<table>
<thead>
<tr>
<th>Title</th>
<th>The Picard group, the figure-eight knot group and Jorgensen groups (Hyperbolic Spaces and Discrete Groups)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Sato, Hiroki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2001年秋 1223: 37-42</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41339">http://hdl.handle.net/2433/41339</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
The Picard group, the figure-eight knot group and

Jørgensen groups

Hiroki Sato

佐藤 昌樹*

Department of Mathematics, Faculty of Science

Shizuoka University

0. Introduction.

In this paper we will state that the Picard group $G_P$ and the figure-eight knot group $G_F$ are two-generator groups and Jørgensen groups. Furthermore we will describe a complete set of relations for $G_P$ as a two-generator group. The detail will appear elsewhere.

1. The Picard group.

**Definition 1.1.** The group

$$G_P := \left\{ \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{Z} + i\mathbb{Z}, ad - bc = 1 \right\}$$

is the Picard group.

*Partly supported by the Grants-in-Aid for Scientific and Co-operative Research, the Ministry of Education, Science, Sports, Culture and Technology, Japan

2000 Mathematics Subject Classification. Primary 32G15; Secondary 20H10, 30F40.
THEOREM A (Magnus [7]) The Picard group $G_P$ is generated by the following four Möbius transformations $S_m, T_m, U_m$ and $V_m$ with corresponding matrices

$$S_m = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad T_m = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad U_m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V_m = \begin{pmatrix} i & -1 \\ 0 & -i \end{pmatrix}.$$ 

THEOREM B (Johnson-Weiss [3]) The Picard group $G_P$ is generated by the following three matrices:

$$B_j = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C_j = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad S_j = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$ 


2. Jørgensen groups.

THEOREM C (Jørgensen [4]). If $\langle A, B \rangle$ is a non-elementary discrete subgroup of Möb, then

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$ 

The lower bound 1 is best possible.

DEFINITION 2.1. Let $A$ and $B$ be Möbius transformations. The Jørgensen number $J(A, B)$ is

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2|.$$ 

DEFINITION 2.2. A non-elementary two-generator discrete subgroup $G$ of Möb is a Jørgensen group if $G$ has generators $A$ and $B$ with $J(A, B) = 1$.

THEOREM D (Jørgensen-Kiikka [5]). Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$, that is, a Jørgensen group. Then $A$ is elliptic of order at least seven or $A$ is parabolic.
Here we only consider the case where $A$ is parabolic, that is, Jørgensen groups of parabolic type. Namely, we consider two-generator groups $G_{ik,\sigma} = \langle A, B_{ik,\sigma} \rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{ik,\sigma} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $k \in \mathbb{R}$ and $\sigma \in \mathbb{C} \setminus \{0\}$.

Let $C$ be the following cylinder: $C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbb{R}\}$.

**Theorem E (Sato [9]).** Every Jørgensen group of type $G_{ik,\sigma}$ lies on the cylinder $C$.

By Theorem E we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma} \rangle$ with $\mu = ik$ ($k \in \mathbb{R}$) and $\sigma = -ie^{i\theta}$ ($0 \leq \theta < 2\pi$). For simplicity we set $B_{k,\theta} := B_{ik,\sigma}$ and $G_{k,\theta} = \langle A, B_{k,\sigma} \rangle$ for $\sigma = -ie^{i\theta}$.

We can see that it suffices to consider the case of ($0 \leq \theta \leq \pi/2$) and $k \geq 0$.

**Theorem F (Jørgensen-Lascurain-Pignataro [6], Sato [9], Sato-Yamada [12]).**

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{k,\theta} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

and let $G_{k,\theta} = \langle A, B_{k,\theta} \rangle$ be the group generated by $A$ and $B_{k,\theta}$, where $k \in \mathbb{R}$ and $\sigma \in \mathbb{C} \setminus \{0\}$. Then

(i) $G_{1/2,\pi/2}$ is a Jørgensen group.

(ii) $G_{\sqrt{3}/2,\pi/6}$ is a Jørgensen group.

See Sato [9,10] for Jørgensen groups of parabolic type.
3. Theorems.

In this section we will state main theorems. We can prove the theorems by using Poincaré's polyhedron theorem (cf. Maskit [8]).

**Theorem 1 (Sato [9,11])**

(i) The Picard group $G_{P}$ is conjugate to $G_{1/2,\pi/2}$, that is, $G_{P} = RG_{1/2,\pi/2}R^{-1}$, where

$$R = \begin{pmatrix} 1 & i/2 \\ 0 & 1 \end{pmatrix}$$

(ii) The following relations form a complete set of relations for $G_{P}$:

$$(B^{-1}ABA^{2}BAB^{-1}A^{2}B^{-1}ABA^{2}BAB^{-1}AB)^{2} = 1$$

$$(B^{-1}ABA^{2}BAB^{-1}A^{2}B^{-1}ABA)^{2} = 1$$

$$(AB^{-1}ABA^{2}BAB^{-1}A^{2}B^{-1}ABA^{2}BAB^{-1}AB)^{2} = 1$$

$$(AB^{-1}ABA^{2}BAB^{-1}A^{2}B^{-1}ABA^{2}BAB^{-1}AB)^{2} = 1$$

$$(B^{-1}ABA)^{3} = 1$$

$$(AB^{-1}ABA)^{2} = 1$$

$$(AB^{-1}ABA^{2}B^{-1}ABA^{2}BAB^{-1}A^{2}B^{-1}ABA^{2}BAB^{-1}AB)^{2} = 1$$

$$(AB^{-1}ABA^{2}B^{-1}ABA^{2}BAB^{-1}A^{2}B^{-1}ABA^{2}BAB^{-1}AB)^{3} = 1,$$

where $B = RB_{1/2,\pi/2}R^{-1}$.

**Corollary.** The Picard group is a two-generator group and a Jørgensen group.

**Theorem 2 (Sato [9,11]).** The figure-eight knot group $G_{F}$ is conjugate to $G_{\sqrt{3}/2,\pi/6}$, that is, $G_{F} = RG_{\sqrt{3}/2,\pi/6}R^{-1}$, where
\[ R = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \]

(ii) *The following relation forms a complete set of relations for* \( G_F \):

\[ ABA^{-1}B^{-1}A = BA^{-1}B^{-1}ABA, \]

where \( B = RB_{\sqrt{3}/2, \pi/6}R^{-1} \).

**Corollary.** *The figure-eight knot group is a two-generator group and a Jørgensen group.*

**References**


Department of Mathematics
Faculty of Science
Shizuoka University
Ohya Shizuoka 422-8529
Japan
e-mail:smhsato@ipc.shizuoka.ac.jp