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On Linear Arrangement Problems on Multidimensional Torus Graphs

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Abstract

A linear layout of a graph G is a one-to-one mapping of the vertices of G into the integers from 1 to the number of vertices of G. Since the range of a linear layout L is regarded as the positions on the number line, the sum of $|L(v)-L(w)|$ over all of the edges vw is called the sum of edge length of L. For every pair of positive integers n and m, the n-dimensional m-torus graph T_m^n is introduced. The vertex-set of T_m^n is the set of all $\mathbb{Z}_m$-valued n-dimensional vectors, and the edge-set is the set of all pairs of vertices $v=(v_1,v_2,\ldots,v_n)$ and $w=(w_1,w_2,\ldots,w_n)$ such that they differ at exactly one index, say j, and either $v_j \equiv w_j+1 \pmod m$ or $w_j \equiv v_j+1 \pmod m$ holds. The natural layout of T_m^n is the linear layout that arranges the vertices in lexicographic order. It is proved that, for every positive integer n, the sum of edge length of the natural layout is the minimum among the ones of all linear layouts of T_3^n. Furthermore, the set of linear layouts of T_3^n whose sum of edge length is the maximum is completely characterized.

Keywords: graph theory, induced subgraph, hypercube, torus, linear layout.

1 Introduction

A linear layout of a graph $G=(V,E)$, where V and E are vertex-set and edge-set, respectively, is a one-to-one mapping $L: V \to \{1,2,\ldots,|V|\}$. If we regard the range of L as the positions on a number line, then L places each vertex of G at the corresponding position on the number line, and each edge of G on the interval between the points corresponding to the end vertices of the edge. We call the value $\sum_{vw\in E}|L(v)-L(w)|$ the sum of edge length of L. Let Q_n denote the n-dimensional hypercube, or n-cube. The vertex-set of Q_n, denoted $V(Q_n)$, is $\{0,1\}^n$, the set of all n-dimensional 0-1 vectors. The edge-set of Q_n, denoted $E(Q_n)$, is the set of all pairs of vertices $v=(v_1,v_2,\ldots,v_n)$ and $w=(w_1,w_2,\ldots,w_n)$ that differ at exactly one index. The natural layout N_n of Q_n is the linear layout defined as

$$N_n((x_1,x_2,\ldots,x_n)) = 1+\sum_{i=1}^{n} 2^{i-1}x_i.$$

In other words, the natural layout of Q_n arranges the vertices in lexicographic order. Nakano et al. showed that the sum of edge length of the natural layout of a hypercube is always the minimum among the ones of all linear layouts of the hypercube[NCM$^+$90].

It is important to analyze topological properties of hypercubes for developing parallel computation technology. Furthermore, when you attempt to encode the RGB data of a picture into binary strings and to transmit them through a communication channel with some transmission error, then you might need to examine the sum of edge length of the linear layout that corresponds to the encoding. The average change of the picture caused by one bit error in the transmission is expected to be in proportion to the sum of edge length. If we can use a larger alphabet than the binary one to encoding the RGB data, then it is probably suitable for the transmission to examine the sum of edge length of the torus graph that corresponds to the encoding.

For every pair of positive integers n and m, the n-dimensional m-torus graph T_m^n is defined as follows: The vertex-set of T_m^n is the set of all $\mathbb{Z}_m$-valued n-dimensional vectors, and the edge-set is the set of all of the pairs of vertices $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ such that they differ at exactly one index, say j, and either $v_j \equiv w_j + 1 \pmod m$ or $w_j \equiv v_j + 1 \pmod m$ holds. The phrase "torus graph" is the general notation for m-torus graphs for all m. The natural layout of T_m^n is the linear layout that arranges the vertices in lexicographic order. We shall prove that, for every positive integer n, the sum of edge length of the natural layout is the minimum among the ones of all linear layouts of T_3^n. In the proof, we basically use the technique of Nakano et al., and develop it into more elaborated one.

Furthermore, we shall completely characterize the set of linear layouts of T_3^n that have the maximum sum of edge length, where n is an arbitrary positive integer. The necessary and sufficient condition for a linear layout L of T_3^n to have the maximum sum of edge length is that, for each triple of vertices of T_3^n that differ at exactly one position of components, each of the three sets of vertices determined by L, $A = \{L^{-1}(1), L^{-1}(2), \ldots, L^{-1}(3^{n-1})\}$, $B = \{L^{-1}(3^{n-1}+1), L^{-1}(3^{n-1}+2), \ldots, L^{-1}(2\cdot 3^{n-1})\}$, and $C = \{L^{-1}(2\cdot 3^{n-1}+1), L^{-1}(2\cdot 3^{n-1}+2), \ldots, L^{-1}(3^n)\}$ contains a vertex in the triple.

