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Notes on discrete subgroups of $PU(1,2;\mathbf{C})$ with Heisenberg translations III

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In a previous paper [8] we have seen that under some conditions Parker's theorem yields the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. In this paper we give a new stable basin region and show the same result as in [8] without the assumption on r. This is a joint work with John R. Parker.

1. First we recall some definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V=V^{1,2}(\mathbf{C})$ denote the vector space $\mathbf{C}^3$, together with the unitary structure defined by the Hermitian form

$$\widetilde{\Phi}(z^*,w^*)=-(\overline{z_0^*}w_1^*+\overline{z_1^*}w_0^*)+\overline{z_2^*}w_2^*$$

for $z^*=(z_0^*,z_1^*,z_2^*), w^*=(w_0^*,w_1^*,w_2^*)$ in V. An automorphism g of V, that is a linear bijection such that $\widetilde{\Phi}(g(z^*),g(w^*))=\widetilde{\Phi}(z^*,w^*)$ for z^*,w^* in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,2;\mathbf{C})$. Let $V_0=\{w^*\in V| \ \ \widetilde{\Phi}(w^*,w^*)=0\}$ and $V_-=\{w^*\in V| \ \ \widetilde{\Phi}(w^*,w^*)<0\}$. It is clear that V_0 and V_- are invariant under $U(1,2;\mathbf{C})$. We denote $U(1,2;\mathbf{C})/(center)$ by $PU(1,2;\mathbf{C})$. Set $V^*=V_-\cup V_0-\{0\}$. Let $\pi:V^*\longrightarrow\pi(V^*)$ be the projection map defined by $\pi(w_0^*,w_1^*,w_2^*)=(w_1,w_2)$, where $w_1=w_1^*/w_0^*$ and $w_2=w_2^*/w_0^*$. We write ∞ for $\pi(0,1,0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^2=\{w=(w_1,w_2)\in\mathbf{C}^2 \ \mid \ \ Re(w_1)>\frac{1}{2}|w_2|^2\}.$$

We can regard an element of $PU(1,2;\mathbf{C})$ as a transformation acting on H^2 and its boundary ∂H^2 (see [6]). Denote $H^2\cup\partial H^2$ by $\overline{H^2}$. We define a new coordinate system in $\overline{H^2}-\{\infty\}$. Our convention slightly differs from Basmajian-Miner [1] and Parker [8]. The $H-$*coordinates* of a point $(w_1,w_2)\in\overline{H^2}-\{\infty\}$ are defined by $(k,t,w_2)_H\in(\mathbf{R}^+\cup\{0\})\times\mathbf{R}\times\mathbf{C}$ such that $k=Re(w_1)-\frac{1}{2}|w_2|^2$ and $t=Im(w_1)$. For simplicity, we write $(t_1,w')_H$ for $(0,t_1,w')_H$. The *Cygan metric* $\rho(p,q)$ for $p=(k_1,t_1,w')_H$ and $q=(k_2,t_2,W')_H$ is given by

$$\rho(p,q)=|\{\frac{1}{2}|W'-w'|^2+|k_2-k_1|\}+i\{t_1-t_2+Im(\overline{w'}W')\}|^{\frac{1}{2}}.$$

We note that the Cygan metric ρ is a generalization of the *Heisenberg metric* δ in ∂H^2

Let $f = (a_{ij})_{1\le i,j\le 3}$ be an element of $PU(1,2;\mathbf{C})$ with $f(\infty) \neq \infty$. We define the *isometric sphere* I_f of f by

$$I_f = \{w = (w_1, w_2) \in \overline{H}^2 \mid |\tilde{\Phi}(W,Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0,1,0)$, $W = (1, w_1, w_2)$ in V^* (see [4]). It follows that the isometric sphere I_f is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k,t,w')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

2. We shall give a modified version of the stable basin theorem in [8]. Let

$$B_r = \{z \in \partial H^2 \mid \delta(z,0) < r\},$$

and let $\overline{B}_s^c = \partial H^2 - \overline{B}_s$. Given r and s with $r < s$, the pair of open sets $(B_r, \overline{B}_s^c)$ is said to be *stable* with respect to a set S of elements in $PU(1,2;\mathbf{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_s^c.$$

