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Notes on discrete subgroups of $PU(1,2;\mathbb{C})$
\hspace{1cm} with Heisenberg translations III

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神谷茂保（岡山理大）

In a previous paper [8] we have seen that under some conditions Parker’s theorem yields the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. In this paper we give a new stable basin region and show the same result as in [8] without the assumption on $r$. This is a joint work with John R. Parker.

1. First we recall some definitions and notation. Let $\mathbb{C}$ be the field of complex numbers. Let $V = V^{1,2}(\mathbb{C})$ denote the vector space $\mathbb{C}^{3}$, together with the unitary structure defined by the Hermitian form

$$\bar{\Phi}(z^{*}, w^{*}) = -(z_{0}^{*}w_{1}^{*} + \overline{z_{1}^{*}}w_{0}^{*}) + \overline{z_{2}^{*}}w_{2}^{*}$$

for $z^{*} = (z_{0}, z_{1}, z_{2}), w^{*} = (w_{0}, w_{1}, w_{2})$ in $V$. An automorphism $g$ of $V$, that is a linear bijection such that $\bar{\Phi}(g(z^{*}), g(w^{*})) = \bar{\Phi}(z^{*}, w^{*})$ for $z^{*}, w^{*}$ in $V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,2;\mathbb{C})$. Let $V_{0} = \{w^{*} \in V| \bar{\Phi}(w^{*}, w^{*}) = 0\}$ and $V_{-} = \{w^{*} \in V| \bar{\Phi}(w^{*}, w^{*}) < 0\}$. It is clear that $V_{0}$ and $V_{-}$ are invariant under $U(1,2;\mathbb{C})$. We denote $U(1,2;\mathbb{C})/(\text{center})$ by $PU(1,2;\mathbb{C})$. Set $V^{*} = V_{-} \cup V_{0} \cup \{0\}$. Let $\pi : V^{*} \rightarrow \pi(V^{*})$ be the projection map defined by $\pi(w_{0}^{*}, w_{1}^{*}, w_{2}^{*}) = (w_{1}, w_{2})$, where $w_{1} = w_{1}^{*}/w_{0}^{*}$ and $w_{2} = w_{2}^{*}/w_{0}^{*}$. We write $\infty$ for $\pi(0,1,0)$. We may identify $\pi(V_{-})$ with the Siegel domain

$$H^{2} = \{w = (w_{1}, w_{2}) \in \mathbb{C}^{2} \mid \Re(w_{1}) > \frac{1}{2}|w_{2}|^{2}\}.$$

We can regard an element of $PU(1,2;\mathbb{C})$ as a transformation acting on $H^{2}$ and its boundary $\partial H^{2}$ (see [6]). Denote $H^{2} \cup \partial H^{2}$ by $\overline{H^{2}}$. We define a new coordinate system in $\overline{H^{2}} - \{\infty\}$. Our convention slightly differs from Basmajian-Miner [1] and Parker [8]. The $H$ – coordinates of a point $(w_{1}, w_{2}) \in \overline{H^{2}} - \{\infty\}$ are defined by $(k, t, w_{2})_{H} \in (\mathbb{R}^{+} \cup \{0\}) \times \mathbb{R} \times \mathbb{C}$ such that $k = \Re(w_{1}) - \frac{1}{2}|w_{2}|^{2}$ and $t = \Im(w_{1})$. For simplicity, we write $(t_{1}, w_{1})_{H}$ for $(0, t_{1}, w_{1})_{H}$. The Cygan metric $\rho(p, q)$ for $p = (k_{1}, t_{1}, w_{1})_{H}$ and $q = (k_{2}, t_{2}, W_{2})_{H}$ is given by

$$\rho(p, q) = |\frac{1}{2}[|W' - w'|^{2} + |k_{2} - k_{1}|] + i(t_{1} - t_{2} + \Im(\overline{w}W'))|^{\frac{1}{2}}.$$

We note that the Cygan metric $\rho$ is a generalization of the Heisenberg metric $\delta$ in $\partial H^{2}$.
Let $f = (a_{ij})_{1 \leq i,j \leq 3}$ be an element of $PU(1, 2; \mathbb{C})$ with $f(\infty) \neq \infty$. We define the isometric sphere $I_f$ of $f$ by

$$I_f = \{ w = (w_1, w_2) \in \overline{H^2} \mid |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))| \},$$

where $Q = (0, 1, 0)$, $W = (1, w_1, w_2)$ in $V^*$ (see [4]). It follows that the isometric sphere $I_f$ is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k, t, w')_H \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R} \times \mathbb{C} \mid \rho(z, f^{-1}(\infty)) = \frac{1}{|a_{12}|} \right\}.$$

2. We shall give a modified version of the stable basin theorem in [8]. Let

$$B_r = \{ z \in \partial H^2 \mid \delta(z, 0) < r \},$$

and let $\overline{B}_r = \partial H^2 - \overline{B}_r$. Given $r$ and $s$ with $r < s$, the pair of open sets $(B_r, \overline{B}_s)$ is said to be stable with respect to a set $S$ of elements in $PU(1, 2; \mathbb{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_s.$$

A loxodromic element $f$ has a unique complex dilation factor $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r, \epsilon)$ denote the family of loxodromic elements $f$ with fixed points in $B_r$ and $\overline{B}_{1/r'}$, and satisfying $|\lambda(f) - 1| < \epsilon$. For positive real numbers $r$ and $r'$ with $r < 1/\sqrt{3}$ and $r' < 1$, we define $\epsilon(r, r')$ by

$$\epsilon(r, r') = \sup\{|\lambda(f) - 1|\},$$

where $|\lambda(f) - 1|$ satisfies the inequality

$$|\lambda(f) - 1| < \sqrt{1 + \left(\frac{1 - (3 + |\lambda(f) - 1|)r^2}{1 - 2r^2}\right)^2 \left(\frac{1 - 3r^2}{1 - r^2}\right)^2 \left(\frac{r'}{r}\right)^2 - 1.} \tag{2.2}$$