2 Preliminaries

The size of a finite set S, namely, the number of elements of S, is denoted by $|S|$. If G is a simple undirected graph, then $V(G)$ and $E(G)$ denote the vertex-set and edge-set of G, respectively. For a graph G and a subset S of $V(G)$, $G[S]$ denotes the subgraph of G induced by S.

A linear layout of a graph G is a one-to-one mapping of the vertices of G into $\{1, 2, \ldots, |V(G)|\}$. Let L be a linear layout of a graph G of order n, where the order of G is the number of the vertices of G. For each edge $e = vw \in E(G)$, $L(e)$ denotes the closed interval $[\min\{L(v), L(w)\}, \max\{L(v), L(w)\}]$ on the number line. The sum of edge length of L is $\sum_{vw\in E(G)} |L(v) - L(w)|$. We write the sum of edge length of L as $SEL(L)$. For an integer $k \in \{1, 2, \ldots, n-1\}$, $C_G(L, k)$ denotes the set of edges $\{e \in E(G) \mid [k, k+1] \subseteq L(e)\}$. If the graph designated by the subscript G is clear in context, then we may leave out the subscript G from the notation $C_G(L, k)$. For an integer $k \in \{1, 2, \ldots, n-1\}$, $H_L(k)$ and $T_L(k)$ denote the following two sets of vertices of G, $\{L^{-1}(1), L^{-1}(2), \ldots, L^{-1}(k)\}$ and $\{L^{-1}(k+1), L^{-1}(k+2), \ldots, L^{-1}(|V(G)|)\}$, respectively. By definition, $C_G(L, k)$ is the set of all edges that connect a vertex in $H_L(k)$ and one in $T_L(k)$. The sum of edge length of L of G is equal to $\sum_{i=1}^{n-1} |C_G(L, i)|$, since each edge vw contributes $|L(v) - L(w)|$ to the sum $\sum_{i=1}^{n-1} |C_G(L, i)|$. Thus, we conclude the following proposition.

Proposition 1 *Let L be a linear layout of a graph G. Then, the following equation holds:*

$$SEL(L) = \sum_{vw \in E(G)} |L(v) - L(w)| = \sum_{i=1}^{n-1} |C_G(L,i)|.$$

Let n be a positive integer. The n-dimensional hypercube, or n-cube, is an undirected graph, denoted by Q_n. The vertex-set and edge-set of Q_n are $\{0,1\}^n$ and $\{((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)) \mid \sum_{i=1}^n |x_i - y_i| = 1\}$, respectively. Let $x = (x_1,x_2,\dots,x_n)$ be a vertex of Q_n. The number of 1's in the components of x, namely, $\sum_{i=1}^n x_i$, is said to be the weight of x, denoted by $w(x)$. The function that assigns the value $1+\sum_{i=1}^n 2^{i-1}x_i$ to each $x = (x_1,x_2,\dots,x_n) \in V(Q_n)$ is said to be the natural layout of Q_n, denoted by $N_n(x)$.

Let n be a positive integer and m an integer greater than 2. The n-dimensional m-torus graph T_m^n is defined in the previous section. The weight of a torus graph is defined in a similar manner to the case for hypercubes. Every element $z \in \mathbb{Z}_m$ can be uniquely represented by the sum of i unit elements, where $i \in \{0,1,2,\dots,m-1\}$. Function $\rho_m : \mathbb{Z}_m \to \mathbb{Z}$ is defined as

$$\rho_m\left(\underbrace{1+1+\cdots+1}_{i}\right) = i,$$

where $i \in \{0,1,2,\dots,m-1\}$. Let $x = (x_1,x_2,\dots,x_n)$ be a vertex of T_m^n. The sum $\sum_{i=1}^n \rho_m(x_i)$ is said to be the weight of x, denoted by $w(x)$. The function that assigns the value $1+\sum_{i=1}^n 3^{i-1}x_i$ to each $x = (x_1,x_2,\dots,x_n) \in V(T_m^n)$ is said to be the natural layout of T_m^n, denoted by $N_n(x)$. We may leave out subscript n from the notation $N_n(x)$, if n is clear in context.

