Characterization of finite Automata by the Images and the Kernels of their Transition Functions¹

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1. Introduction

By an automaton $A$, we mean here a 3-tuple $(X, A, \delta)$, where $X$ is a finite set (the set of states), $A$ is a finite alphabet (the set of inputs) and $\delta$ is a mapping of $X \times A$ into $X$ (the transition function).

As usual, $A^*$ and $A^+$ denotes the free monoid and free semigroup generated by $A$, respectively, and $\delta$ is extended from $X \times A$ to $X \times A^*$. In this case, $\delta(x, s)$ is denoted simply by $xs$ for $x \in X, s \in A^*$.

Let $\rho = \{(s, t) \in A^* \times A^* : xs = xt$ for every $x \in X\}$. Then $\rho$ is a congruence on $A^*$ and $A^*/\rho$ is a finite transformation semigroup on $X$ by defining the action of $s\rho \in A^*/\rho$ on $X$ as $x(s\rho) = xs$. The semigroup $A^*/\rho$ is called the characteristic semigroup of $A$. Let $\mathcal{V}$ be a class of semigroups not necessarily a variety. Then an automaton $A$ is called a $\mathcal{V}$-type if $A^*/\rho \in \mathcal{V}$.

For $s \in A^*$, let $\text{im} s = \{xs : x \in X\} = Xs$ and $\ker s = \{(x, y) \in X \times X : xs = ys\}$, which are called the image and the kernel of $s$, respectively. Then $\ker s$ is an equivalence on $X$.

Let $\mathcal{V}$ and $\mathcal{U}$ be two classes of semigroups. Then the direct product of $\mathcal{V}$ and $\mathcal{U}$ is defined by $\mathcal{V} \times \mathcal{U} = \{V \times U : V \in \mathcal{V}, U \in \mathcal{U}\}$. Let $U \in \mathcal{U}$ and let $S$ be a semigroup. If for each $s \in V$, there exists $U_s \in U$ such that $S = \sqcup\{U_s : s \in U\}$ and $U_s \cdot U_t = \{u_s u_t : v_s \in U_s, v_t \in U_t\} \subseteq U_{st}$, then we say that $S$ belongs to $\mathcal{V}(U)$, where $\sqcup$ denotes a disjoint union.

As our start, we consider the following classes of semigroups: $\mathcal{G} = \{\text{groups}\}$, $\mathcal{LZ} = \{\text{left zero semigroups} \ [st = s]\}$, $\mathcal{RZ} = \{\text{right zero semigroups} \ [st = t]\}$ and $\mathcal{SL} = \{ \text{semilattices} \ [st = ts, s^2 = s]\}$. We first characterize, for the classes $\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}$ and $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$, their types automata by the images and the kernels of their transition functions. By using the results, we characterize, for $\mathcal{V} \in \{\mathcal{SL}, \mathcal{LZ}, \mathcal{RZ}\}$ and $\mathcal{U} \in \{\mathcal{G}, \mathcal{LZ} \times \mathcal{G}, \mathcal{G} \times \mathcal{RZ}, \mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}\}$, $\mathcal{V}(\mathcal{U})$-type automaton by the same way.

¹ This is an abstract and the details will be published eleswhere
For a example, we show later that an automaton \( \mathcal{A} \) is a \( \mathcal{SL}(\mathcal{LZ} \times \mathcal{G}) \)-type if and only if \( \text{im } st = \text{im } s \cap \text{im } t \) for every \( s, t \in A^+ \).

## 2. Preliminaries

For a set \( Y \), \( |Y| \) denotes the cardinality of \( Y \), for an equivalence \( \lambda \), \( x \lambda \) denotes the \( \lambda \)-class containing \( x \) and \( |\lambda| \) denotes the number of \( \lambda \)-classes. i.e., \( |\lambda| = |\{x \lambda : x \in X\}| \), and for \( s \in A^+ \), \( |s| \) denotes the length of \( s \). Then clearly \( |\text{im } s| = |\ker s| \) for every \( s \in A^+ \).

For \( s \in A^* \), let fix \( s = \{x \in X : xs = x\} \). Let \( E(A^+) = \{e \in A^+ : (e, e^2) \in \rho\} \). Then it is easy to see that \( e \in E(A^+) \) if and only if \( \text{im } e = \text{fix } e \). and that \( (e, f) \in \rho \) for \( e, f \in E(A^+) \) if and only if \( \text{im } e = \text{im } f \) and \( \ker e = \ker f \). Since \( A^+/\rho \) is finite, for every \( s \in A^+ \), there exists a positive integer \( m \) such that \( s^m \in E(A^+) \).

**Lemma 1.** Let \( s, t \in A^+ \). Then

1. If \( |\ker st| = |\ker t| \), then \( \text{im } st = \text{im } t \).
2. If \( |\text{im } st| = |\text{im } s| \), then \( \ker st = \ker s \).

