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An Automaton for Deciding Whether a Given Set of Words is a Code.

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For a given finite set X of words on an alphabet A, it is well-know that there is an algorithm for deciding whether the set X is a code (see [1]). In this paper, we define the ambiguous word which has more than two factorizations in X^* and we construct an automaton $\mathcal{A}_X$ such that the set $L(\mathcal{A}_X)$ recognized by $\mathcal{A}_X$ is the set of all ambiguous words in $\mathcal{A}_X$. We show that a given set X of words on an alphabet A is a code if and only if that the set $L(\mathcal{A}_X)$ recognized by $\mathcal{A}_X$ is empty.

For a finite set X of words on an alphabet A, we denote by $P(X)$ the set $\{p \in X \mid p$ is a proper prefix of some word in $X\}$. Let c_1 be the cardinality of $P(X)$. Then, there is an injection φ from $P(X)$ into the set of natural numbers such that $1 \leq \varphi(p) \leq c_1$ for every $p \in P(X)$. We also denote by $S(X)$ the set $\{s \in A^+ \mid s$ is a proper suffix of some word in $X\}$. There is an injection ψ from $S(X)$ into the set of natural numbers such that $c_1 + 1 \leq \psi(s)$ for every $s \in S(X)$.

Now, we construct an automaton $\mathcal{A}_X$ over A inductively. The edges and states of $\mathcal{A}_X$ are defined by the following rules. Let i be the unique initial state of $\mathcal{A}_X$. Every element of $\varphi(P(X))$ is state of $\mathcal{A}_X$ and, for every word p in $P(X)$,

$$i \xrightarrow{p} \varphi(p)$$

is a path in $\mathcal{A}_X$. As every word p in $P(X)$ is a proper prefix of some word x in X, the word $u = p^{-1}x$ is a suffix of x, that is, u is in $S(X)$. Then, $\psi(u)$ is a state of $\mathcal{A}_X$ and

$$\varphi(p) \xrightarrow{u} \psi(u)$$

is a path in $\mathcal{A}_X$. If $\psi(u)$ is a state of $\mathcal{A}_X$ and if, for some word v in $S(X)$, the concatenation

uv is written of the form

$$uv = x_1 \cdots x_m \quad (x_1, \cdots, x_m \text{ is in } \mathcal{A}_X \text{ and } v \text{ is a proper suffix of } x_m)$$

then, $\psi(v)$ is a state of $\mathcal{A}_X$ and

$$\psi(u) \xrightarrow{v} \psi(v)$$

is a path in $\mathcal{A}_X$. Since $\varphi(P(X)) \cup \psi(S(X))$ and $S(X)$ are finite, this procedure has finite steps. Let Q be the set of all states of $\mathcal{A}_X$. The set of terminal states of $\mathcal{A}_X$ is the set $\psi(S(X) \cap X^*) \cap Q$.

A word w is said to be *ambiguous* if there exist words $x_1, \cdots, x_m, y_1, \cdots, y_n$ of X such that

$$w = x_1 \cdots x_m = y_1 \cdots y_n \text{ and } x_1 \cdots x_i \neq y_1 \cdots y_j \ (i = 1, \cdots, m-1, \ \ j = 1, \cdots, n-1).$$

Theorem. For a given set X on an alphabet A, the set $L(\mathcal{A}_X)$ recognized by the automaton $\mathcal{A}_X$ of X is the set of all ambiguous words in X^*.

Proof. Let $w \in L(\mathcal{A}_X)$. There exist $x_1 \in P(X), x_2, \cdots, x_r \in S(X), x_r \in S(X) \cap X^*$ such that $w = x_1 x_2 \cdots x_r$ and a succesible path

$$i \xrightarrow{x_1} q_1 \xrightarrow{x_2} q_2 \longrightarrow \cdots \longrightarrow q_{r-1} \xrightarrow{x_r} q_r$$

where $q_1 = \varphi(x_1), q_2 = \psi(x_2), \cdots, q_r = \psi(x_r)$ and q_r is a terminal state. Since $q_1 = \varphi(x_1)$ is in $\varphi(P(X))$, q_1 is not a terminal state. If $r = 2$, then

$$i \xrightarrow{x_1} q_1 \xrightarrow{x_2} q_2$$

is succesible. By the definition of $\mathcal{A}_X$ the word $w = x_1 x_2$ itself is in X. Thus, w is ambiguous. Let $r > 2$. By the definition of $\mathcal{A}_X$, $x_1, x_{k-1}x_k, x_r$ $(k = 2, \cdots, r)$ are words of X^*, thus $w = x_1 x_2 \cdots x_r$ has two factorizations:

$$w = y_1 y_2 \cdots y_m = z_1 z_2 \cdots z_n \quad (y_1, \cdots, y_m, z_1, \cdots, z_n \in X).$$