3 The Minimality of the Sum of Edge Length of the Natural Layouts of Hypercubes

We shall review the point of the proof of the result of Nakano et al., which asserts the minimality of the sum of edge length of the natural layouts of hypercubes. We believe that there are two points in the proof. The first point is that they estimated the sum of edge length of L of Q_n by using the size of the subgraphs Q_n induce by $H_L(i)$ for $i \in \{1,2,\dots,2^n-1\}$. The second point is that the induction hypothesis that they employed is sufficiently strong. It was proved by induction on i that the size of the subgraph induced by $H_N(i)$ is maximum among the subgraphs induced by i vertices of Q_n.

It is sufficient to prove the minimality of the sum of edge length of the natural layout N of Q_n that, for each $i \in 1,2,\dots,2^n-1$, the subgraph of Q_n induced by the vertices $H_N(i) = \{N^{-1}(1), N^{-1}(2), \dots, N^{-1}(i)\}$ is maximum among the subgraphs of Q_n that consist of i vertices. From Proposition 1 in the previous section, the sum of edge length of a linear layout L of Q_n, namely $\sum_{vw\in E(G)} |L(v) - L(w)|$, is equal to $\sum_{i=1}^{n-1} |C_{Q_n}(L,i)|$.

Since Q_n is an n-regular graph, the sum of degrees of all vertices in $H_L(i)$ does not depend on L. We, therefore, have

$$|C_{Q_n}(L,i)| = ni - 2|E(Q_n[H_L(i)])|.$$

Thus, it holds that if, for each $i \in \{1, 2, \ldots, 2^n - 1\}$, $|E(Q_n[H_N(i)])|$ is the maximum size of a subgraph of Q_n induced by i vertices, then the sum of edge length of N is the maximum of the sum of edge length of a linear layout of Q_n.

Let $f_n(k)$ denote the number of edges of the subgraph of Q_n induced by $H_N(k)$, namely $f_n(k) = |E(Q_n[H_N(k)])|$. Since the following proposition holds, we shall therefore leave out the subscript n from f_n.

Proposition 2 *For any positive integers n_1, n_2, and k with $1 \leq k \leq 2^{n_1}$ and $n_1 < n_2$, $Q_{n_1}[H_N(k)]$ is isomorphic to $Q_{n_2}[H_N(k)]$.*

We have two recursive definition of $f(k)$. In one definition, a subgraph of Q_n is decomposed according to the first components of the vertices of the subgraph. We call such decomposition "the first index decomposition." In the other definition, a subgraph of Q_n is decomposed according to the j-th components of vertices of the subgraph, where j is the maximum integer such that there are two distinct j-th components of the vertices of the subgraph. We call such decomposition "the largest index decomposition." The recursive step of the former definition of $f(k)$ is

$$f(k) = f(\lfloor k/2 \rfloor) + f(\lfloor (k+1)/2 \rfloor) + \lfloor k/2 \rfloor, \tag{1}$$

and the recursive step of the latter one is

$$f(k) = f(2^{\lceil \log_2 (k/2) \rceil}) + f(k - 2^{\lceil \log_2 (k/2) \rceil}) + k - 2^{\lceil \log_2 (k/2) \rceil}. \tag{2}$$

Both of the bases of the two definition are the same as

$$f(0) = f(1) = 0. \tag{3}$$

The following proposition ensure that the expressions above define $f(k)$ correctly.

Proposition 3 *Let n, j, and k be positive integers with $j \leq n$ and $k < 2^n$. Let S_0 and S_1 denote $\{(x_1, x_2, \ldots, x_n) \in H_N(k) \mid x_j = 0\}$ and $\{(x_1, x_2, \ldots, x_n) \in H_N(k) \mid x_j = 1\}$, respectively. Then, induced subgraphs $Q_n[S_0]$ and $Q_n[S_1]$ are isomorphic to $Q_n[H_N(|S_0|)]$ and $Q_n[H_N(|S_1|)]$, respectively.*