A loxodromic element f has a unique complex dilation factor $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r,\varepsilon)$ denote the family of loxodromic elements f with fixed points in B_r and $\overline{B}_{1/r'}^c$, and satisfying $|\lambda(f) - 1| < \varepsilon$. For positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, we define $\varepsilon(r,r')$ by

$$\varepsilon(r,r') = \sup\{|\lambda(f) - 1|\}, \tag{2.1}$$

where $|\lambda(f) - 1|$ satisfies the inequality

$$|\lambda(f) - 1| < \sqrt{1 + \left(\frac{1 - (3 + |\lambda(f) - 1|)r^2)}{1 - 2r^2}\right)^2 \left(\frac{1 - 3r^2}{1 - r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - 1. \tag{2.2}$$

A triple of non-negative numbers (r, r', ε) is said to be a ***basin point*** provided that $r < 1/\sqrt{3}$, $r' < 1$ and $\varepsilon < \varepsilon(r,r')$. In particular, if $r' \le r$, we call (r,r',ε) a ***stable basin point.*** Call the set of all such points the ***stable basin region.*** For simplicity, we abbreviate (r,r,ε) to (r,ε). Figure 1 shows our new stable basin region, which contains regions in [1] and [8]. Some stable basin points are tabulated in Table 2.

Exactly the same arguments except for using the following Lemma 2.1 instead of Proposition 3.3 in [1] shows our new stable basin theorem.

Lemma 2.1. *Let $b, c > 0$ be given. If f is a complex dilation and its complex dilation factor satisfies $|\lambda(f) - 1| \le \sqrt{1 + (b/c)^2} - 1$, then $f(p) \in B_b(p)$ for $p \in \overline{B}_c$.*

Theorem 2.2 (cf. [8; Stable Basin Theorem]). *Given positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, the pair of open sets $(B_{r'}, \overline{B}^c_{1/r'})$ is stable with respect to the family $S(r, \varepsilon(r, r'))$, where $\varepsilon(r, r')$ is given by (2.2).*

Remark 2.3. By arguing as in Corollary 6.14 in [1], we may find the boundary of the stable basin region by equating both sides of inequality (2.2) and solving for $|\lambda(f) - 1|$ in terms of r. If we use Basmajian and Miner's inequality (6.2) in [1], this involves solving a polynomial of degree 6. Using our inequality (2.2), we have

$$|\lambda(f) - 1| < \frac{a_2 a_1 \sqrt{a_3^4 b_1 + a_2^2 a_1^2 + 2a_3^3 r^2 b_1} - a_2^2 a_1^2 - a_3^3 b_1 r^2}{a_2^2 a_1^2 - a_3^2 b_1 r^4},$$

where $a_j = 1 - jr^2$ and for $j = 1, 2, 3$ and $b_1 = (r'/r)^2$.

3. We begin with recalling Parker's theorem on the discreteness of subgroups of $PU(1, 2; \mathbf{C})$.

Theorem 3.1 ([9; Theorem 2.1]). *Let g be a Heisenberg translation with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \overline{a} \\ a & 0 & 1 \end{pmatrix},$$

where $Re(s) = \frac{1}{2}|a|^2$. Let f be any element of $PU(1, 2; \mathbf{C})$ with isometric sphere of radius R_f. If

$$R_f^2 > \delta(gf^{-1}(\infty), f^{-1}(\infty))\delta(gf(\infty), f(\infty)) + 2|a|^2,$$

then the group $< f, g >$ generated by f and g is not discrete.

In Theorem 4.5 of [8] we have shown that if $r < 0.484$, then Theorem 3.1 leads to the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. By using a more precise estimate on the Heisenberg distance between fixed points of f in terms of R_f and $\lambda(f)$, we have the following same result without the assumption on r.

Theorem 3.2. *Fix a stable basin point (r, ε). Let g be the same element as in Theorem 3.1. Let f be a loxodromic element with fixed point 0 and q, and satisfying $|\lambda(f) - 1| < \varepsilon$. If $\delta(0, q) > \frac{\delta(0, g(0))}{r^2}(1 + r^2 + \sqrt{1 + r^2})$, then the group $< f, g >$ generated by f and g is not discrete.*

To prove our theorem, we need the following lemmas.

