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Characterization of finite Automata by the Images and the Kernels of their Transition Functions

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1. Introduction

By an automaton $\mathcal{A}$, we mean here a 3-tuple $(X, A, \delta)$, where $X$ is a finite set (the set of states), $A$ is a finite alphabet (the set of inputs) and $\delta$ is a mapping of $X \times A$ into $X$ (the transition function).

As usual, $A^*$ and $A^+$ denotes the free monoid and free semigroup generated by $A$, respectively, and $\delta$ is extended from $X \times A$ to $X \times A^*$. In this case, $\delta(x, s)$ is denoted simply by $xs$ for $x \in X, s \in A^*$.

Let $\rho = \{(s, t) \in A^* \times A^* : xs = xt$ for every $x \in X\}$. Then $\rho$ is a congruence on $A^*$ and $A^*/\rho$ is a finite transformation semigroup on $X$ by defining the action of $s \rho \in A^*/\rho$ on $X$ as $x(s \rho) = xs$. The semigroup $A^*/\rho$ is called the characteristic semigroup of $\mathcal{A}$. Let $\mathcal{V}$ be a class of semigroups not necessarily a variety. Then an automaton $\mathcal{A}$ is called a $\mathcal{V}$-type if $A^*/\rho \in \mathcal{V}$.

For $s \in A^*$, let $\mathcal{I}m \ s = \{xs : x \in X\} = Xs$ and $\mathcal{K}er \ s = \{(x, y) \in X \times X : xs = ys\}$, which are called the image and the kernel of $s$, respectively. Then $\mathcal{K}er \ s$ is an equivalence on $X$.

Let $\mathcal{V}$ and $\mathcal{U}$ be two classes of semigroups. Then the direct product of $\mathcal{V}$ and $\mathcal{U}$ is defined by $\mathcal{V} \times \mathcal{U} = \{V \times U : V \in \mathcal{V}, U \in \mathcal{U}\}$. Let $U \in \mathcal{U}$ and let $S$ be a semigroup. If for each $s \in V$, there exists $U_s \in U$ such that $S = \sqcup\{U_s : s \in U\}$ and $U_s \cdot U_t = \{u_s u_t : u_s \in U_s, u_t \in U_t\} \subseteq U_{st}$, then we say that $S$ belongs to $\mathcal{V} \mathcal{U}$, where $\sqcup$ denotes a disjoint union.

As our start, we consider the following classes of semigroups: $\mathcal{G} = \{\text{groups}\}$, $\mathcal{LZ} = \{\text{left zero semigroups } [st = s]\}$, $\mathcal{RZ} = \{\text{right zero semigroups } [st = t]\}$ and $\mathcal{SL} = \{\text{semilattices } [st = ts, s^2 = s]\}$. We first characterize, for the classes $\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}$ and $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$, their types automata by the images and the kernels of their transition functions. By using the results, we characterize, for $\mathcal{V} \in \{\mathcal{SL}, \mathcal{LZ}, \mathcal{RZ}\}$ and $\mathcal{U} \in \{\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}, \mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}\}$, $\mathcal{V} \mathcal{U}$-type automaton by the same way.

\footnote{This is an abstract and the details will be published elsewhere}
For a example, we show later that an automaton $\mathcal{A}$ is a $SL(\mathcal{L} \times \mathcal{G})$-type if and only if $\im st = \im s \cap \im t$ for every $s, t \in A^+$.  

2. Preliminaries

For a set $Y$, $|Y|$ denotes the cardinality of $Y$, for an equivalence $\lambda$, $x \lambda$ denotes the $\lambda$-class containing $x$ and $|\lambda|$ denotes the number of $\lambda$-classes. i.e., $|\lambda| = |\{x \lambda : x \in X\}|$, and for $s \in A^+$, $|s|$ denotes the length of $s$. Then clearly $|\im s| = |\ker s|$ for every $s \in A^+$.

For $s \in A^*$, let fix $s = \{x \in X : xs = x\}$. Let $E(A^+) = \{e \in A^+ : (e, e^2) \in \rho\}$. Then it is easy to see that $e \in E(A^+)$ if and only if $\im e = \fix e$. and that $(e, f) \in \rho$ for $e, f \in E(A^+)$ if and only if $\im e = \im f$ and $\ker e = \ker f$. Since $A^+/\rho$ is finite, for every $s \in A^+$, there exists a positive integer $m$ such that $s^m \in E(A^+)$. 

Lemma 1. Let $s, t \in A^+$. Then

1. If $|\ker st| = |\ker t|$, then $\im st = \im t$.
2. If $|\im st| = |\im s|$, then $\ker st = \ker s$.

Lemma 2. Let $s \in A^+$. Then the following are equivalent:

1. $\im s \cap x \ker s \neq \emptyset$ for every $x \in X$.
2. $\im s^m = \im s$ and $\ker s^m = \ker s$ for every $m \in \mathbb{N}^+$.
3. There exists $e \in E(A^+)$ such that $\im s = \im e, \ker s = \ker e, (s, se) \in \rho$ and $(s, es) \in \rho$.