Let $g(k)$ denote the function from non-negative integers to non-negative integers recursively defined by

$$g(0) = g(1) = 0,$$

and

$$g(k) = \max_{1 \leq i \leq k/2} \{g(i) + g(n-i) + i\}.$$

Then, it follows from the definitions of g that, for all $k \in \{0, 1, \ldots, 2^n\}$, the size of every subgraph of Q_n whose order is k is not greater than $g(k)$. The reason is as follows. Every non-null subgraph G of Q_n induced by S can be decomposed into three parts by partitioning S into S_0 and S_1. Two of the three parts are subgraphs induced by S_0 and S_1, respectively, and the other part is the set of edges that connect a vertex in S_0 and one in S_1. The partition of S into S_0 and S_1 is performed according to the j-th components of the vertices, where j is determined by a particular edge $e = vw$ of G. By definition, v and w have only one position j at which the components of v and w are different. We partition S into S_0 and S_1 so that the j-th component of each vertex in S_i is i for both $i \in \{0, 1\}$. Then, the number of edges that connect a vertex in S_0 and one in S_1 is the size of the smaller one in S_0 and S_1.

In fact, the function $f(k)$ coincides with $g(k)$ for every non-negative integer k. Nakano et al. proved the fact by induction on k, where the induction hypothesis is $f(k) \geq g(k)$. Notice that $f(k) \geq g(k)$ is equivalent to $f(k) = g(k)$ by definition. It holds that the size of any subgraph H of Q_n is not greater than $f(|V(H)|)$. This statement seems to be too weak for the induction hypothesis, although it is directly necessary for the proof.

Nakano et al. decomposed $Q_n[H_N(k)]$ by the first index decomposition, and employed expression (1) in the definition of $f(k)$. However, their selection of the decomposition method seems not to be essential to the proof, since we can complete the proof by employing the largest index decomposition[Mou96, HMJ99]. We omit the detailed story reluctantly.

4 The Minimality of the Sum of Edge Length of the Natural Layouts of Torus Graphs

In this section, we shall show briefly that the sum of edge length of the natural layout of a 3-torus graph is the maximum of the sum of edge length of all of the linear layouts of the 3-torus graph.

Theorem 4 *Let n be an integer greater than or equal to 2. Then, the following equation holds:*

$$SEL(N_n) = \min_{L \in \mathcal{L}_n} SEL(L),$$

where N_n denotes the natural layout of T_3^n and $\mathcal{L}_n$ denotes the set of all linear layouts of T_3^n.

The outline of the proof of Theorem 4 advances in almost parallel with the proof for hypercubes.

By an argument similar to the corresponding one in the previous section, in which Proposition 1 and the fact that T_3^n is $2n$-regular graph are used, we have that the following lemma is sufficient to prove Theorem 4. Notice that $H_N(i) = \{N^{-1}(1), N^{-1}(2), \ldots, N^{-1}(i)\}$ is defined in Section 2.

Lemma 5 *For each $i \in 1, 2, \ldots, 3^n - 1$,*

$$|E(T_3^n[H_N(i)])| = \max_H |E(T_3^n[H])|,$$

where H in the right-hand side of the equation above ranges over all of the subsets of $V(T_3^n) = \mathbb{Z}_3^n$ whose size is i.

The following lemmas can be proved easily. We omit the proofs reluctantly.

Lemma 6 *For any positive integers n_1, n_2, and k with $1 \leq k \leq 3^{n_1}$ and $n_1 < n_2$, $T_3^{n_1}[H_N(k)]$ is isomorphic to $T_3^{n_2}[H_N(k)]$.*

Lemma 7 *Let n be a positive integer. Let j be a positive integer less than or equal to n, and k a positive integer less than 3^n. Let S_0, S_1, and S_2 denote $\{(x_1, x_2, \ldots, x_n) \in H_N(k) \mid x_j = 0\}$, $\{(x_1, x_2, \ldots, x_n) \in H_N(k) \mid x_j = 1\}$, and $\{(x_1, x_2, \ldots, x_n) \in H_N(k) \mid x_j = 2\}$, respectively. Then, induced subgraphs $T_3^n[S_0]$, $T_3^n[S_1]$, and $T_3^n[S_2]$ are isomorphic to $T_3^n[H_N(|S_0|)]$, $T_3^n[H_N(|S_1|)]$, and $T_3^n[H_N(|S_2|)]$, respectively.*

Let $f(k)$ denote $|E(T_3^n[H_N(k)])|$ for arbitrary n greater than 1. By Lemma 6, the value $|E(T_3^n[H_N(k)])|$ does not depend on n. By observing that, for any distinct i and j, any two edges e_1 and e_2 of T_3^n that connect a vertex in $H_N(|S_i|)$ and one in $H_N(|S_j|)]$ are not adjacent, we have a recursive definition of $f(k)$ as follows:

$$f(0) = f(1) = 0, f(2) = 1, \tag{4}$$

and

$$f(k) = f\left(\left\lfloor \frac{k}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{k+1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{k+2}{3} \right\rfloor\right) + 2\left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k+1}{3} \right\rfloor. \tag{5}$$

The following lemma can be proved easily. We omit the proof.