We show that $y_1 y_2 \cdots y_i \neq z_1 z_2 \cdots z_j$ for all $i = 1, \cdots, m$, $j = 1, \cdots, n$. Suppose that $y_1 y_2 \cdots y_i = z_1 z_2 \cdots z_j$ for some i, j. There exists x_t such that $y_1 y_2 \cdots y_i = z_1 z_2 \cdots z_j$ is a prefix of $x_1 x_2 \cdots x_t$ and that $y_1 y_2 \cdots y_i = z_1 z_2 \cdots z_j$ is not a prefix of $x_1 x_2 \cdots x_{t-1}$. By the definition of $\mathcal{A}_X$, we may assume that y_i is a subwords of $x_{t-1} x_t$, then x_t is a suffix of y_i. However, z_j must be a subwords of $x_t x_{t+1}$. It is impossible.

Let w be ambiguous and $w = y_1y_2\cdots y_m = z_1z_2\cdots z_n$ $(y_1,\cdots,y_m,z_1,\cdots,z_n \in X)$, $y_1y_2\cdots y_i \neq z_1z_2\cdots z_j$ for all $i=1,\cdots,m$, $j=1,\cdots,n$. We may assume that z_1 is a proper prefix of y_1. Since w is ambiguous, there exist $i_1, i_2, \cdots, j_1, j_2, \cdots$ such that the following conditions are satisfied:

y_1 is a proper prefix of $z_1z_2\cdots z_{j_1}$ and not a prefix of $z_1z_2\cdots z_{j_1-1}$

$z_1z_2\cdots z_{j_1}$ is a proper prefix of $y_1\cdots y_{i_2}$ and not a prefix of $y_1\cdots y_{i_2-1}$

...

Let $x_1 = z_1^{-1}y_1$, $x_2 = y_1^{-1}z_1z_2\cdots z_{j_1}$, $x_3 = (z_1z_2\cdots z_{j_1})^{-1}y_1y_2\cdots y_{i_2}$, $\cdots$. If z_n is a proper prefix of y_m and if $z_kz_{k+1}\cdots z_m$ is a suffix of y_m and $z_{k-1}z_k\cdots z_m$ is not, then we set $x_r = (z_1z_2\cdots z_{k-1})^{-1}y_1y_2\cdots y_m = z_k\cdots z_n$. If y_m is a proper prefix of z_n and if $y_ky_{k+1}\cdots y_m$ is a suffix of z_n and $y_{k-1}y_k\cdots y_m$ is not, then we set $x_r = (y_1y_2\cdots y_{k-1})^{-1}z_1z_2\cdots z_n = y_k\cdots y_m$. Then, we have a succesible path

$$i \xrightarrow{x_1} q_1 \xrightarrow{x_2} q_2 \longrightarrow \cdots \longrightarrow q_{r-1} \xrightarrow{x_r} q_r$$

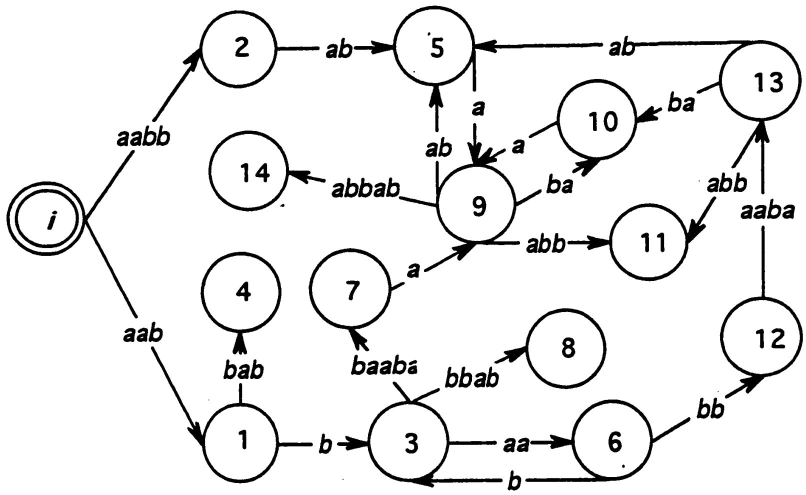
where $q_1 = \varphi(x_1)$, $q_2 = \psi(x_2), \cdots, q_r = \psi(x_r)$. q.e.d.

The proposition yields the following corollary immediately.

Corollary. A given set X on an alphabet A is a code if and only if the set $L(\mathcal{A}_X)$ recognized by the automaton $\mathcal{A}_X$ of X is empty.

