Bounded cohomology of subgroups of 
mapping class groups

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I give a talk on a joint work with Mladen Bestvina [3]. 
When $G$ is a discrete group, a quasi-homomorphism on $G$ is a function $h : G \to \mathbb{R}$ such that
\[ \Delta(h) := \sup_{\gamma_1, \gamma_2 \in G} |h(\gamma_1 \gamma_2) - h(\gamma_1) - h(\gamma_2)| < \infty. \]
The number $\Delta(h)$ is the defect of $h$. We denote by $QH(G)$ the vector space of all quasi-homomorphisms $G \to \mathbb{R}$ modulo the subspace of bounded functions, and by $\overline{QH}(G)$ the vector space of all quasi-homomorphisms $G \to \mathbb{R}$ modulo the subspace of functions within uniform distance to a homomorphism.

Let $S$ be a compact orientable surface of genus $g$ and $p$ punctures. We consider the associated mapping class group $Mod(S)$ of $S$. This group acts on the curve complex $X$ of $S$ defined by Harvey [7] and successfully used in the study of mapping class groups by Harer [6], [5]. For our purposes, we will restrict to the 1-skeleton of Harvey's complex, so that $X$ is a graph whose vertices are isotopy classes of essential, non-parallel, nonperipheral, pairwise disjoint simple closed curves in $S$ (also called curve systems) and two distinct vertices are joined by an edge if the corresponding curve systems can be realized simultaneously by pairwise disjoint curves. In certain sporadic cases $X$ as defined above is 0-dimensional (this happens when there are no curve systems consisting of two curves, i.e. when $g = 0$, $p \leq 4$ and when $g = 1$, $p \leq 1$). In the theorem below these cases are excluded. The mapping class group $Mod(S)$ acts on $X$ by $f \cdot a = f(a)$.

H. Masur and Y. Minsky proved the following remarkable result.
Theorem 1 [9] The curve complex $X$ is $\delta$-hyperbolic. An element of $\text{Mod}(S)$ acts hyperbolically on $X$ if and only if it is pseudo-Anosov.

Using their result, we show the following theorem. H. Endo and D. Kotschick [2] have shown using 4-manifold topology and Seiberg-Witten invariants that $\overline{QH}(\text{Mod}(S)) \neq 0$ when $S$ is hyperbolic.

Theorem 2 [3] Let $G$ be a subgroup of $\text{Mod}(S)$ which is not virtually abelian. Then $\dim \overline{QH}(G) = \infty$.

The following is a version of superrigidity for mapping class groups. It was conjectured by N.V. Ivanov and proved by Kaimanovich and Masur [8] in the case when the image group contains independent pseudo-Anosov homeomorphisms and it was extended to the general case by Farb and Masur [4] using the classification of subgroups of $\text{Mod}(S)$ as above. Our proof is different in that it does not use random walks on mapping class groups, but instead uses the work of M. Burger and N. Monod [1] on bounded cohomology of lattices.

Corollary 3 [3] Let $\Gamma$ be an irreducible lattice in a connected semi-simple Lie group $G$ with no compact factors and of rank $> 1$. Then every homomorphism $\Gamma \to \text{Mod}(S)$ has finite image.

参考文献