A triple of non-negative numbers $(r, r', \epsilon)$ is said to be a basin point provided that $r < 1/\sqrt{3}$, $r' < 1$ and $\epsilon < \epsilon(r, r')$. In particular, if $r' \leq r$, we call $(r, r', \epsilon)$ a stable basin point. Call the set of all such points the stable basin region. For simplicity, we abbreviate $(r, r, \epsilon)$ to $(r, \epsilon)$. Figure 1 shows our new stable basin region, which contains regions in [1] and [8]. Some stable basin points are tabulated in Table 2.

Exactly the same arguments except for using the following Lemma 2.1 instead of Proposition 3.3 in [1] shows our new stable basin theorem.

**Lemma 2.1.** Let $b, c > 0$ be given. If $f$ is a complex dilation and its complex dilation factor satisfies $|\lambda(f) - 1| \leq \sqrt{1 + (b/c)^2} - 1$, then $f(p) \in B_b(p)$ for $p \in \overline{B}_c$. 


Theorem 2.2 (cf. [8; Stable Basin Theorem]). Given positive real numbers \( r \) and \( r' \) with \( r < 1/\sqrt{3} \) and \( r' < 1 \), the pair of open sets \((B_{r}, B_{r}')\) is stable with respect to the family \( S(r, \varepsilon(r, r')) \), where \( \varepsilon(r, r') \) is given by (2.2).

Remark 2.3. By arguing as in Corollary 6.14 in [1], we may find the boundary of the stable basin region by equating both sides of inequality (2.2) and solving for \(|\lambda(f) - 1|\) in terms of \( r \). If we use Basmajian and Miner's inequality (6.2) in [1], this involves solving a polynomial of degree 6. Using our inequality (2.2), we have

\[
|\lambda(f) - 1| < \frac{a_{2}a_{1}\sqrt{a_{3}^{4}b_{1} + a_{2}^{2}a_{1}^{2} + 2a_{3}^{3}r^{2}b_{1}} - a_{2}^{2}a_{1}^{2} - a_{3}^{3}b_{1}r^{2}}{a_{2}^{2}a_{1}^{2} - a_{3}^{2}b_{1}r^{4}},
\]

where \( a_{j} = 1 - jr^{2} \) and for \( j = 1, 2, 3 \) and \( b_{1} = (r'/r)^{2} \).

3. We begin with recalling Parker's theorem on the discreteness of subgroups of \( PU(1, 2; \mathbb{C}) \).

Theorem 3.1 ([9; Theorem 2.1]). Let \( g \) be a Heisenberg translation with the form

\[
g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & a \\ a & 0 & 1 \end{pmatrix},
\]

where \( \text{Re}(s) = \frac{1}{2}|a|^{2} \). Let \( f \) be any element of \( PU(1, 2; \mathbb{C}) \) with isometric sphere of radius \( R_{f} \). If

\[
R_{f}^{2} > \delta(gf^{-1}(\infty), f^{-1}(\infty))\delta(gf(\infty), f(\infty)) + 2|a|^{2},
\]

then the group \( \langle f, g \rangle \) generated by \( f \) and \( g \) is not discrete.

In Theorem 4.5 of [8] we have shown that if \( r < 0.484 \), then Theorem 3.1 leads to the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. By using a more precise estimate on the Heisenberg distance between fixed points of \( f \) in terms of \( R_{f} \) and \( \lambda(f) \), we have the following same result without the assumption on \( r \).

Theorem 3.2. Fix a stable basin point \((r, \varepsilon)\). Let \( g \) be the same element as in Theorem 3.1. Let \( f \) be a loxodromic element with fixed point 0 and \( g \), and satisfying \(|\lambda(f) - 1| < \varepsilon \). If \( \delta(0, q) > \frac{\delta(0, g(0))}{R_{f}}(1 + r^{2} + \sqrt{1 + r^{2}}) \), then the group \( \langle f, g \rangle \) generated by \( f \) and \( g \) is not discrete.

To prove our theorem, we need the following lemmas.

Lemma 3.3. Let \( f \) be a loxodromic element with fixed points 0 and \( q \), satisfying \(|\lambda(f) - 1| < \varepsilon \). Then

\[
\left( \frac{\delta(0, q)}{R_{f}} \right)^{2} \leq \frac{2\varepsilon - \varepsilon^{2}}{1 - \varepsilon}.
\]
Lemma 3.4. For a stable basin point \((r, \epsilon)\),

\[
\frac{1 + r^2 + \sqrt{1 + r^2}}{r^2} > \left( \frac{2\epsilon - \epsilon^2}{1 - \epsilon} \right)^{\frac{1}{2}} \left( 2 + \left( 8 + \frac{M(\epsilon)}{2} \right)^{\frac{1}{3}} \right),
\]

where \(M(\epsilon) = (1 + \epsilon)^{\frac{1}{4}} + (1 + \epsilon)^{-\frac{1}{4}}\).

Figure 1. Graph of \(\epsilon(r, r)\)

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<th>(r)</th>
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Table 2
References

8. S. Kamiya, On discrete subgroups of $PU(1, 2; C)$ with Heisenberg translations, J. London Math. Soc. (2) 62, 827-842 (2000).

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