**Lemma 2.** Let \( s \in A^+ \). Then the following are equivalent:

1. \( \text{im } s \cap x \ker s \neq \emptyset \) for every \( x \in X \).
2. \( \text{im } s^m = \text{im } s \) and \( \ker s^m = \ker s \) for every \( m \in \mathbb{N}^+ \).
3. There exists \( e \in E(A^+) \) such that \( \text{im } s = \text{im } e \), \( \ker s = \ker e \), \((s, se) \in \rho \) and \((s, es) \in \rho \).

**Lemma 3.** The following are equivalent:

For every \( s, t \in A^+ \),

1. \( \text{im } s \cap x \ker t \neq \emptyset \) for every \( x \in X \),
2. \( \text{im } st = \text{im } t \).
3. \( \ker st = \ker s \).

Let \( \mathcal{A} = (X, A, \delta) \) be an automaton, and let \( Y = \cup \{\text{im } a : a \in A\} \) and \( \kappa = \cap \{\ker a : a \in A\} \). Then we have \( Y = \cup \{\text{im } s ; s \in A^+ \} \) and \( \kappa = \cap \{\ker s : s \in A^+ \} \). In fact, if \( s \in A^+ \), then \( s = s'a = bs'' \) for some \( a, b \in A, s's'' \in A^* \), so that \( \text{im } s'a \subseteq \text{im } a \) and \( \ker b \subseteq \ker bs'' \).

Since \( Ys \subseteq Y \) for every \( s \in A^+ \), the restriction \( s_Y \) of \( s \) to \( Y \) can be defined. Let \( A_Y = \{a_Y : a \in A\} \). Then the automaton \( \mathcal{A}_Y = (Y, A_Y, \delta) \) is called the subautomaton of \( \mathcal{A} \) with respect to \( Y \).

Let \( s, t \in A^+ \) and \( x \in X \). Since \( (xs)t_Y = (xs)t \), the action of \( st_Y \) on \( X \) is defined by \( x(st_Y) = x(st) \).
Let \( \kappa \) be as above. Define the action of \( s \in A^+ \) on \( X/\kappa \) by \((x\kappa)s = (xs)\kappa\). Then the action is well-defined. In fact, if \( x\kappa = y\kappa \), then \((x,y) \in \kappa \subseteq \ker s \), so that \( xs = ys \). When the action of \( s \) is on \( X/\kappa \), \( s \) is denoted by \( s_\kappa \). Let \( A_\kappa = \{a_\kappa : a \in A\} \). Then the automaton \( A_\kappa = (X/\kappa, A_\kappa, \delta) \) is called the automaton induced from \( A \) by \( \kappa \).

Let \( s, t \in A^+ \) and \( x \in X \). Then clearly \((x\ker s)s = xs \). Since \( \kappa \subseteq \ker s \), we have \((x\kappa)s = xs \), so that \(((x\kappa)s_\kappa)t = ((xs)\kappa)t = (xs)t \). Thus the action of \( s_\kappa t \) on \( X \) is defined by \( x(s_\kappa t) = x(st) \).

For an automaton \( A = (X, A, \delta) \), let \( \text{Im}(A^+) = \{\text{im} s : s \in A^+\} = \{Y_i : i \in I\} \), i.e., for each \( i \in I \), \( Y_i = \text{im} s \) for some \( s \in A^+ \) and \( \text{im} s \in \text{Im}(A^+) \) for every \( s \in A^+ \), and let \( \text{Ker}(A^+) = \{\ker s : s \in A^+\} = \{\kappa_\mu : \mu \in M\} \), \( \text{Im}(A^+) = \{\text{im} s_\kappa : s \in A^+\} = \{Z_i : i \in I'\} \) and \( \text{Ker}(A^+) = \{\ker s_Y : s \in A^+\} = \{\kappa_\mu : \mu \in M'\} \). In this case, if \( \text{im} s \cap \ker s \neq \emptyset \) holds for every \( s \in A^+ \) and \( x \in X \), then \( \text{Im}(A^+) = \text{Im}(E(A^+)) \) and \( \text{Ker}(A^+) = \text{Ker}(E(A^+)) \).

3. Main Results

A semigroup in \( \mathcal{LZ} \times \mathcal{G} \) is called a left group whose class is denoted simply by \( \mathcal{LG} \), i.e., \( \mathcal{LG} = \mathcal{LZ} \times \mathcal{G} \).

**Theorem 1.** Let \( A = (X, A, \delta) \) be an automaton. Then the following are equivalent:

1. There exists a subset \( Y \) of \( X \) such that \( \text{im} a = Y \) and \( Y \cap \ker a \neq \emptyset \) for every \( a \in A \) and \( x \in X \).
2. There exists a subset \( Y \) of \( X \) such that \( \text{im} s = Y \) for every \( s \in S \).
3. \( A \) is a left group type.