Lemma 3. The following are equivalent:

For every $s, t \in A^+$,

1. $\im s \cap x \ker t \neq \emptyset$ for every $x \in X$,
2. $\im st = \im t$.
3. $\ker st = \ker s$.

Let $A = (X, A, \delta)$ be an automaton, and let $Y = \cup\{\im a : a \in A\}$ and $\kappa = \cap\{\ker a : a \in A\}$. Then we have $Y = \cup\{\im s ; s \in A^+\}$ and $\kappa = \cap\{\ker s : s \in A^+\}$. In fact, if $s \in A^+$, then $s = s'a = bs''$ for some $a, b \in A, s's'' \in A^*$, so that $\im s' a \subseteq \im a$ and $\ker b \subseteq \ker b s''$.

Since $Y s \subseteq Y$ for every $s \in A^+$, the restriction $s_Y$ of $s$ to $Y$ can be defined. Let $A_Y = \{a_Y : a \in A\}$. Then the automaton $A_Y = (Y, A_Y, \delta)$ is called the subautomaton of $A$ with respect to $Y$.

Let $s, t \in A^+$ and $x \in X$. Since $(xs)t_Y = (xs)t$, the action of $st_Y$ on $X$ is defined by $x(st_Y) = x(st)$.
Let \( \kappa \) be as above. Define the action of \( s \in A^+ \) on \( X/\kappa \) by \( (x\kappa)s = (xs)\kappa \). Then the action is well-defined. In fact, if \( x\kappa = y\kappa \), then \( (x,y) \in \kappa \subseteq \ker s \), so that \( xs = ys \). When the action of \( s \) is on \( X/\kappa \), \( s \) is denoted by \( s_\kappa \). Let \( A_\kappa = \{a_\kappa : a \in a\} \). Then the automaton \( A_\kappa = (X/\kappa,A_\kappa,\delta) \) is called the automaton induced from \( A \) by \( \kappa \).

Let \( s,t \in A^+ \) and \( x \in X \). Then clearly \((x\ker s)s = xs\). Since \( \kappa \subseteq \ker s \), we have \((x\kappa)s = xs\), so that \(((x\kappa)s_\kappa)t = ((xs)\kappa)t = (xs)t\). Thus the action of \( s_\kappa t \) on \( X \) is defined by \( x(s_\kappa t) = x(st) \).

For an automaton \( A = (X,A,\delta) \), let \( \text{Im}(A^+) = \{\text{im} s : s \in A^+\} = \{Y_i : i \in I\} \), i.e., for each \( i \in I \), \( Y_i = \text{im} s \) for some \( s \in A^+ \) and \( \text{im} s \in \text{Im}(A^+ \) for every \( s \in A^+ \), and let \( \text{Ker}(A^+) = \{\ker s : s \in A^+\} = \{\kappa_\mu : \mu \in M\} \), \( \text{Im}(A^+) = \{\text{im} s_\kappa : s \in A^+\} = \{Z_i : i \in I'\} \) and \( \text{Ker}(A^+) = \{\ker s_\gamma : s \in A^+\} = \{\kappa_\mu : \mu \in M'\} \). In this case, if \( \text{im} s \cap \text{ker} s \neq \emptyset \) holds for every \( s \in A^+ \) and \( x \in X \), then \( \text{Im}(A^+) = \text{Im}(E(A^+)) \) and \( \text{Ker}(A^+) = \text{Ker}(E(A^+)) \).

3. Main Results

A semigroup in \( \mathcal{LZ} \times \mathcal{G} \) is called a left group whose class is denoted simply by \( \mathcal{L} \), i.e., \( \mathcal{L} = \mathcal{LZ} \times \mathcal{G} \).

**Theorem 1.** Let \( A = (X,A,\delta) \) be an automaton. Then the following are equivalent:

1. There exists a subset \( Y \) of \( X \) such that \( \text{im} a = Y \) and \( Y \cap x\ker a \neq \emptyset \) for every \( a \in A \) and \( x \in X \).
2. There exists a subset \( Y \) of \( X \) such that \( \text{im} s = Y \) for every \( s \in S \).
3. \( A \) is a left group type.

From Theorem 1 we obtain the following results

**Corollary 1.1.** An automaton \( A = (X,A,\delta) \) is a \( \mathcal{SL}(\mathcal{L}) \)-type if and only if \( \text{im} st = \text{im} s \cap \text{im} t \) for every \( s,t \in A^+ \).

**Corollary 1.2.** An automaton \( A = (X,A,\delta) \) is a \( \mathcal{RZ}(\mathcal{L}) \)-type if and only if \( \text{im} st = \text{im} t \) for every \( s,t \in A^+ \).

