Stokes' theorem, self-adjointness of the Laplacian and Hodge's theorem for hyperbolic 3-cone-manifolds

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§1. Introduction

By a hyperbolic 3-cone-manifold, we will mean an orientable (not necessarily volume-finite) Riemannian 3-manifold $C$ of constant sectional curvature $-1$ with cone-type singularity along a 1-dimensional graph $\Sigma$ which consists of geodesic segments in $C$. The subset $M := C - \Sigma$ has a smooth, incomplete hyperbolic structure whose metric completion is identical to the singular hyperbolic structure on $C$. The hyperbolic 3-manifold $M$ is incomplete near $\Sigma$.

In this paper, we will inform that Stokes' theorem for smooth $L^2$-forms on the incomplete hyperbolic manifold $M$ holds. The proof can be performed by following the argument described in Hodgson-Kerckhoff [5]. (In [5], Stokes' theorem in the case where each component of the singular locus $\Sigma$ is homeomorphic to $S^1$ and the complement of an open tubular neighborhood of $\Sigma$ is compact was shown.) Then from Stokes' theorem, by using a result of Gaffney [3], it is shown that there is a maximal extension of the Laplacian on $M$ which is self-adjoint on its adequately defined domain. Thus, we have an extension of Hodge theory to hyperbolic 3-cone-manifolds whose singular loci are smooth 1-manifolds. Let $E$ denote the flat vector bundle of local Killing vector fields on the hyperbolic 3-manifold $M$. Then, if the singular locus $\Sigma$ of the hyperbolic 3-cone-manifold $C$ is a smooth 1-dimensional manifold, for any $E$-valued 1-form $\tilde{\omega}$ which represents an infinitesimal deformation of the hyperbolic structure on $M$ around $\Sigma$ and which satisfies some conditions related with the domain of the Laplacian ($\tilde{\omega}$ is called to be "in standard form"), there is a closed and co-closed $E$-valued 1-form $\omega$ which is equivalent to $\tilde{\omega}$ in the de Rham cohomology group $H^1(M; E)$. The 1-form $\omega$ is a representative with specific control on the asymptotic behavior near the singular locus.
§2. Stokes' theorem and self-adjointness of the Laplacian for hyperbolic 3-cone-manifolds

First we will give the definition of hyperbolic 3-cone-manifolds. Consider a smooth 3-dimensional manifold $N$, which has a path metric given by a gluing of the faces of finitely many geodesic polyhedra possibly with ideal vertices in the 3-dimensional hyperbolic space $\mathbb{H}^3$. The gluing is performed by orientation reversing isometries of $\mathbb{H}^3$. It is permitted that the polyhedra have "faces" on the sphere at infinity $S^2_\infty$ which are not glued to another such "faces". We assume that the link of a vertex is piecewise linear homeomorphic to a sphere and the link of an ideal vertex is piecewise homeomorphic to a torus, an open annulus or an open disk. We also assume that the path metric on $N$ is complete. The manifold $N$ with the metric above is called a hyperbolic 3-cone-manifold.

The singular locus $\Sigma$ of a hyperbolic 3-cone-manifold consists of the points with no neighborhood isometric to a ball in $\mathbb{H}^3$. It is a union of totally geodesic closed simplices of dimension 1. At each point of $\Sigma$ in an open 1-simplex, there is a cone angle which is the sum of dihedral angles of polyhedra containing the point. The subset $N - \Sigma$ has a smooth riemannian metric of constant curvature $-1$, but this metric is incomplete near $\Sigma$ if $\Sigma \neq \phi$.

Let $C$ be a (not necessarily volume-finite) hyperbolic 3-cone-manifold with singular locus $\Sigma$. Let $M := C - \Sigma$ be a smooth (but incomplete) hyperbolic 3-manifold. A tubular neighborhood of a singular point of $C$, which is not a vertex, has the metric

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2,$$

by using the cylindrical coordinate. There are finitely many vertices of $\Sigma$.

We have a developing map of $M$ from its universal covering space $\tilde{M}$,

$$\mathcal{D}_C : \tilde{M} \to \mathbb{H}^3,$$

and a holonomy representation,

$$\rho_C : \pi_1(M) \to \text{PSL}_2(\mathbb{C}).$$

They are called a developing map and a holonomy representation of the cone-manifold $C$.

