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Author(s)
Komeda, Jiryo

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Weierstrass semigroups of a pair of points whose first non-gaps are three

Jiryo Komeda (米田二良)
(Collaboration with Seon Jeong Kim)
Kanagawa Institute of Technology (神奈川工科大学)

1 Introduction

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. Let $C$ be a complete nonsingular curve of genus $g \geq 2$ over an algebraically closed field $k$ of characteristic 0, which is called a curve in this talk, and $K(C)$ the field of rational functions on $C$.

Definition 1.1. For a point $P$ of $C$, we set

$$H(P) := \{ \alpha \in \mathbb{N} \mid \text{there exists } f \in K(C) \text{ with } (f)_{\infty} = \alpha P \},$$

which is called the Weierstrass semigroup of the point $P$. An integer $n$ is called the first non-gap of $P$ if it is the minimum positive integer in $H(P)$.

Definition 1.2. For distinct points $P$ and $Q$ of $C$, we set

$$H(P, Q) := \{ (\alpha, \beta) \in \mathbb{N} \times \mathbb{N} \mid \text{there exists } f \in K(C) \text{ with } (f)_{\infty} = \alpha P + \beta Q \},$$

which is called the Weierstrass semigroup of the pair $(P, Q)$ of points.

Fact 1.3. (Kim [2]) If $C$ is a hyperelliptic curve, i.e., a double covering of the projective line, then the semigroup $H(P, Q)$ is determined explicitly.

Fact 1.4. (Kim-Komeda [4]) If $C$ is of genus 3, then the semigroup $H(P, Q)$ is determined explicitly.

Aim 1.5. Let $P$ and $Q$ be distinct points of $C$ of genus $g \geq 4$. If the first non-gaps of $P$ and $Q$ are three, we determine the semigroup $H(P, Q)$ explicitly. Moreover, we give examples of two pointed curves $(C, P, Q)$ whose semigroups are the given
2 Possible Weierstrass semigroups of genus $\geq 5$

First, let us review some Kim's results. Let $C$ be a curve of genus $g$ and $P$ its point.

Definition 2.1. We set

$$G(P) := \mathbb{N} \backslash H(P) = \{l_1 < l_2 < \ldots < l_g\}.$$ 

The integer $l_g$ is called the last gap at $P$.

Let $Q$ be another point of $C$ which is distinct from $P$. For each $l \in G(P)$, the integer $\min\{\beta|(l, \beta) \in H(P, Q)\}$ must be equal to some element in $G(Q)$, say $\sigma(l)$, and this correspondence $\sigma$ gives a bijective map between the sets $G(P)$ and $G(Q)$.

Definition 2.2. We set

$$\Gamma(P, Q) := \{(l, \sigma(l))|l \in G(P)\}.$$ 

Fact 2.3. (Kim [2]) The semigroup $H(P, Q)$ is completely determined by the bijective correspondence $\sigma$, i.e.,

$$G(P, Q) = \bigcup_{l \in G(P)} \{(l, \beta)|\beta = 0, 1, \ldots, \sigma(l) - 1\} \cup \{(\alpha, \sigma(l))|\alpha = 0, 1, \ldots, l - 1\},$$

where we set

$$G(P, Q) = \mathbb{N} \times \mathbb{N} \backslash H(P, Q).$$

Thus, it suffices to determine the graph $\Gamma(P, Q)$ of $\sigma$ for describing the semigroup $H(P, Q)$.

We consider the case where the first non-gaps of distinct points $P$ and $Q$ are three.

Theorem 2.4. $\sigma(l_g) = 1$ or $\sigma(l_g) = 2$, i.e., $(l_g, 1) \in \Gamma(P, Q)$ or $(l_g, 2) \in \Gamma(P, Q)$.