Lemma 8 *Let a, b, c, and k be non-negative integers. Then,*

$$a = \lfloor k/3 \rfloor, \quad b = \lfloor (k+1)/3 \rfloor, \quad \textit{and} \quad c = \lfloor (k+2)/3 \rfloor,$$

if and only if

$$k = a + b + c \quad \textit{and} \quad a \leq b \leq c \leq a + 1.$$

Let $g(k)$ denote the function from non-negative integers to non-negative integers recursively defined by

$$g(0) = g(1) = 0, g(2) = 1, \tag{6}$$

and

$$g(k) = \max_{h,i,j}\{g(h) + g(i) + g(j) + 2h + i\}, \tag{7}$$

where h, i, and j range over the integers with $0 \leq h \leq i \leq j \leq k-1$ and $h + i + j = k$. By an argument similar to the corresponding one in the previous section, we have that, for all $k \in \{1, 2, \ldots, 3^n\}$, the size of every subgraph of T_3^n whose order is k is not greater than $g(k)$.

The following lemma is the core of the proof of Theorem 4.

Lemma 9 *Let k be a positive integer. Let h_0, h_1, h_2, i_0, i_1, i_2, j_0, j_1, and j_2 be non-negative integers. If*

$$h_0 \le h_1 \le h_2 \le h_0 + 1, \quad i_0 \le i_1 \le i_2 \le i_0 + 1, \quad j_0 \le j_1 \le j_2 \le j_0 + 1,$$

$$h_0 + h_1 + h_2 \le i_0 + i_1 + i_2 \le j_0 + j_1 + j_2 < k,$$

and

$$h_0 + h_1 + h_2 + i_0 + i_1 + i_2 + j_0 + j_1 + j_2 = k,$$

then there exist nine integers h'_0, h'_1, h'_2, i'_0, i'_1, i'_2, j'_0, j'_1, and j'_2 such that

$$(h'_0, h'_1, h'_2, i'_0, i'_1, i'_2, j'_0, j'_1, j'_2) \quad \textit{is a permutation of} \quad (h_0, h_1, h_2, i_0, i_1, i_2, j_0, j_1, j_2),$$

$$h'_0 \le i'_0 \le j'_0, \quad h'_1 \le i'_1 \le j'_1, \quad h'_2 \le i'_2 \le j'_2,$$

$$h'_0 + i'_0 + j'_0 \le h'_1 + i'_1 + j'_1 \le h'_2 + i'_2 + j'_2 \le h'_0 + i'_0 + j'_0 + 1,$$

and

$$4h_0 + 3h_1 + 2h_2 + 3i_0 + 2i_1 + i_2 + 2j_0 + j_1 \le 4h'_0 + 3h'_1 + 2h'_2 + 3i'_0 + 2i'_1 + i'_2 + 2j'_0 + j'_1.$$

Proof. Assume that the nine components satisfy the premise of the lemma. If $h_0 + h_1 + h_2 = i_0 + i_1 + i_2 = j_0 + j_1 + j_2$ holds, then set the nine variables with a prime mark as $h'_0 = h_0$, $i'_0 = h_1$, $j'_0 = h_2$, $h'_1 = i_0$, $i'_1 = i_1$, $j'_1 = i_2$, $h'_2 = j_0$, $i'_2 = j_1$, *and* $j'_2 = j_2$. Then, the conclusion of the lemma holds. In what follows, we, therefore, assume that

$$h_0 + h_1 + h_2 \ne i_0 + i_1 + i_2 \quad \text{or} \quad i_0 + i_1 + i_2 \ne j_0 + j_1 + j_2 \tag{8}$$

holds.