From Theorem 1 we obtain the following results

**Corollary 1.1.** An automaton \( A = (X, A, \delta) \) is a \( \mathcal{SL}(\mathcal{LG}) \)-type if and only if \( \text{im} st = \text{im} s \cap \text{im} t \) for every \( s, t \in A^+ \).

**Corollary 1.2.** An automaton \( A = (X, A, \delta) \) is a \( \mathcal{RZ}(\mathcal{LG}) \)-type if and only if \( \text{im} st = \text{im} t \) for every \( s, t \in A^+ \).

A semigroup in \( \mathcal{G} \times \mathcal{RZ} \) is called a right group whose class is denoted by \( \mathcal{RG} \), i.e., \( \mathcal{RG} = \mathcal{G} \times \mathcal{RZ} \).

**Theorem 2.** Let \( A = (X, A, \delta) \) be an automaton. Then the following are equivalent:
(1) There exists an equivalence $\kappa$ on $X$ such that $\ker a = \kappa$ and $\im a \cap x\kappa \neq \emptyset$ for every $a \in A$.

(2) There exists an equivalence $\kappa$ on $X$ such that $\ker s = \kappa$ for every $s \in A^+$.

(3) $A$ is a right group type.

**Corollary 2.1.** An automaton $A = (X, A, \delta)$ is a $SL(RG)$-type if and only if $\ker st = \ker s \vee \ker t$ for every $s, t \in A^+$.

**Corollary 2.2.** An automaton $A = (X, A, \delta)$ is a $LZ(RG)$-type if and only if $\im st = \im t$ for every $s, t \in A^+$.

From Corollaries 1.2 and 2.2 we obtain:

**Corollary 2.3.** An automaton $RZ(LG)$-type if and only if it is $LZ(RG)$-type.

**Remark.** It can be easily show that $LZ(LG) = LZ(G) = LZ$ and $RZ(RG) = RZ(G) = RG$.

**Theorem 3.** Let $A = (X, A, \delta)$ be an automaton. Then the following are equivalent:

1. There exist a subset $Y$ of $X$ and an equivalence $\kappa$ on $X$ such that $\im a = Y$ and $\ker a = \kappa$ for every $a \in A$ and $Y \cap x\kappa \neq \emptyset$ for every $x \in X$.

2. There exist a subset $Y$ of $X$ and an equivalence $\kappa$ on $X$ such that $\im s = Y$ and $\ker s = \kappa$ for every $s \in A^+$,

3. $A$ is a group-type.

A semigroup in $SL(G)$ is called a Cliford semigroup,

**Corollary 3.1.** An automaton $A = (X, A, \delta)$ is a Cliford smigroup type if and only if $\im st = \im s \cap \im t$ and $\ker st = \ker s \vee \ker t$ for every $s, t \in A^+$.

**Theorem 4.** Let $A = (X, A, \delta)$ be an automaton, and Let $Y = \cup\{\im a : a \in A\}, \kappa = \cap\{\ker a : a \in A\}$. Suppose that $\im s \cap x\ker s \neq \emptyset$ for every $s \in A^+, x \in X$. Then the following are equivalent:

1. $A$ is a $LZ \times G \times RZ$-type.
2. $\ker s_Y = \ker t_Y$ for every $s, t \in A^+$.
3. $\im s_\kappa = \im t_\kappa$ for every $s, t \in A^+$.

**Corollary 4.1.** With the assumption of Theorem 4, the following are equivalent:

1. $A$ is a $SL(LZ \times G \times RZ)$-type.
(2) $\ker sy \cdot ty = \ker sy \vee \ker ty$ for every $s, t \in A^+$.  
(3) $\text{im} s_\kappa t_\kappa = \text{im} s_\kappa \cap \text{im} t_\kappa$ for every $s, t \in A^+$.

Suppose that an automaton $A$ is a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$-type. As is seen in the proof of Theorem 4, $A^+ / \rho = \{(i, g, \mu) : i \in I, g \in G, \mu \in M\}$. For $i \in I$ and $\mu \in M$, let $A_i / \rho = \{(i, g, \mu) : g \in G, \mu \in M\}$ and $A_\mu = \{(i, g, \mu) : i \in I, g \in G\}$, respectively. Then $A_i / \rho \in \mathcal{RG}$ and $A_\mu / \rho \in \mathcal{LG}$. For $s \rho = (i, g, \nu), t \rho = (j, h, \mu)$, since $(st) \rho = (i, gh, \mu)$, by Theorems 1 and 2, we have $\ker st = \ker s$ and $\text{im} st = \text{im} t$. Thus we obtain:

**Corollary 4.2.** If an automaton $A$ is a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$-type, then it is a $\mathcal{RZ}(\mathcal{LG})$-type. The converse is not true.

There is a simple example that a $\mathcal{RZ}(\mathcal{LG})$-type automaton which is not a $\mathcal{LZ} \times \mathcal{G} \times \mathcal{RZ}$-type.

**References**