Proof. Assume that $(l_g, \beta) \in \Gamma(P, Q)$ with $\beta \geq 4$. Then

$$(\alpha_1, 1) \in \Gamma(P, Q), 1 \leq \alpha_1 < l_g \text{ and } (\alpha_2, 2) \in \Gamma(P, Q), 1 \leq \alpha_2 < l_g.$$ 

Now

$$\beta \equiv i \mod 3 \text{ for some } i = 1, 2 \text{ and } (\alpha_i, \beta) = (\alpha_i, i) + (0, 3k) \text{ for some } k.$$ 

Since $(\alpha_i, i) \in \Gamma(P, Q)$ and $(0, 3k) \in H(P, Q)$, we have $(\alpha_i, \beta) \in H(P, Q)$. But $(l_g, \beta) \in \Gamma(P, Q)$ and $\alpha_i < l_g$. This contradicts Fact 2.3. Q.E.D.
From now on we assume that $g \geq 5$. By the theory of curves we can find some integer $n$ with 
\[
\frac{g-1}{3} \leq n \leq \frac{g}{2}
\]
such that 
\[
S(H(P)) = \{3, 3n + 2, 3(g - n) + 1\}
\]
or 
\[
S(H(P)) = \{3, 3n + 1, 3(g - n) + 2\},
\]
where $S(H(P)) = \{3, s_1, s_2\}$, which is called the standard basis for $H(P)$, with 
\[
s_i = \text{Min}\{s \in H(P)|s \equiv i \text{ mod 3}\} \text{ for } i = 1, 2.
\]
Moreover, since there exists $f \in K(C)$ such that $(f) = 3P - 3Q$, the standard basis $S(H(Q))$ must be one of the above two.

**Definition 2.5.** The point $P$ is said to be of the $n$-th kind if 
\[
S(H(P)) = \{3, 3n + 2, 3(g - n) + 1\}
\]
or 
\[
S(H(P)) = \{3, 3n + 1, 3(g - n) + 2\},
\]

**Definition 2.6.** The point $P$ of the $n$-th kind is said to be of type I (resp. II) if 
\[
n \neq \frac{g}{2} \text{ and } 3n + 2 \in S(H(P)) \text{ (resp. } 3n + 1 \in S(H(P))\).
\]
We note that If $n = \frac{g}{2}$, then $S(H(P)) = \{3, 3n + 1, 3n + 2\}$.

Using the types of the points $P$ and $Q$ we can determine whether 
\[
(l_g, 1) \in \Gamma(P,Q) \text{ or } (l_g, 2) \in \Gamma(P,Q).
\]

**Theorem 2.7.** Let $P$ and $Q$ be two distinct points of the $n$-th kind and $n \neq \frac{g}{2}$.

i) If $P$ and $Q$ are of type II, then 
\[
(l_g, 2) \in \Gamma(P,Q) \text{ and } (3n - 2, 1) \in \Gamma(P,Q).
\]

ii) If $P$ (resp. $Q$) is of type I (resp. II), then 
\[
(l_g, 1) \in \Gamma(P,Q) \text{ and } (3n - 2, 2) \in \Gamma(P,Q).
\]

iii) If $P$ and $Q$ are of type I, then 
\[
(l_g, 1) \in \Gamma(P,Q) \text{ and } (3n - 1, 2) \in \Gamma(P,Q).
\]
Moreover, if \((\alpha_1, 1)\) and \((\alpha_2, 2)\) belong to \(\Gamma(P, Q)\), then
\[
\Gamma(P, Q) = \left\{ (\alpha_1 - 3k, 1 + 3k) | 0 \leq k < \frac{\alpha_1}{3} \right\} \cup \left\{ (\alpha_2 - 3k, 2 + 3k) | 0 \leq k < \frac{\alpha_2}{3} \right\}.
\]

In some case with \(n = \frac{g}{3}\) we have no candidate of the semigroup \(H(P, Q)\).

**Proposition 2.8.** Let \(P\) and \(Q\) be points of the \(\frac{g}{3}\)-th kind. If they are of type II, then \(P = Q\). In this case, \(H(P)\) is generated by 3 and \(g + 1\) with \(g \equiv 0 \mod 3\).

There are two possibilities in the case \(n = \frac{g}{2}\).

