Locally connected tree-like invariant continua under Kleinian groups

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This note deals with invariant continua under Kleinian groups. Here, a continuum is a compact connected subset of the Riemann sphere $S^2$, and a Kleinian group is a discrete subgroup of Möbius transformations of $S^2$.

Let $L$ be a continuum on $S^2$. By definition, $L$ is locally connected at a point $y \in L$ if, for any neighborhood $U$ of $y$, there exists a smaller neighborhood $V \subset U$ such that $L \cap V$ is connected. We say that $L$ is locally connected if it is locally connected at any point $y \in L$. We say that a continuum $L$ is tree-like if the complement $S^2 - L$ is connected and if the interior of $L$ is empty. Locally connected, tree-like continua are characterized by the following property. See [1, Section 10].

**Proposition.** Let $L \subset S^2$ be a locally connected, tree-like continuum. Then, for any points $x$ and $y$ in $L$, there exists a unique arc $\overline{xy}$ in $L$ that connects $x$ and $y$.

A point $\xi$ on a locally connected, tree-like continuum $L$ is called an endpoint if there exists no arc $\lambda$ in $L$ such that $\xi$ is an interior point of $\lambda$ with respect to the relative topology on $\lambda$. This is equivalent to saying that $L - \{\xi\}$ is connected.

Let $\Gamma$ be a Kleinian group. A loxodromic fixed point of $\Gamma$ is a point that is fixed by a loxodromic element of $\Gamma$. The limit set $\Lambda(\Gamma)$ for $\Gamma$ is the closure of the set of all loxodromic fixed points of $\Gamma$. We say that $\xi \in S^2$ is a point of approximation (or a conical limit point) for $\Gamma$ if there exists a sequence of elements $\gamma_n \in \Gamma$ and distinct points $x$ and $y$ on $S^2$ such that $\gamma_n(\xi)$ converge to $x$ and $\gamma_n(z)$ converge to $y$ locally uniformly for $z \in S^2 - \{\xi\}$. See [3, p.22]. Points of approximation belong to the limit set. If all the points in the limit set $\Lambda(\Gamma)$ are points of approximation, then the Kleinian group $\Gamma$ is convex cocompact. A Schottky group is a convex cocompact, free Kleinian group.

Abikoff [1, Lemma 1] proved that any loxodromic fixed point of a Kleinian group $\Gamma$ with the locally connected, tree-like limit set $\Lambda(\Gamma)$ is its endpoint. In this note, we extend this result in the following form.

**Theorem.** Let $\Gamma$ be a Kleinian group and $L$ a locally connected, tree-like continuum that is invariant under $\Gamma$. Then any point of approximation for $\Gamma$ is an endpoint.
Proof. Suppose that a point $\xi$ of approximation for $\Gamma$ is not an endpoint of $L$. Then there exists an arc $\lambda$ in $L$ such that $\xi$ is in its interior. Let $z_1$ and $z_2$ be the endpoints of $\lambda$. Since $\xi$ is a point of approximation, there exists a sequence of elements $\gamma_n \in \Gamma$ and distinct points $x$ and $y$ on $S^2$ such that $\gamma_n(\xi)$ converge to $x$ and $\gamma_n(z_i)$ converge to $y$ for $i = 1, 2$. The $\Gamma$-invariance of $L$ implies that $\gamma_n(z_i) = \gamma_n(z_i)\gamma_n(\xi)$ lies in $L$ as well as $y$ belongs to $L$.

Let $V$ be an open neighborhood of $y$ such that $x$ is not contained in the closure of $V$ and that $L \cap V$ is connected. For a sufficiently large $n$, $\gamma_n(z_i)$ is contained in $V$ but $\gamma_n(\xi)$ is not. Since $\gamma_n(z_i)$ can be connected with $y$ in $L \cap V$, we take an arc $\overline{y\gamma_n(z_i)}$ there. Then $\overline{y\gamma_n(z_i)\gamma_n(\xi)}$ for $i = 1, 2$ are distinct arcs in $L$ connecting $y$ and $\gamma_n(\xi)$. However, this contradicts the uniqueness of the arc in $L$ as in the previous proposition. $\square$

**Corollary.** Let $\Gamma$ be a convex cocompact Kleinian group and $L$ a locally connected, tree-like continuum that is invariant under $\Gamma$. Then $L - \Lambda(\Gamma)$ is connected.

Maskit [2] considered this problem for the case that $L$ is the limit set for a degenerate Kleinian group $G$ and $\Gamma$ is a Schottky subgroup of $G$. His arguments did not involve any assumption on local connectivity for $L$, however, a certain property for $L$ seems to have been necessary to complete the proof. It is conjectured that the limit set for a degenerate Kleinian group is locally connected (cf. [1]), however, only partial solutions have so far been obtained.

**References**


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