Let $\Omega^p(M)$ denote the space of smooth, real-valued $p$-forms of $M$ and $\Omega^*(M)$ denote the space of smooth, real-valued forms on $M$. Let $\hat{d}$ be the usual exterior derivative of smooth real-valued forms on $M$:

$$\hat{d} : \Omega^p(M) \to \Omega^{p+1}(M).$$

Let $\hat{*}$ be the Hodge star operator defined by using the riemannian metric $g$ on $M$:

$$g(\phi, \hat{*} \psi) dM = \phi \wedge \psi$$
for any real-valued $p$-form $\phi$ and $(3-p)$-form $\psi$. Let $\hat{\delta}$ be the adjoint of $\hat{d}$:
$$\hat{\delta} : \Omega^p(M) \to \Omega^{p-1}(M).$$

Let $\hat{\Delta}$ be the Laplacian on smooth real-valued forms for the riemannian manifold $M$:
$$\hat{\Delta} = \hat{d}\hat{\delta} + \hat{\delta}\hat{d}.$$

We will use $<,>$ to denote an $L^2$ inner product on real-valued forms:
$$< \xi, \eta > = \int_M \xi \wedge \hat{*} \eta = \int_M g(\xi, \eta) \, dM.$$

It is seen that Stokes' theorem for smooth $L^2$-forms on the incomplete hyperbolic manifold $M$ can be proved as in Hodgson-Kerckhoff [5]. The proof is performed by using Cheeger's method in [1].

**Theorem 1 (Stokes' theorem).** Let $C$ be a hyperbolic 3-cone-manifold with singular locus $\Sigma$. Let $M := C - \Sigma$ be the smooth, incomplete hyperbolic 3-manifold. Then Stokes' theorem holds:
$$\int_M \hat{d}\alpha \wedge \hat{*}\beta = \int_M \alpha \wedge \hat{*}\delta\beta,$$
for smooth $L^2$-forms $\alpha, \beta$ on $M$ such that $\hat{d}\alpha, \hat{\delta}\beta$ are $L^2$-forms on $M$.

If we define the domains of $\hat{d}$ and $\hat{\delta}$ by
$$\mathrm{dom}\hat{d} = \{ \alpha \in \Omega^*(M) ; \alpha \text{ and } \hat{d}\alpha \text{ are } L^2 \},$$
$$\mathrm{dom}\hat{\delta} = \{ \beta \in \Omega^*(M) ; \beta \text{ and } \hat{\delta}\beta \text{ are } L^2 \},$$
then Theorem 1 says that $< \hat{d}\alpha, \beta > = < \alpha, \hat{\delta}\beta >$ holds for all $\alpha \in \mathrm{dom}\hat{d}, \beta \in \mathrm{dom}\hat{\delta}$.

The strong closure $\overline{\hat{d}}$ of $\hat{d}$ is defined as follows (see [1]): $\overline{\hat{d}}\alpha = \eta$ means that $\alpha$ is an $L^2$-form and there exist $\alpha_i \in \mathrm{dom}\hat{d}$ ($i \in \mathbb{N}$) such that $\alpha_i \to \alpha$, $\hat{d}\alpha_i \to \eta$. The domain of $\overline{\hat{d}}$ is defined by
$$\mathrm{dom}\overline{\hat{d}} = \{ \alpha ; \alpha \text{ and } \overline{\hat{d}}\alpha \text{ are } L^2\text{-forms on } M \}.$$ 

In the same manner, the strong closure $\overline{\hat{\delta}}$ of $\hat{\delta}$ and its domain $\mathrm{dom}\overline{\hat{\delta}}$ are defined.

The theorem above means that the manifold $M$ has a negligible boundary (see [3],[4]). Then, by the result of Gaffney [3], for our manifold $M$, the Hilbert space closure $\overline{\hat{\Delta}}$ of $\hat{\Delta}$ is self-adjoint.

**Theorem 2 (self-adjointness of $\overline{\hat{\Delta}}$).** Let $C$ be a hyperbolic 3-cone-manifold with singular locus $\Sigma$. Let $M := C - \Sigma$ be the smooth, incomplete hyperbolic 3-manifold. Let $\overline{\hat{\Delta}}$ be the
Hilbert space closure of the Laplacian for the riemannian manifold $M$ so that

the domain of $\bar{\Delta} = \{ \alpha \in \text{dom } \bar{d} \cap \text{dom } \bar{\delta}; \ \bar{\delta} \alpha \in \text{dom } \bar{d}, \ \bar{d} \alpha \in \text{dom } \bar{\delta} \}$.  