**Proposition 2.9.** Let \(P\) and \(Q\) be distinct points of the \(\frac{g}{2}\)-th kind, i.e.,
\[
S(H(P)) = S(H(Q)) = \{3, 3n + 1, 3n + 2\}.
\]
Then
\[
(3n - 1, 1) \in \Gamma(P, Q) \text{ or } (3n - 1, 2) \in \Gamma(P, Q).
\]
If \((3n - 1, 1) \in \Gamma(P, Q)\) (resp. \((3n - 1, 2) \in \Gamma(P, Q)\)), then
\[
\Gamma(P, Q) = \{ (\alpha, 3n - \alpha) | \alpha \in G(P) \}
\]
(resp. \(\{ (3k - 2, (3n - 1) - (3k - 2)) | k = 1, \ldots, n \} \)
\[
\cup \{ (3k - 1, (3n + 1) - (3k - 1)) | k = 1, \ldots, n \} \}).
\]

3 The existence of two pointed curves

In the previous section we determined the possible Weierstrass semigroups \(H\) of a pair of points on a curve of genus \(\geq 5\) whose first non-gaps are three. In this section for each such a semigroup \(H\) we give two pointed curves \((C, P, Q)\) such that \(H(P, Q) = H\).

Let \(C\) be the curve whose function field \(K(C) = k(x, y)\) is defined by the equation
\[
y^3 = (x - c_1) \cdots (x - c_i) (x - c_{i+1})^2 \cdots (x - c_{i+i_2})^2,
\]
where \(c_1, \ldots, c_{i+i_2}\) are distinct elements of \(k\) and \(i_1 + 2i_2\) is not divisible by 3. We note that the genus \(g\) of the curve \(C\) is \(i_1 + i_2 - 1\) by Riemann-Hurwitz formula.

Let \(\pi : C \to P^1\) be the morphism corresponding to the inclusion \(k(x) \subset K(C)\), i.e., \(\pi(P) = (1 : x(P))\), where \(P^1\) denotes the projective line. We set
\[
\{P_\infty\} = \pi^{-1}(0 : 1) \text{ and } \{P_s\} = \pi^{-1}(1 : c_s) \text{ for } s = 1, \ldots, i_1 + i_2.
\]
Then we have \( S(H(P_\infty)) = \{3, i_1 + 2i_2, 2i_1 + i_2\} \) (For example, see Kim-Komeda [3]). If \( i_1 + 2i_2 \equiv 1 \) mod 3,
\[
S(H(P_s)) = \begin{cases} 
\{3, i_1 + 2i_2 + 1, 2i_1 + i_2 - 1\} & \text{if } 1 \leq s \leq i_1 \\
S(H(P_\infty)) = \{3, i_1 + 2i_2, 2i_1 + i_2\} & \text{if } i_1 + 1 \leq s \leq i_1 + i_2
\end{cases}
\]
If \( i_1 + 2i_2 \equiv 2 \) mod 3,
\[
S(H(P_s)) = \begin{cases} 
S(H(P_\infty)) = \{3, i_1 + 2i_2 - 1, 2i_1 + i_2 + 1\} & \text{if } 1 \leq s \leq i_1 \\
\{3, i_1 + 2i_2, 2i_1 + i_2\} & \text{if } i_1 + 1 \leq s \leq i_1 + i_2
\end{cases}
\]

From now on we assume that \( g \geq 5 \). By the above formula we get the following examples:

**Example 3.1.** i) Let \( \frac{g}{3} < n < \frac{g}{2} \). If \( i_1 = 2g - 3n + 1 > n + 1 \) and \( i_2 = 3n - g > 0 \), then the points \( P_\infty \) and \( P_s \ (i_1 + 1 \leq s \leq i_1 + i_2) \) are of the \( n \)-th kind of type II.

ii) Let \( \frac{g}{3} \leq n \leq \frac{g-1}{2} \). If \( i_1 = 2g - 3n + 1 > n + 3 \) and \( i_2 = 3n - g \geq 0 \), then the points \( P_\infty \) is of the \( n \)-th kind of type II and the points \( P_s \ (1 \leq s \leq i_1) \) are of the \( n \)-th kind of type I.

iii) Let \( \frac{g-1}{3} \leq n \leq \frac{g-1}{2} \). If \( i_1 = 2g - 3n + 1 \geq n + 2 \) and \( i_2 = 3n - g + 1 \geq 0 \), then the points \( P_\infty \) and \( P_s \ (1 \leq s \leq i_1) \) are of the \( n \)-th kind of type I.