A semigroup in \( \mathcal{G} \times \mathcal{RZ} \) is called a right group whose class is denoted by \( \mathcal{R} \), i.e., \( \mathcal{R} = \mathcal{G} \times \mathcal{RZ} \).

**Theorem 2.** Let \( A = (X,A,\delta) \) be an automaton. Then the following are equivalent:

There exists an equivalence $\kappa$ on $X$ such that $\ker a = \kappa$ and $\im a \cap x\kappa \neq \emptyset$ for every $a \in A$.

There exists an equivalence $\kappa$ on $X$ such that $\ker s = \kappa$ for every $s \in A^+$.

$A$ is a right group type.

**Corollary 2.1.** An automaton $A = (X, A, \delta)$ is a $\mathcal{SL}(\mathcal{RG})$-type if and only if $\ker st = \ker s \lor \ker t$ for every $s, t \in A^+$.

**Corollary 2.2.** An automaton $A = (X, A, \delta)$ is a $\mathcal{LZ}(\mathcal{RG})$-type if and only if $\im st = \im t$ for every $s, t \in A^+$.

From Corollaries 1.2 and 2.2 we obtain:

**Corollary 2.3.** An automaton $\mathcal{RZ}(\mathcal{LG})$-type if and only if it is $\mathcal{LZ}(\mathcal{RG})$-type.

**Remark.** It can be easily show that $\mathcal{LZ}(\mathcal{LG}) = \mathcal{LZ}(\mathcal{G}) = \mathcal{LZ}$ and $\mathcal{RZ}(\mathcal{RG}) = \mathcal{RZ}(\mathcal{G}) = \mathcal{RZ}$.

**Theorem 3.** Let $A = (X, A, \delta)$ be an automaton. Then the following are equivalent:

1. There exist a subset $Y$ of $X$ and an equivalence $\kappa$ on $X$ such that $\im a = Y$ and $\ker a = \kappa$ for every $a \in A$ and $Y \cap x\kappa \neq \emptyset$ for every $x \in X$.
2. There exist a subset $Y$ of $X$ and an equivalence $\kappa$ on $X$ such that $\im s = Y$ and $\ker s = \kappa$ for every $s \in A^+$,
3. $A$ is a group-type.

A semigroup in $\mathcal{SL}(\mathcal{G})$ is called a Clifford semigroup.

**Corollary 3.1.** An automaton $A = (X, A, \delta)$ is a Clifford semigroup type if and only if $\im st = \im s \cap \im t$ and $\ker st = \ker s \lor \ker t$ for every $s, t \in A^+$.

**Theorem 4.** Let $A = (X, A, \delta)$ be an automaton, and Let $Y = \bigcup \{\im a : a \in A\}, \kappa = \bigcap \{\ker a : a \in A\}$. Suppose that $\im s \cap x\ker s \neq \emptyset$ for every $s \in A^+, x \in X$. Then the following are equivalent:

1. $A$ is a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$-type.
2. $\ker s\gamma = \ker t\gamma$ for every $s, t \in A^+$.
3. $\im s\kappa = \im t\kappa$ for every $s, t \in A^+$.

**Corollary 4.1.** With the assumption of Theorem 4, the following are equivalent:

1. $A$ is a $\mathcal{SL}(\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ})$-type.
(2) \( \ker s_Y t_Y = \ker s_Y \setminus \ker t_Y \) for every \( s, t \in A^+ \).
(3) \( \im s \cap \im t = \im s \cap \im t \) for every \( s, t \in A^+ \).

Suppose that an automaton \( A \) is a \( \mathcal{L}Z \times \mathcal{G} \times \mathcal{R}Z \)-type. As is seen in the proof of Theorem 4, \( A^+/\rho = \{(i, g, \mu) : i \in I, g \in G, \mu \in M\} \). For \( i \in I \) and \( \mu \in M \), let \( A_i/\rho = \{(i, g, \mu) : g \in G, \mu \in M\} \) and \( A_{i\mu} = \{(i, g, \mu) : i \in I, g \in G\} \), respectively. Then \( A_i/\rho \in \mathcal{R}G \) and \( A_{i\mu}/\rho \in \mathcal{L}G \). For \( s\rho = (i, g, \nu), t\rho = (j, h, \mu) \), since \( (st)\rho = (i, gh, \mu) \), by Theorems 1 and 2, we have \( \ker st = \ker s \) and \( \im st = \im t \). Thus we obtain:

**Corollary 4.2.** If an automaton \( A \) is a \( \mathcal{L}Z \times \mathcal{G} \times \mathcal{R}Z \)-type, then it is a \( \mathcal{R}Z(\mathcal{L}G) \)-type. The converse is not true.

There is a simple example that a \( \mathcal{R}Z(\mathcal{L}G) \)-type automaton which is not a \( \mathcal{L}Z \times \mathcal{G} \times \mathcal{R}Z \)-type.

**References**