Given a 3×3 matrix A, we define four positive integers $x_0(A)$, $x_1(A)$, $y_0(A)$, and $y_1(A)$ as follows:

$$x_0(A) = \max\{i \in \{1,2,3\} \mid \sum_{j=1}^{3} A(i,j) = \min\{\sum_{j=1}^{3} A(k,j) \mid k \in \{1,2,3\}\},$$

$$x_1(A) = \min\{i \in \{1,2,3\} \mid \sum_{j=1}^{3} A(i,j) = \max\{\sum_{j=1}^{3} A(k,j) \mid k \in \{1,2,3\}\},$$

$$y_1(A) = \max\{j \in \{1,2,3\} \mid A(x_0(A), j) < A(x_1(A), j)\}, \quad \text{and}$$

$$y_0(A) = \min\{j \in \{1,2,3\} \mid A(x_1(A), j) = A(x_1(A), y_1(A))\}.$$

Moreover, for two pairs of integers (i,j) and (i',j') whose components are all belong to $\{1,2,3\}$, let $A[(i,j),(i',j')]$ denote the 3×3 matrix obtained by swapping the (i,j) and (i',j') components of A, that is,

$$A[(i,j),(i',j')](i,j) = A(i',j'), A[(i,j),(i',j')](i',j') = A(i,j),$$

and if $(x,y) \neq (i,j)$ and $(x,y) \neq (i',j')$ then

$$A[(i,j),(i',j')](x,y) = A(x,y).$$

Furthermore, $P(A)$ denote the following property of 3×3 matrix A:

$$A(i,1) \leq A(i,2) \leq A(i,3) \quad \text{for each } i \in \{1,2,3\} \quad \text{and}$$

$$A(1,1) + A(1,2) + A(1,3) \leq A(2,1) + A(2,2) + A(2,3) \leq A(3,1) + A(3,2) + A(3,3).$$

We define functions λ and μ both defined on 3×3 matrices as

$$\lambda(A) = A(3,1) + A(3,2) + A(3,3) - A(1,1) - A(1,2) - A(1,3)$$

and

$$\mu(A) = 4A(1,1) + 3(A(1,2) + A(2,1)) + 2(A(1,3) + A(2,2) + A(3,1)) + A(2,3) + A(3,2).$$

Let M denote the 3×3 matrix defined by

$$M(1,1) = h_0, \quad M(1,2) = i_0, \quad M(1,3) = j_0, \quad M(2,1) = h_1, \quad M(2,2) = i_1,$$

$$M(2,3) = j_1, \quad M(3,1) = h_2, \quad M(3,2) = i_2, \quad \text{and} \quad M(3,3) = j_2.$$

Notice that the premise of the lemma implies that $P(M)$ holds.

At this point, we are ready to describe the permutation of the components of M that brings values assigned to the nine variables with a prime mark. We define matrix M' as follows. If $\lambda(M) \leq 1$, then $M' = M$. Otherwise, let M_1 denote $M[(x_0(M), y_1(M)), (x_1(M), y_0(M))]$. If $\lambda(M) = 2$, then $M' = M_1$. Otherwise, i.e. $\lambda(M) = 3$, then, $M' = M_1[(x_0(M_1), y_1(M_1)), (x_1(M_1), y_0(M_1))]$. In general, for any 3×3 matrix A with $P(A)$ and $\lambda(A) \geq 2$, we observe that $P(A[(x_0(A), y_1(A)), (x_1(A), y_0(A))])$, $\lambda(A[(x_0(A), y_1(A)), (x_1(A), y_0(A))]) \leq \lambda(A) - 1$, and $\mu(A[(x_0(A), y_1(A)), (x_1(A), y_0(A))]) \geq \mu(A)$ hold. Thus, we have $P(M')$, $\lambda(M') \leq 1$, and $\mu(M) \leq \mu(M')$ in any case. The following setting of the nine variables with a prime mark satisfies the conclusion of the lemma:

$$h_0' = M'(1,1), \quad i_0' = M'(1,2), \quad j_0' = M'(1,3), \quad h_1' = M'(2,1), \quad i_1' = M'(2,2),$$

$$j_1' = M'(2,3), h_2' = M'(3,1), \quad i_2' = M'(3,2), \quad \text{and} \quad j_2' = M'(3,3).$$

We have thus proved the lemma.