Then $\bar{\Delta} = \bar{d} \bar{\delta} + \bar{\delta} \bar{d}$, and $\bar{\Delta}$ is a closed, non-negative, self-adjoint and elliptic operator.

§3. Hodge theorem for hyperbolic 3-cone-manifolds

Let $C$ be the hyperbolic 3-cone-manifold with singular locus $\Sigma$ and $M = C - \Sigma$ be the hyperbolic 3-manifold considered in §2. Let $G$ denote the group consisting of orientation preserving isometries of $H^3$. The group $G$ can be naturally identified with $\text{PSL}_2(C)$. Let $\mathcal{G}$ denote the Lie algebra of $G$ and $\text{Ad}$ the adjoint representation of $G$ on $\mathcal{G}$. Associated to the hyperbolic structure $\rho_C$ is a flat $\mathcal{G}$ vector bundle $E$ over $M$:

$$E = \mathcal{M} \times_{\text{Ad} \rho_C} \mathcal{G}.$$ 

Let $\Omega^p(M; E)$ denote the space consisting of smooth $E$-valued $p$-forms on $M$. Let $d$ be a covariant exterior derivative

$$d : \Omega^p(M; E) \rightarrow \Omega^{p+1}(M; E),$$

which is given by the flat connection on $E$. Then the $p$th de Rham cohomology group $H^p(M; E)$ of $M$ with coefficients in $E$ is defined by $d$.

There is a natural metric on $E$ as follows. For each $x \in M$, the fiber $E_x$ of the bundle $E$ decomposes as a direct sum $\mathcal{P} \oplus \mathcal{K}$, where $\mathcal{P}$ consists of the infinitesimal pure translations at $x$ and $\mathcal{K}$ consists of the infinitesimal rotations at $x$. Since an infinitesimal pure translation at $x$ corresponds to a tangent vector to $M$ at $x$, $\mathcal{P}$ is identified with the tangent space $T_x M$ of $M$ at $x$. Then we give $\mathcal{P}$ the metric induced from the riemannian metric on $M$. Similarly, since an element of $\mathcal{K}$ operates linearly and isometrically on the tangent space, a metric on $\mathcal{K}$ comes from identifying it with a subspace of $o(3)$ with its usual metric. In fact, $\mathcal{K}$ is identified with the total space $o(3)$. Then we give a metric on $\mathcal{P} \oplus \mathcal{K}$ by regarding the direct sum as an orthogonal direct sum. Let $h$ denote the metric on $E$ given as above.

Let $\star$ denote the Hodge star operator on $\Omega^*(M; E)$ defined by using the riemannian metric $h$ on $E$ and the Hodge star operator $\hat{\star}$ on $\Omega^*(M)$:

$$\alpha \wedge \star \beta = (a\xi) \wedge (b \hat{\star} \eta) = (ab) (\xi \wedge \hat{\star} \eta) = h(a, b) g(\xi, \eta) \, dM,$$

for any $\alpha = a\xi, \beta = b\eta$ ($a, b \in \Omega^0(M; E), \xi, \eta \in \Omega^*(M)$). For two forms $\alpha = a\xi, \beta = b\eta \in \Omega^*(M; E)$, put

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star \beta = \int_M h(a, b) g(\xi, \eta) dM.$$
This is an $L^2$ inner product on $\Omega^*(M; E)$. We define
\[ \delta : \Omega^p(M; E) \to \Omega^{p-1}(M; E) \]
by putting
\[ \delta \alpha = (-1)^{3(p+1)+1} \ast d \ast \alpha \]
for any $\alpha \in \Omega^p(M; E)$. Then the associated Laplacian $\Delta$ is defined by
\[ \Delta := d\delta + \delta d. \]

Let $\nabla$ denote the Levi-Civita connection on $E$ with respect to the metric $h$, and $D$ denote a covariant exterior derivative induced by the connection $\nabla$:
\[ \nabla : \Omega^0(M; E) \to \Omega^1(M; E), \]
\[ D : \Omega^p(M; E) \to \Omega^{p+1}(M; E). \]

Put
\[ D^\ast \alpha = (-1)^{3(p+1)+1} \ast D \ast \alpha, \]
for all $\alpha \in \Omega^p(M; E)$. Let $\{e_1, e_2, e_3\}$ be any orthonormal frame for $TM$ and $\{\omega^1, \omega^2, \omega^3\}$ be the dual co-frame. Let $i()$ denote the interior product on forms. Then $D$ and $D^\ast$ are described as in the following:
\[ D = \sum_{j=1}^{3} \omega^j \wedge \nabla_{e_j}, \]
\[ D^\ast = -\sum_{j=1}^{3} i(e_j) \nabla_{e_j}. \]