Example 1. Let $A = \{a. b\}$ and let $X = \{aab, aabb, aabbab, aba, baa, bbaaba\}$. We construct $\mathcal{A}_X$ and we show that $L(\mathcal{A}_X)$ is empty, therefore, X is a code.

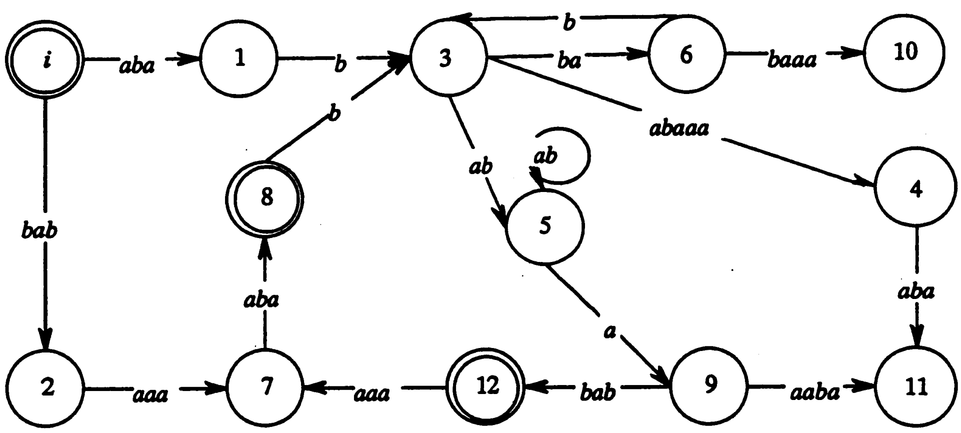
It is clear that $P(X) = \{aab, aabb\}$. We define a bijection $\varphi : P(X) \rightarrow \{1, 2\}$ by $\varphi(aab) = 1, \varphi(aabb) = 2$. Since aab is a prefix of $aabb$, $aabbab$ and $aabb$ is a prefix of $aabbab$, then b, bab , ab are in $S(X)$. We can define an injection ψ from $S(X)$ into the set of natural numbers such that $\psi(b) = 3, \psi(bab) = 4, \psi(ab) = 5$. Since b is a prefix of baa , $bbaaba$ and ab is a prefix of aba , then aa , $baaba$, a are in $S(X)$. But, bab is not prefix of any word of X, therefore the state $\psi(bab) = 4$ is not coaccessible. Continuing this process, we have the following automaton :



where $\psi(aa) = 6, \psi(baaba) = 7, \psi(bbab) = 8, \psi(a) = 9, \psi(ba) = 10, \psi(abb) = 11,$
$\psi(bb) = 12, \psi(aaba) = 13, \psi(abbab) = 14$. Since $S(X) \cap X = \{aba\}$ and $\psi(bab)$ is not a state of $\mathcal{A}_X$, there is no terminal state in $\mathcal{A}_X$, thus $L(\mathcal{A}_X)$ is empty and X is a code.

Example 2. Let $A = \{a. b\}$ and let $X = \{aaaaba, aba, abab, bab, babaaa, bba\}$. We construct $\mathcal{A}_X$. In this case $L(\mathcal{A}_X)$ is not empty, therefore, X is not a code.

It is clear that $P(X) = \{aba, bab\}$. We define a bijection $\varphi : P(X) \rightarrow \{1, 2\}$ by $\varphi(aba) = 1, \varphi(bab) = 2$. We have the following automaton :



The set of all states Q is $\{\psi(aba)=1, \psi(bab)=2, \psi(b)=3, \psi(abaaa)=4, \psi(ab)=5, \psi(ba)=6, \psi(aaa)=7, \psi(aba)=8, \psi(a)=9, \psi(baaa)=10, \psi(aaba)=11, \psi(bab)=12\}$. The set of all terminal states is $S(X) \cap X \cap Q = \{\psi(aba)=8,\ \psi(bab)=12\}$. Since $\psi(aba)=8$ is a terminal state, a path

$$i \xrightarrow{aba} 1 \xrightarrow{b} 3 \xrightarrow{ba} 6 \xrightarrow{b} 3 \xrightarrow{ab} 5 \xrightarrow{ab} 5 \xrightarrow{a} 9 \xrightarrow{bab} 12 \xrightarrow{aaa} 7 \xrightarrow{aba} 8$$

is successible, thus $w = ababbababababaaaaba$ is accepted by $\mathcal{A}_X$ and w is ambiguous. In fact, w has two factorizations

$$w = (aba)(bba)(bab)(aba)(babaaa)(aba) = (abab)(bab)(abab)(abab)(aaaaba)$$

in X.

Reference

[1] J. Berstel & D. Perrin, Theory of codes, Academic Press, 1985.