Finitely Generated Idempotent-free
Semilattice-Indecomposable Semigroups with Relations I

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A semigroup $S$ is called $S$-indecomposable if $S$ has no semilattice homomorphic image except the trivial semilattice. We assume $S \neq S^2$, $|S\backslash S^2| < \infty$ and $S$ is generated by $S\backslash S^2$. Let $B = S\backslash S^2 = \{a_1, \ldots, a_k\}$. The purpose of this paper is to report the structure of idempotent-free $S$-indecomposable semigroup $S$ generated by $B$ with relation as defined below. Let $Z_+$ be the set of all positive integers. We assume

\[(1) \quad a_1^{m_1} = \cdots = a_k^{m_k} \quad \text{for some} \quad m_1, \ldots, m_k \in Z_+.
\]

In particular we study here the free semigroup satisfying (1), that is, every such semigroup is a homomorphic image of the free one. The condition (1) is so strong that the property of $S$-indecomposability is derived from (1).

**Lemma 1.** If $S$ is a semigroup generated by $B$ and satisfies (1), then $S$ is $S$-indecomposable.

Since $|B| < \infty$ the condition (1) is equivalent to (1') below.

\[(1') \quad \text{For each pair} \quad a_i, a_j \in B \quad \text{there exist} \quad n_i, n_j \in Z_+ \quad \text{such that} \quad a_i^{n_i} = a_j^{n_j}.
\]

If $B$ satisfies (1'), equivalently (1), we say $S$ is *power jointly generated* by $B$.

Let $S$ be an idempotent-free semigroup which is power-jointly generated by $B$ with (1), and let $F$ be the free semigroup over $B$. There is a homomorphism $f : F \to S$ which satisfies the following conditions

i) $X \in F, \quad a \in B, \quad f(X) = f\{(a)\} \Rightarrow X = \{a\}$.

ii) $f(a_1)^{m_1} = \cdots = f(a_k)^{m_k}$.

Let $\rho$ denote the congruence on $F$ generated by the set of binary relations

$\{(a_i^{m_i}, a_j^{m_j}) : a_i, a_j \in B\}$. 

Then $S$ is a homomorphic image of $F/\rho$ keeping every element of $B$ fixed. In this paper we study $F/\rho$. For simplicity of notation, let $S = F/\rho$, so $X\rho Y$ in $F$ if and only if $X = Y$ in $S$.

From (1) we immediately have

**Lemma 2.** $a_i^{\lambda m_i}a_j^s = a_j^{s}a_i^{\lambda m_i}$ for $i, j = 1, \ldots, k$, for any $\lambda \in \mathbb{Z}_+$. 

Let $X \in S$. $X$ has the form $X = a_{i_1}^{s_1} \cdots a_{i_s}^{s_s}$ where $a_{i_j} \in B$, $(j = 1, \ldots, s)$ $x_j' \in \mathbb{Z}_+$,

(2) $a_{i_j} \neq a_{i_{j+1}}$, $(j = 1, \ldots, s - 1)$.

We rewrite $x_j' = x_i + \lambda_i m_i$ where $0 < x_i \leq m_i$, $\lambda_i \in \mathbb{Z}_+ = \mathbb{Z}_+ \cup \{0\}$. Let $M = a_{i_1}^{m_1} = \cdots = a_{i_s}^{m_s}$. By using Lemma 2 repeatedly we have

(3) $X = a_{i_1}^{s_1} \cdots a_{i_s}^{s_s} a_{i_s}^{\lambda m_s} \cdots a_{i_2}^{s_2} M^\lambda$ where $\lambda = \lambda_1 + \cdots + \lambda_s$.

Likewise $Y = a_{i_1}^{y_1} \cdots a_{i_s}^{y_s} a_{i_s}^{\mu m_s} \cdots a_{i_2}^{y_2} M^\mu$ where $\mu = \mu_1 + \cdots + \mu_s$.

Consider the product $XY$. Again by using Lemma 2 we have:

If $i_s \neq j_1$ $XY = a_{i_1}^{s_1} \cdots a_{i_s}^{s_s} a_{j_1}^{y_1} \cdots a_{j_s}^{y_s} M^{\lambda + \mu}$.

If $i_s = j_1$ and $x_s + y_1 \leq 2m_s$, then $XY = a_{i_1}^{s_1} \cdots a_{i_{s-1}}^{s_{s-1}} a_{i_s}^{s_s} a_{j_2}^{y_2} \cdots a_{j_s}^{y_s} M^{\lambda + \mu}$ where $0 < z_s \leq m_i$ and $z_s \equiv x_s + y_1 \pmod{m_i}$.

If $i_s = j_1$ and $x_s + y_1 > 2m_s$, then $XY = a_{i_1}^{s_1} \cdots a_{i_{s-1}}^{s_{s-1}} a_{i_s}^{s_s} a_{j_2}^{y_2} \cdots a_{j_s}^{y_s} M^{\lambda + \mu + 1}$ where $0 < z_s \leq m_i$ and $z_s \equiv x_s + y_1 \pmod{m_i}$.

Let $P$ denote the set of finite sequences $V$ of elements of $B$, $V = a_{i_1} \cdots a_{i_s}$ satisfying $a_{i_j} \neq a_{i_{j-1}}$, $j = 1, \ldots, s - 1$.