□

The following lemma is sufficient to prove Lemma 5. Thus, we conclude that Theorem 4

Lemma 10 *For every k, $f(k) = g(k)$.*

Proof. We prove the lemma by induction on k. The base clearly holds. Let k be an integer greater than 2. Assume that $f(l) = g(l)$ holds for all non-negative integers l less than k. Let h, i, and j be integers such that $0 \leq h \leq i \leq j \leq k-1$, $h+i+j = k$, and $g(k) = g(h) + g(i) + g(j) + 2h + i = f(h) + f(i) + f(j) + 2h + i$. For $l \in \{0,1,2\}$, let h_l, i_l, and j_l denote integers $\lfloor (h+l)/3 \rfloor$, $\lfloor (i+l)/3 \rfloor$, and $\lfloor (j+l)/3 \rfloor$, respectively. Then, we have the following sequence of equations by the definitions of f and g, the induction hypothesis, and Lemma 9.

$$\begin{aligned}
g(k) &= f(h_0) + f(i_0) + f(j_0) + 2h_0 + i_0 + f(h_1) + f(i_1) + f(j_1) + 2h_1 + i_1 \\
&\quad + f(h_2) + f(i_2) + f(j_2) + 2h_2 + i_2 + 2h_0 + 2i_0 + 2j_0 + h_1 + i_1 + j_1 \\
&= g(h_0) + g(i_0) + g(j_0) + g(h_1) + g(i_1) + g(j_1) + g(h_2) + g(i_2) + g(j_2) \\
&\quad + 4h_0 + 3h_1 + 2h_2 + 3i_0 + 2i_1 + i_2 + 2j_0 + j_1 \\
&\leq g(h'_0) + g(i'_0) + g(j'_0) + g(h'_1) + g(i'_1) + g(j'_1) + g(h'_2) + g(i'_2) + g(j'_2) \\
&\quad + 4h'_0 + 3h'_1 + 2h'_2 + 3i'_0 + 2i'_1 + i'_2 + 2j'_0 + j'_1 \\
&\leq g(h'_0 + i'_0 + j'_0) + g(h'_1 + i'_1 + j'_1) + g(h'_2 + i'_2 + j'_2) \\
&\quad + 2(h'_0 + i'_0 + j'_0) + h'_1 + i'_1 + j'_1 \\
&= f(h'_0 + i'_0 + j'_0) + f(h'_1 + i'_1 + j'_1) \\
&\quad + f(h'_2 + i'_2 + j'_2) + 2(h'_0 + i'_0 + j'_0) + h'_1 + i'_1 + j'_1 \\
&= f(h'_0 + i'_0 + j'_0 + h'_1 + i'_1 + j'_1 + h'_2 + i'_2 + j'_2) = f(k).
\end{aligned}$$

We have thus proved the lemma. □

5 Linear Layouts of Torus Graphs that Maximize the Sum of Edge Length

For a vertex $x = (x_1, x_2, \ldots, x_n)$ of T_3^n and an integer $j \in \{1, 2, \ldots, n\}$, $\Delta(x, j)$ denotes the triangle subgraph of T_3^n induced by $\{(y_1, y_2, \ldots, y_n) \in \mathbb{Z}_3^n \mid y_1 = x_1, y_2 = x_2, \ldots, y_{j-1} = x_{j-1}, y_{j+1} = x_{j+1}, \ldots, y_n = x_n\}$. Notice that the edge-set of T_3^n can be partitioned into $n3^{n-1}$ such triangles.

Theorem 11 *Let n be a positive integer. A linear layout L of T_3^n maximizes the sum of edge length if and only if, for all $x \in V(T_3^n)$ and all $j \in \{1, 2, \ldots, n\}$,*

$$V(\Delta(x, j)) \cap \{L^{-1}(1), L^{-1}(2), \ldots, L^{-1}(3^{n-1})\} \neq \emptyset, \tag{9}$$

$$V(\Delta(x, j)) \cap \{L^{-1}(3^{n-1}+1), L^{-1}(3^{n-1}+2), \ldots, L^{-1}(2 \cdot 3^{n-1})\} \neq \emptyset, \tag{10}$$

and

$$V(\Delta(x, j)) \cap \{L^{-1}(2 \cdot 3^{n-1}+1), L^{-1}(2 \cdot 3^{n-1}+2), \ldots, L^{-1}(3^n)\} \neq \emptyset. \tag{11}$$

Proof. Let X_L, Y_L, and Z_L denote $\{L^{-1}(1), L^{-1}(2), \ldots, L^{-1}(3^{n-1})\}$, $\{L^{-1}(3^{n-1}+1), L^{-1}(3^{n-1}+2), \ldots, L^{-1}(2\cdot 3^{n-1})\}$, and $\{L^{-1}(2\cdot 3^{n-1}+1), L^{-1}(2\cdot 3^{n-1}+2), \ldots, L^{-1}(3^n)\}$, respectively. We show that L maximizes $|C(L,k)|$ for all $k \in \{1,2,\ldots,3^n-1\}$ if and only if each of X_L, Y_L, and Z_L intersects all of the triangles.