Lemma 3.3. *Let f be a loxodromic element with fixed points 0 and q, satisfying $|\lambda(f) - 1| < \varepsilon$. Then*

$$\left(\frac{\delta(0, q)}{R_f}\right)^2 \leq \frac{2\varepsilon - \varepsilon^2}{1 - \varepsilon}.$$

Lemma 3.4. *For a stable basin point (r, ε),*

$$\frac{1+r^2+\sqrt{1+r^2}}{r^2} > \left(\frac{2\varepsilon-\varepsilon^2}{1-\varepsilon}\right)^{\frac{1}{2}} \left(2+\left(8+\frac{M(\varepsilon)}{2}\right)^{\frac{1}{2}}\right),$$

where $M(\varepsilon) = (1+\varepsilon)^{\frac{1}{2}} + (1+\varepsilon)^{-\frac{1}{2}}$.

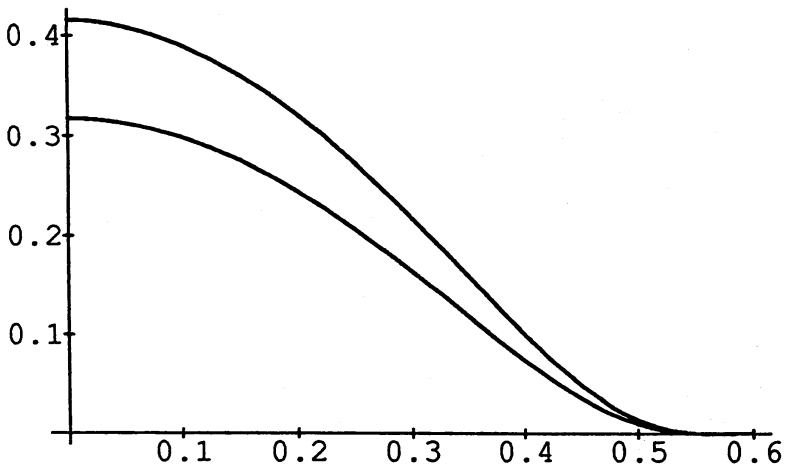

Figure 1. Graph of $\varepsilon(r, r)$

r\r'	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0.05	0.408	1.217	2.1258	3.064	4.013	4.964	5.916	6.866	7.813	8.757
0.1	0.111	0.390	0.755	1.161	1.586	2.019	2.456	2.894	3.331	3.766
0.15	0.046	0.175	0.361	0.581	0.820	1.072	1.329	1.589	1.850	2.111
0.2	0.023	0.091	0.193	0.321	0.466	0.622	0.785	0.953	1.122	1.292
0.25	0.012	0.050	0.108	0.184	0.273	0.371	0.475	0.584	0.696	0.810
0.3	0.007	0.027	0.060	0.104	0.157	0.217	0.283	0.352	0.424	0.499
0.35	0.003	0.014	0.032	0.056	0.086	0.120	0.158	0.200	0.243	0.289
0.4	0.001	0.006	0.015	0.027	0.041	0.059	0.078	0.100	0.123	0.148
0.45	0.000	0.002	0.005	0.010	0.015	0.022	0.030	0.039	0.048	0.059
0.5	0.000	0.000	0.001	0.002	0.003	0.004	0.006	0.008	0.010	0.013

Table 2

References

1. A. Basmajian and R. Miner, Discrete subgroups of complex hyperbolic motions, Invent. Math. 131, 85-136 (1998).
2. A.F. Beardon, The Geometry of Discrete Groups, Springer-Verlag, New York, 1983.
3. L. R. Ford, Automorphic Functions (Second Edition), Chelsea, New York, 1951.
4. W. M. Goldman, Complex hyperbolic geometry, Oxford University Press, 1999.
5. S. Kamiya, Notes on non-discrete subgroups of $\tilde{U}(1,n;F)$, Hiroshima Math. J. 13, 501-506, (1983).
6. S. Kamiya, Notes on elements of $U(1,n;C)$, Hiroshima Math. J. 21, 23-45, (1991).
7. S. Kamiya, Parabolic elements of $U(1,n;C)$, Rev. Romaine Math. Pures et Appl. 40, 55-64, (1995).
8. S. Kamiya, On discrete subgroups of $PU(1,2;\mathbf{C})$ with Heisenberg translations, J. London Math. Soc. (2) 62, 827-842 (2000).
9. J. Parker, Uniform discreteness and Heisenberg translations, Math. Z. 225, 485-505 (1997).

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