The binary operation on $P$ is defined by

$$(a_{i_1} \cdots a_{i_s}) * (a_{j_1} \cdots a_{j_t}) = \begin{cases} a_{i_1} \cdots a_{i_s} a_{j_1} \cdots a_{j_t} & \text{if } i_s \neq j_1 \\ a_{i_1} \cdots a_{i_s} a_{j_2} \cdots a_{j_t} & \text{if } i_s = j_1 \end{cases}$$

that is, if $i_s \neq j_1$, the product is juxtaposition, if $i_s = j_1$ then one of $a_{i_s}$ and $a_{j_1}$ is omitted.

**Proposition 1.** $P$ is a semigroup and $S$ is homomorphic onto $P$ under the mapping $a_{i_1}^{s_1} \cdots a_{i_s}^{s_s} \rightarrow a_{i_1} \cdots a_{i_s}$.

$P$ is regarded as the set of finite sequences $i_1 \cdots i_s$ of elements of $B = \{1, \ldots, k\}$ subject to $i_j \neq i_{j+1}$, $j = 1, \ldots, s - 1$, $s \geq 1$. In the form (3): $X = a_{i_1}^{s_1} \cdots a_{i_s}^{s_s} M^{\lambda}$, we
rewrite $x_j$ by $x_{ij} (j = 1, \ldots, s)$

(3') $X = a_{i_{1}}^{x_{i_{1}}} \cdots a_{i_{s}}^{x_{i_{s}}} M^{\lambda}$.

The sequence $x_{i_{1}} \cdots x_{i_{s}}$ is regarded as a mapping from a sequence $i_{1} \cdots i_{s}$ of elements of $\{1, \ldots, k\}$ to a sequence $x_{i_{1}} \cdots x_{i_{s}}$ such that $x_{ij} \in Z_{m_{ij}}$ (i.e. an element modulo $m_{ij}$) and $0 < x_{ij} \leq m_{ij}$, $j = 1, \ldots, s$, $s = 1, \ldots, k$. Let $\varphi : i_{1} \cdots i_{s} \to x_{i_{1}} \cdots x_{i_{s}}$, $\psi : j_{1} \cdots j_{t} \to y_{j_{1}} \cdots y_{j_{t}}$ and let $\Phi$ denote the set of all such $\varphi$'s and define the binary operation $\varphi \psi$ on $\Phi$ as follows:

- If $i_{s} \neq j_{1}$, $(i_{1} \cdots i_{s}) \ast (j_{1} \cdots j_{t}) = i_{1} \cdots i_{s}j_{1} \cdots j_{t} \to x_{i_{1}} \cdots x_{i_{s}}y_{j_{1}} \cdots y_{j_{t}}$.
- If $i_{s} = j_{1}$, $(i_{1} \cdots i_{s}) \ast (j_{1} \cdots j_{t}) = i_{1} \cdots i_{s-1}i_{s}j_{2} \cdots j_{t} \to x_{i_{1}} \cdots x_{i_{s-1}}x_{i_{s}}y_{j_{2}} \cdots y_{j_{t}}$, where $z_{s} = x_{i_{s}} + y_{j_{1}} (\mod m_{i_{s}})$, $0 < z_{s} \leq m_{i_{s}}$.

**Proposition 2.** $\Phi$ is a semigroup, and $S$ is homomorphic onto $\Phi$ under the mapping $X = a_{i_{1}}^{x_{i_{1}}} \cdots a_{i_{s}}^{x_{i_{s}}} M^{\lambda} \to \varphi$ where $\varphi : i_{1} \cdots i_{s} \to x_{i_{1}} \cdots x_{i_{s}}$.

Define a mapping $g : \Phi \times \Phi \to Z_{+}^{0}$ as follows:

$$g(\varphi, \psi) = \begin{cases} 1 & \text{if } i_{s} = j_{1} \text{ and } x_{i_{s}} + y_{j_{1}} > m_{i_{s}} \\ 0 & \text{otherwise} \end{cases}$$

Let $\Gamma = \{(\varphi, \lambda) : \varphi \in \Phi, \lambda \in Z_{+}^{0}\}$ and define the binary operation on $\Gamma$ as follows:

$$(\varphi, \lambda)(\psi, \mu) = (\varphi\psi, \lambda + \mu + g(\varphi, \psi)).$$

Note that $g$ satisfies the condition:

$$g(\varphi, \psi) + g(\varphi\psi, \xi) = g(\varphi, \psi\xi) + g(\psi, \xi) \text{ for all } \varphi, \psi, \xi \in \Phi.$$ 

Now we have the main theorem

**Theorem.** $\Gamma$ is a semigroup and $S$ is isomorphic onto $\Gamma$ under the mapping $X = a_{i_{1}}^{x_{i_{1}}} \cdots a_{i_{s}}^{x_{i_{s}}} M^{\lambda} \to (\varphi, \lambda)$ where $\varphi : i_{1} \cdots i_{s} \to x_{i_{1}} \cdots x_{i_{s}}$.

The idea of constructing $S$-indecomposable semigroups from a certain free semigroup was initiated by the author in case of finite nil semigroups 1958 [2], and also the idea was used in case of finitely generated $Z$-semigroups [3].

The representation of $S$ by means of $\Gamma$ is similar to $N$-semigroups (i.e. idempotent-free cancellative commutative archimedean semigroups) [1].
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REFERENCES