In the case of Proposition 2.9 we get the following examples:

**Example 3.2.** Let \( n \geq 3 \). If \( i_1 = n + 1 \) and \( i_2 = n \), then \( g = 2n \) and the points \( P_\infty \) and \( P_s \) are of the \( \frac{g}{2} \)-th kind. Then \( S(H(P_\infty)) = S(H(P_s)) = \{3, 3n + 1, 3n + 2\} \).

Moreover,
\[
\Gamma(P_\infty, P_s) \ni (3n - 1, 1) \text{ for } 1 \leq s \leq i_1
\]
and
\[
\Gamma(P_\infty, P_s) \ni (3n - 1, 2) \text{ for } i_1 + 1 \leq s \leq i_1 + i_2.
\]

4 Weierstrass semigroups of genus 4

In this section we treat the curves \( C \) of genus 4 with point \( P \) whose first non-gap is 3. Then we have \( S(H(P)) = \{3, 5, 10\} \) or \( \{3, 7, 8\} \).

**Remark 4.1.** Let \( S(H(P)) = \{3, 5, 10\} \). If \( Q \) is another point of \( C \) whose first non-gap is three, then \( S(H(Q)) = \{3, 5, 10\} \) and there exists \( f \in \mathcal{K}(C) \) such that \( (f) = 3P - 3Q \).
Proposition 4.2. Let $P$ and $Q$ be two distinct points such that $S(H(P)) = S(H(Q)) = \{3, 5, 10\}$. Then $\Gamma(P, Q) = \{(1, 7), (2, 2), (4, 4), (7, 1)\}$. For example, such pointed curves are given by

$$y^3 = (x - c_1) \cdots (x - c_5), \quad P = P_\infty \text{ and } Q = P_s \text{ for } s = 1, \ldots, 5$$

where we use the notations in Section 3.

Proposition 4.3. Let $S(H(P)) = \{3, 7, 8\}$ and $Q$ another point of $C$ whose first non-gap is three. Suppose that there exists $f \in K(C)$ such that $(f) = 3P - 3Q$.

i) $(5, 1) \in \Gamma(P, Q)$ or $(5, 2) \in \Gamma(P, Q)$.

ii) If $(5, 1) \in \Gamma(P, Q)$, then $\Gamma(P, Q) = \{(5, 1), (4, 2), (2, 4), (1, 5)\}$. For example, such pointed curves are given by

$$y^3 = (x - c_1)(x - c_2)(x - c_3)(x - c_4)^2(x - c_5)^2, \quad P = P_\infty \text{ and } Q = P_s \text{ for } s = 1, 2, 3.$$ 

iii) If $(5, 2) \in \Gamma(P, Q)$, then $\Gamma(P, Q) = \{(5, 2), (4, 1), (1, 4), (2, 5)\}$. Such pointed curves are given by the same equations as above, $P = P_\infty$ and $Q = P_s$ for $s = 4, 5$.

Proposition 4.4. Let $S(H(P)) = \{3, 7, 8\}$ and $Q$ another point of $C$ whose first non-gap is three. Suppose that there is no $f \in K(C)$ such that $(f) = 3P - 3Q$. Then $(5, 1) \in \Gamma(P, Q)$ and $\Gamma(P, Q) = \{(5, 1), (4, 4), (2, 2), (1, 5)\}$. Such curves $C$ are also given by the equations

$$y^3 = (x - c_1)(x - c_2)(x - c_3)(x - c_4)^2(x - c_5)^2.$$

Using the result of Kato [1] we get our desired points $P$ and $Q$.

References