Put
\[ T := \sum_{j=1}^{3} \omega^j \wedge \text{ad}(E_j), \]
\[ T^\ast := \sum_{j=1}^{3} i(e_j) \text{ad}(E_j), \]
where $E_j$ is the element in the fiber over any point on $M$, which is the infinitesimal translation in the direction $e_j$ at that point, and $\text{ad}(E_j)$ sends an element $Y$ in the fiber to $[E_j, Y]$. Then we have
\[ d = D + T, \]
\[ \delta = D^\ast + T^\ast. \]

This shows a relationship between the flat structure on $E$, which is defined by the hyperbolic structure on $M$, and the natural metric $h$ on $E$, which is defined by using the local geometry on $M$. (See Matsushima-Murakami [8] for the formulation above.)
As described above, at each point \( x \in M \), the fiber \( E_x \) is decomposed into the orthogonal direct sum \( \mathcal{P} \oplus \mathcal{K} \). Then the vector bundle \( E \) is decomposed into an orthogonal direct sum of two sub-bundles which we also denote as \( \mathcal{P} \) and \( \mathcal{K} \):

\[
E = \mathcal{P} \oplus \mathcal{K}.
\]

This decomposition induces a decomposition:

\[
\Omega^p(M; E) = \Omega^p(M; \mathcal{P}) \oplus \Omega^p(M; \mathcal{K}).
\]

The bundle \( \mathcal{P} \) is naturally identified with the tangent bundle \( TM \) of \( M \). The Levi-Civita connection \( \nabla \) restricted to \( \mathcal{P} \)-valued forms is the Levi-Civita connection on \( M \). On \( \mathcal{K} = o(3) \subset Hom(TM, TM) \), it is again the Levi-Civita connection induced by the one on \( TM \).

The operators \( D \) and \( D^* \) preserve the decomposition, while \( T \) and \( T^* \) map \( \Omega^*(M; \mathcal{P}) \) to \( \Omega^*(M; \mathcal{K}) \) and vice versa:

\[
\begin{array}{ccc}
\Omega^*(M; \mathcal{P}) & \oplus & \Omega^*(M; \mathcal{K}) \\
D, D^* & \downarrow & T, T^* \\
\Omega^*(M; \mathcal{P}) & \oplus & \Omega^*(M; \mathcal{K}), & \Omega^*(M; \mathcal{K}) & \oplus & \Omega^*(M; \mathcal{P}).
\end{array}
\]

The Lie algebra \( \mathcal{G} = sl_2(\mathbb{C}) \) has a natural complex structure which is related to the decomposition \( E = \mathcal{P} \oplus \mathcal{K} \) by \( \mathcal{K} = \mathcal{P} \). The multiplication by \( i \) in the Lie algebra induces a bundle isomorphism from \( \mathcal{P} \) to \( \mathcal{K} \), which respects the local geometry of \( M \). For example, if \( t \) denotes an infinitesimal translation, then \( it \) is an infinitesimal rotation around the axis of \( t \), and \( t \) and \( it \) are orthogonal. Now we will think of \( \Omega^*(M; \mathcal{P}) \) and \( \Omega^*(M; \mathcal{K}) \) as the real and imaginary parts of \( \Omega^*(M; E) \):

\[
\Omega^*(M; E) = \text{Re} \, \Omega^*(M; E) \oplus \text{Im} \, \Omega^*(M; E) \\
= \Omega^*(M; \mathcal{P}) \oplus \Omega^*(M; \mathcal{K}) \\
= \Omega^*(M; \mathcal{P}) \oplus i \Omega^*(M; \mathcal{P}).
\]

An \( E \)-valued \( p \)-form \( \alpha \) is a pair of a real part \( \alpha_{\text{real}} \) and an imaginary part \( \alpha_{\text{imag}} \). The real part \( \alpha_{\text{real}} \) is a \( \mathcal{P} \)-valued \( p \)-form on \( M \). If \( v \) is a \( \mathcal{P} \)-valued 0-form (namely a tangent vector field) on \( M \), then \( (dv)_{\text{real}} \) is \( Dv \in \Omega^1(M; \mathcal{P}) = \Omega^1(M; TM) = Hom(TM, TM) \), which is also equal to \( \nabla v \), and \( (dv)_{\text{imag}} \) is \( Tv \in \Omega^1(M; \mathcal{K}) = i \Omega^1(M; \mathcal{P}) \).