Let k be an integer in $\{1,2,\ldots,3^{n-1}\}$. We readily have that $|C(L,k)|$ is not greater than the sum of degree of the vertices in $\{L^{-1}(1), L^{-1}(2), \ldots, L^{-1}(k)\}$. We also have that all of the edges incident to a vertex in $\{L^{-1}(1), L^{-1}(2), \ldots, L^{-1}(k)\}$ contribute 1 to $|C(L,k)|$ if and only if, for every triangle Δ, X_L contains exactly one vertex of Δ. We therefore have that L maximizes $|C(L,k)|$ for all k in $\{1,2,\ldots,3^{n-1}\}$ if and only if, for every triangle Δ, X_L contains exactly one vertex of Δ. In a symmetric manner, we also have that L maximizes $|C(L,k)|$ for all k in $\{2\cdot 3^{n-1}, 2\cdot 3^{n-1}+1, 2\cdot 3^{n-1}+2, \ldots, 3^n-1\}$ if and only if, for every triangle Δ, Z_L contains exactly one vertex of Δ.

We readily have that any triangle contributes at most 2 to $|C(L,k)|$. We also have that a triangle contributes 2 to $|C(L,k)|$ for every $k \in Y_L$ if the triangle intersects both of X_L and Z_L.

Thus, we conclude that the lemma holds. □

For example, the three conditions below imply conditions (9), (10), and (11):

$$(\forall x \in X_L)(\forall y \in X_L)(w(x) \equiv w(y) \pmod 3), \tag{12}$$

$$(\forall x \in Y_L)(\forall y \in Y_L)(w(x) \equiv w(y) \pmod 3), \tag{13}$$

and

$$(\forall x \in Z_L)(\forall y \in Z_L)(w(x) \equiv w(y) \pmod 3). \tag{14}$$

However, we can find a large number of linear layouts that maximizes the sum of edge length and that does not satisfy the three conditions above, since if L maximizes the sum of edge length, then $L \circ \varphi_j[a,b]$ also maximizes the sum of edge length for every $j \in \{1,2,\ldots,n\}$ and for every a and b in $\mathbb{Z}_3$ with $a \neq b$, where $L \circ \varphi_j[a,b](v) = L(\varphi_j[a,b](v))$ and function $\varphi_j[a,b]$ is defined as follows: for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_3^n$,

$$\varphi_j[a,b](x) = (x_1,\ldots,x_{j-1},b,x_{j+1},\ldots,x_n) \quad \text{if } x_j = a,$$

$$\varphi_j[a,b](x) = (x_1,\ldots,x_{j-1},a,x_{j+1},\ldots,x_n) \quad \text{if } x_j = b,$$

$$\text{and} \quad \varphi_j[a,b](x) = x \quad \text{otherwise.}$$

For example, let L denote the linear layout of T_3^2 defined by

$$L((0,0)) = 2,\ \ L((1,0)) = 4,\ \ L((2,0)) = 9,\ \ L((0,1)) = 5,\ \ L((1,1)) = 8,$$
$$L((2,1)) = 1,\ \ L((0,2)) = 7,\ \ L((1,2)) = 3,\ \ L((2,2)) = 6.$$

Then, $L' = L \circ \varphi_2[0,2]$ is described by

$$L'((0,0)) = 7,\ \ L'((1,0)) = 3,\ \ L'((2,0)) = 6,\ \ L'((0,1)) = 5,\ \ L'((1,1)) = 8,$$
$$L'((2,1)) = 1,\ \ L'((0,2)) = 2,\ \ L'((1,2)) = 4,\ \ L'((2,2)) = 9.$$

Linear layout L maximizes the sum of edge length, since L satisfies the three conditions (12), (13), and (14). Furthermore, linear layout L' also maximize the sum of edge length, although L' satisfies none of the conditions (12), (13), and (14).

6 Concluding Remarks

Let G denote the subgraph of T_5^2 induced by $\{(0,0),(1,0),(1,1),(0,1)\}$. Since $|E(T_5^2[H_N(4)])| = |E(T_5^2[\{(0,0),(1,0),(2,0),(3,0)\}])| = 3$ and $|E(T_5^2[G])| = 4$, we cannot replace 3's in Lemma 5 with 5's. However, we conjecture that the 3's in Lemma 5 can be replaced with 4's.

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