1 Let $X = \{a,b\}$. Consider $A = \{a\}$ and $B = \{bb\}$. Then both $A$ and $B$ are $mCPN$ languages over $\{a\}$ and $\{b\}$, respectively. Hence $A \oplus B = \{a,ba,bb\}$ is an $mCPN$ language over $X$.

Proposition 2 Let $A, B \subseteq X^+$ be finite $mCPN$ languages over $X$. Then $A \oplus B$ is an $mCPN$ language over $X$ if and only if $A = B = X$.

Remark 2 For the class of infinite $mCPN$ languages over $X$, the situation is different. For instance, let $X = \{a,b\}$ and let $A = B = b^*a$. Then $A \oplus B = b^*a$ and $A, B$ and $A \oplus B$ are $mCPN$ languages over $X$.

Proposition 3 Let $A, B \subseteq X^+$ be $mCPN$ languages over $X$. Then there exist an alphabet $Y$, $D \subseteq Y^+$: an $mPCN$ language over $Y$ and a homomorphism $h$ of $Y^*$ onto $X^*$ such that $A \oplus B = h(D)$.

Definition 2 Let $A \subseteq X^+$. By $m(A)$, we denote the language $\{v \in A | \forall u, v \in A, \forall x \in X^+, v = u x \Rightarrow x = 1\}$. Obviously, $m(A)$ is a prefix code over $X$. Let $A, B \subseteq X^+$. By $A \otimes B$, we denote the language $m(A \cup B)$.

Proposition 4 Let $A, B$ be $mCPN$ languages over $X$. Then, $A \otimes B$ is an $mCPN$ language over $X$.

Example 2 It is obvious that $a^*b$ and $(a \cup b)^3$ are $mCPN$ languages over $\{a,b\}$. Hence $a^*b \otimes (a \cup b)^3 = \{b, ab, aaa, aab\}$ is an $mCPN$ language over $\{a,b\}$.

Remark 3 Proposition 4 does not hold for the class of $CPN$ languages over $X$. The reason is the following: Suppose that $A \otimes B$ is a $CPN$ language over $X$ for any two $CPN$ languages $A$ and $B$ over $X$. Then we can show that, for a given finite $CPN$ language $A$ over $X$, there exists a finite $mCPN$ language $B$ over $X$ such that $A \subseteq B$ as follows. Let $A \subseteq X^+$ be a finite $CPN$ language over $X$ which is not an $mCPN$ language. Let $n = max\{|u| | u \in A\}$. Consider $X^n$ which is an $mCPN$ language over $X$. By assumption, $A \otimes X^n$ becomes a $CPN$ language (in fact, an $mCPN$ language) over $X$. By the definition of the operation $\otimes$, it can be also proved that $A \subseteq A \otimes X^n$. Notice that there exists a finite $CPN$ language $A$ over $X$ such that there exists no $mCPN$ language $B$ over $X$ with $A \subseteq B$. Hence, Proposition 4 does not hold for the class of all $CPN$ languages over $X$. 
Remark 4 The set of all \( mCPN \) languages over \( X \) forms a semigroup under \( \otimes \). Moreover, the operation \( \otimes \) has the following properties:

1. \( A \otimes B = B \otimes A \)
2. \( A \otimes A = A \)
3. \( A \otimes X = X \)

Consequently, the set of all \( mCPN \) languages over \( X \) forms a commutative band with zero under \( \otimes \).

Definition 3 Let \( A \subseteq X^+ \) be a \( CPN \) language over \( X \). By \( r(A) \) we denote the value \( \min\{|P||D=(P,X,\delta,\mu_0),\mathcal{L}(D)=A\} \).

Remark 5 Let \( A \subseteq X^+ \) be a finite \( CPN \) language over \( X \). Then \( r(A) \leq |A| \). Moreover, let \( A, B \subseteq X^+ \) be \( mCPN \) languages over \( X \). Then \( r(A \otimes B) \leq r(A) + r(B) \). In the above, if \( |A|, |B| \) are finite, then \( |A \otimes B| \leq \max(|A|, |B|) \).

We define three language classes as follows: \( \mathcal{L}_{CPN} = \{A \subseteq X^+|A \text{ is a CPN language over } X\} \), \( \mathcal{L}_{mCPN} = \{A \subseteq X^+|A \text{ is an mCPN language over } X\} \), \( \mathcal{L}_{NmCPN} = \{A \subseteq X^+|A: \text{an mCPN language over } X, \exists D = (P,X,\delta,\mu_0), \forall p \in P, \forall a \in X, \#(p \rightarrow a) \leq 1, \mathcal{L}(D) = A\} \). Then it is obvious that we have the following inclusion relations: \( \mathcal{L}_{NmCPN} \subseteq \mathcal{L}_{mCPN} \subseteq \mathcal{L}_{CPN} \). It is also obvious that \( \mathcal{L}_{mCPN} \neq \mathcal{L}_{mCPN} \).

Problem 1 \( \mathcal{L}_{mCPN} \neq \mathcal{L}_{NmCPN} \)?

Proposition 5 Let \( A \in \mathcal{L}_{CPN} \) and let \( r(A) = k \). Then there exist \( A_1, A_2, \ldots, A_k \in \mathcal{L}_{CPN} \) such that \( r(A_i) = 1, i = 1, 2, \ldots, k \) and \( A = A_1 \otimes A_2 \otimes \ldots \otimes A_k \). Moreover, in the above, if \( A \in \mathcal{L}_{NmCPN} \), then \( A_1, A_2, \ldots, A_k \) are in \( \mathcal{L}_{NmCPN} \) and context-free.

For \( mCPN \) languages with rank 1, we have the following:

Proposition 6 Let \( A \subseteq X^+ \) be a finite \( mCPN \) language with \( r(A) = 1 \) over \( X \). Then \( A \) is a full uniform code over \( X \).

Proposition 7 Let \( A \subseteq X^+ \) be an \( mCPN \) language with \( r(A) = 1 \) over \( X \) and let \( k \) be a positive integer. Then \( A^k \) is an \( mCPN \) language with \( r(A^k) = 1 \) over \( X \).

Proposition 8 Let \( A \in \mathcal{L}_{NmCPN} \) and let \( r(A) = k \). Then there exist \( A_1, A_2, \ldots, A_k \in \mathcal{L}_{NmCPN} \) such that \( r(A_i) = 1, i = 1, 2, \ldots, k \) and \( A = \)}
$A_1 \otimes A_2 \otimes \ldots \otimes A_k$. Let $n_1, n_2, \ldots, n_k$ be positive integers. Then $A_1^{n_1} \otimes A_2^{n_2} \otimes \ldots \otimes A_k^{n_k} \in \mathcal{L}_{NmCPN}$.

Finally, we can prove the following main theorem by two different ways, i.e. the first one is an indirect proof and the second one is a direct proof.

**Theorem 2** Let $C \subseteq X^+$ be a CPN language over $X$. Then $C$ is a context-sensitive language over $X$. 