By using the orthonormal frame \( \{e_k, e_i, e_j\} \) and the dual co-frame \( \{\omega^k, \omega^i, \omega^j\} \), we can describe a canonical isomorphism between skew-symmetric elements of \( Hom(TM, TM) \) and vector fields:

\[
Hom(TM, TM)_{\text{skew}} \ni e_l \otimes \omega^j - e_j \otimes \omega^l \to e_k \in \Omega^0(M; TM).
\]
If $v$ is a tangent vector field on $M$, $Dv$ is an element of $\text{Hom}(TM, TM)$. The skew-symmetric part $(Dv)_{\text{skew}}$ of $Dv$ is called the curl of $v$, and is denoted by $\text{curl } v$. By the isomorphism above, $\text{curl } v$ is regarded as a vector field on $M$. Note that this vector field is the half of the usual curl considered in elementary vector calculus. The trace of $Dv$ is called the divergence of $v$, and is denoted by $\text{div } v$. The traceless, symmetric part of $Dv$ is called the strain of $v$, and is denoted by $\text{str } v$.

If $v$ is a locally defined tangent vector field on $M$, then we can consider a local section of the bundle $E$, which is defined by $s_v = v - i \text{ curl } v$. Call it the canonical lift of $v$.

Let $\sigma$ be any closed smooth $E$-valued 1-form on $M$. Choosing a point $x \in M$, we can locally define a section $f_x \sigma$ of the bundle $E$ by integrating $\sigma$ along paths beginning at $x$, which is called the associated local section. Note that we are using the flat connection on $E$ to identify the fibers at different points along the path in order to do the integration. Since $\sigma$ is closed, the value of the integral depends only on the homotopy class of the path; a well-defined section is determined on any simply connected subset of $M$. Then $d f_x \sigma = \sigma$ on such a subset. In general, the section will not extend to a global section on $M$.

In the rest of the paper, we assume that the singular locus $\Sigma$ of the cone-manifold $C$ is a smooth 1-manifold:

$$\Sigma \approx \mathbf{R} \cup \ldots \cup \mathbf{R} \cup \mathbf{S}^1 \cup \ldots \cup \mathbf{S}^1.$$

Some examples of hyperbolic 3-cone-manifolds with infinite volume, whose singular loci are homeomorphic to $\mathbf{R}$, are illustrated in [9].

In a tubular neighborhood $U_k$ of each component $\Sigma_k$ of $\Sigma$, we use cylindrical coordinates, $(r, \theta, z)$. Then the hyperbolic metric on $U_k$ is $dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$. We will use the orthonormal frame $\{e_1, e_2, e_3\}$ of $TM$ adapted to this coordinate system:

$$e_1 := \frac{\partial}{\partial r}, \quad e_2 := \frac{1}{\sinh r} \frac{\partial}{\partial \theta}, \quad e_3 := \frac{1}{\cosh r} \frac{\partial}{\partial z}.$$

Then the dual co-frame $\{\omega^1, \omega^2, \omega^3\}$ is

$$\omega^1 = dr, \quad \omega^2 = \sinh r \, d\theta, \quad \omega^3 = \cosh r \, dz.$$

An $E$-valued 1-form can be interpreted as a complex-valued section of $\mathcal{P} \otimes T^*M \cong TM \otimes T^*M \cong \text{Hom}(TM, TM)$. Then an $E$-valued 1-form can be described as a matrix in $M_3(\mathbf{C})$ whose $(i, j)$ entry is the coefficient of $e_i \otimes \omega^j$.

The form in (1) below is a closed and co-closed form which represents an infinitesimal deformation which does not change the real part of the complex length of an element of
the fundamental group of $U_k$ which is so called the meridian of $U_k$. The meridian is the class of the fundamental group which wraps around $\Sigma_k$ once and bounds a singular disk with cone angle equal to that of $\Sigma_k$. The infinitesimal deformation preserves the property that the meridian is elliptic. Then it gives a small deformation of the cone-manifold $U_k$ to a cone-manifold. The infinitesimal deformation also has the remarkable property that it decreases the cone angle.

$$\tilde{\omega}_{(1)} = \begin{pmatrix} \cosh^2 r & 0 & 0 \\ -\frac{1}{\cosh^2 r} \sinh^2 r & 0 & 0 \\ 0 \frac{1}{\sinh r} \cosh^2 r & \frac{-i}{\cosh^2 r \sinh r} \end{pmatrix}$$

(1)

The form in (2) below is a closed and co-closed form which represents an infinitesimal deformation which leaves the holonomy of the meridian (hence the cone angle) unchanged. If $\Sigma_k$ is homeomorphic to $S^1$, this deformation stretches the length of $\Sigma_k$.

$$\tilde{\omega}_{(2)} = \begin{pmatrix} -\frac{1}{\cosh^2 r} & 0 & 0 \\ 0 & -1 & \frac{-i}{\cosh r} \frac{\sinh r}{\cosh r} \\ 0 & \frac{-i}{\cosh r} \frac{\sinh r}{\cosh^2 r} & \frac{1}{\cosh^2 r} \end{pmatrix}$$

(2)

Definition (in standard form). Let $\tilde{\omega}$ be a smooth, closed, $E$-valued 1-form on $M$ such that $\delta \tilde{\omega}, d(\delta \tilde{\omega}), \delta d(\delta \tilde{\omega})$ are $L^2$. We say that the 1-form $\tilde{\omega}$ is in standard form if the following conditions are satisfied:

- The associated local section $\int_x \tilde{\omega}$ is the canonical lift of its real part:
  $$\int_x \tilde{\omega} = (\int_x \tilde{\omega})_{\text{real}} - i \text{curl} (\int_x \tilde{\omega})_{\text{real}}$$
  for any $x \in M$.

- In a tubular neighborhood $U_k$ of a component $\Sigma_k$ of the singular locus $\Sigma$,
  $$\tilde{\omega} = h_1 \tilde{\omega}_{(1)} + h_2 \tilde{\omega}_{(2)}$$
  for some $h_1, h_2 \in C$.

Theorem 3 (Hodge theorem for hyperbolic 3-cone-manifolds). Let $C$ be a hyperbolic 3-cone-manifold with singular locus $\Sigma$. Let $M := C - \Sigma$ be the smooth, incomplete hyperbolic 3-manifold. Assume that $\Sigma$ is a disjoint union of smooth 1-manifolds; $\Sigma \approx R \cup \ldots \cup R \cup S^1 \cup \ldots \cup S^1$. Let $\tilde{\omega} \in \Omega^1(M; E)$ be a smooth, $E$-valued 1-form which is in standard form. Then there exists a smooth, closed and co-closed $E$-valued 1-form $\omega$, which is cohomologous to $\tilde{\omega}$ and whose associated local section $\int_x \omega$ is the canonical lift of a divergence-free, harmonic vector field. Moreover, there is a unique such form satisfying the condition that $\tilde{\omega} - \omega = ds$ where $s$ is a globally defined $L^2$ section of $E$. 
Outline of the proof. We want to solve the equation $\Delta s = \delta \overline{\omega}$ for a globally defined section $s$ of $E$. Since the associated local section $\int_{x} \overline{\omega}$ is the canonical lift of its real part, $\delta \overline{\omega}$ is also the canonical lift of its real part. Thus, it suffices to solve $\Delta v = (\delta \overline{\omega})_{\text{real}}$ for a globally defined vector field $v$ on $M$. Let $\zeta \in \Omega^{1}(M)$ be a smooth, real-valued 1-form which is the dual to the vector field $(\delta \overline{\omega})_{\text{real}}$. Then, by using a Weitzenböck formula, we can see that it suffices to solve

$$(\hat{\Delta} + 4)\tau = \zeta,$$

for a smooth, real-valued 1-form $\tau \in \Omega^{1}(M)$. Now we apply the self-adjointness of the closure $\overline{\Delta}$ of the Laplacian $\hat{\Delta}$ on $\Omega^{*}(M)$. Since $\zeta$ is in the domain of $\overline{\Delta} + 4$, then by Theorem 2, there is a unique solution $\tau \in$ the domain of $\overline{\Delta} + 4$. Since $\zeta$ is smooth, then, by the usually regularity theory for elliptic operators, $\tau$ is also smooth. Therefore, we can find a globally defined smooth section $s$ of $E$ which satisfies $\Delta s = \delta \tilde{\omega}$. Then put $\omega := \tilde{\omega} - ds$. It is easy to see that $\omega$ and $s$ satisfy the condition described in the theorem. $\square$

If each component $\Sigma_{k}$ of the singular locus $\Sigma$ is homeomorphic to $S^{1}$ and $M - \cup_{k} U_{k}$ is compact, each cohomology class has a representative in standard form (see Lemma 3.3 in [5]).

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