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Hyperbolic spatial graphs arising from torus knots

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Abstract

トーラス結び目の強可逆性を使って、三次元球面内に空間グラフを作ります。そして、出来たグラフが「双曲的」、即ちその外部に、境界が全測地的となる双曲構造が入る事も示します。

Key words: hyperbolic manifold, spatial graph, torus knot.
1991 Mathematics Subject Classifications: Primary: 57M50; secondary: 57N10.

1 Introduction

In this paper, a *graph* G means a finite, one-dimensional CW-complex without isolated vertices. An embedding $f: G \longrightarrow M$ of G into a three-dimensional manifold M is called a *spatial embedding* of G, and its image $f(G)$ is called a *spatial graph.* Especially if G is homeomorphic to a circle (resp. union of circles), then the spatial graph $f(G)$ is called a *knot* (resp. *link*). A spatial graph is said to be *planar* (or *unknotted*) if it can be drawn in a plane without edges crossing, and otherwise *knotted.*

Let S be a union of (topologically) circular components of $f(G)$, and we denote by $C(f(G))$ a manifold obtained from M by removing regular neighborhoods of S and the interior of regular neighborhoods of $f(G) - S$. Then we define a spatial graph $f(G)$ in M to be *hyperbolic* if $C(f(G))$ admits a complete hyperbolic structure with each toric boundary component (i.e., it comes from S) becoming cusp neighborhood and each non-toric one (i.e., it comes from $f(G) - S$) becoming totally geodesic. We note that any hyperbolic spatial graph in the three-dimensional sphere $\mathbf{S}^3$ is knotted, since unknotted handle bodies do not admit complete hyperbolic structure with totally geodesic boundary.

One of the first example of hyperbolic spatial graph is constructed by W. P. Thurston (see [Th, Example 3.3.12]). This graph has two vertices with three edges in $\mathbf{S}^3$ and had already known as Kinoshita's theta-curve (see [Ki]). Lately this graph is generalized to Suzuki's Brunnian graph θ_n of order n (see [Sc, Su]. Kinoshita's theta-curve is Suzuki's Brunnian graph of order 3), and L. Paoluzzi and B. Zimmermann proved in [PZ] that the graph θ_n is hyperbolic for any $n \geq 3$. Actually θ_2 is so-called the trefoil knot, which is a typical non-hyperbolic knot, and θ_n (or $C(\theta_n)$) is obtained by "$n/2$-fold" cyclic branched covering of θ_2 (or $C(\theta_2)$), branched over the axis of the symmetry of order 2. We here explain how to obtain θ_n from the trefoil knot.

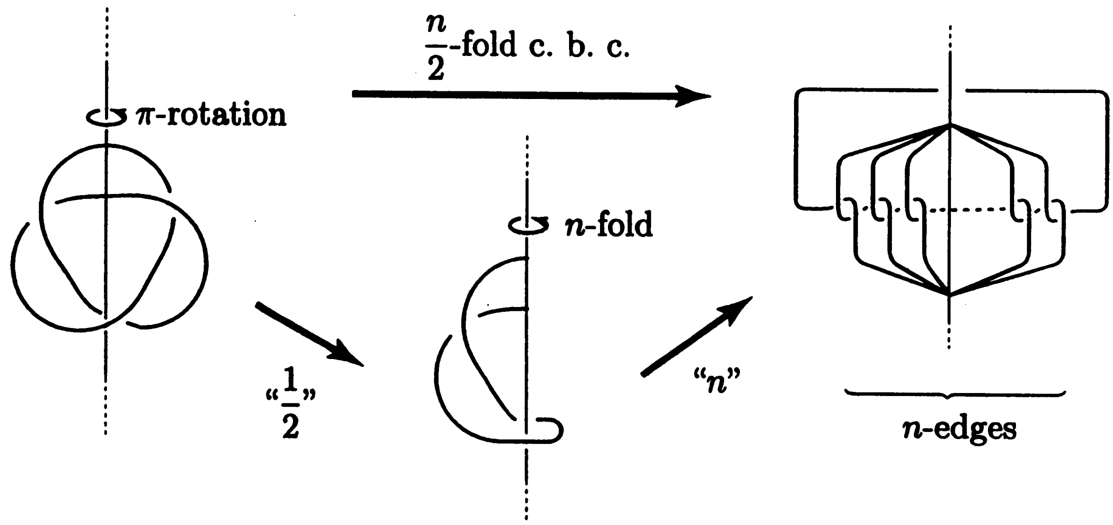

Figure 1: $n/2$-fold cyclic branched covering over a strongly invertible axis of $T(2,3)$

The trefoil knot is a knotted circle in $\mathbf{S}^3$ like the left side figure of Figure 1, and has a symmetry of order 2, which means that there is an automorphism, say φ, of $\mathbf{S}^3$ preserving the knot and φ^2 being the identity of $\mathbf{S}^3$. Then we take the quotient of $\mathbf{S}^3$ by φ, and obtain a graph in it, consisting of a trivial circle arising from the axis of the action and an arc from the knot. We might say the quotient space is a "1/2" of $\mathbf{S}^3$. We then take the n-fold cyclic branched covering of the quotient space, branched over the (axial) circle. Thus we obtain the graph θ_n, the right side figure of Figure 1, after slightly moving the arcs. So, by this construction, we could say θ_n (or $C(\theta_n)$) is obtained by "$n/2$-fold" cyclic branched covering of the trefoil knot θ_2 (or $C(\theta_2)$), branched over the axis of the strongly invertible action.

A (p,q)-torus knot $T(p,q)$ is obtained by looping a string through the hole of a standard torus p times with q revolutions before joining its ends, where p and q are relatively prime. We recall that $T(p,q)$ is unknotted if and only if $|p| \geq 2$ and $|q| \geq 2$. An l component torus link $T(lp,lq)$ is a union of l string torus knots $T(p,q)$ running parallel to them.

The trefoil knot θ_2 is the torus knot $T(2,3)$. It is easy to see that the automorphism φ defined above also acts on $\left(\mathbf{S}^3, T(lp,lq)\right)$ as order 2 symmetry

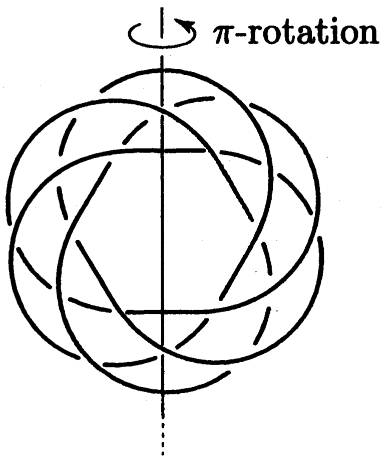

Figure 2: Torus link $T(4,6)$ with the axis of an order 2 symmetry

(see Figure 2). Thus, for any knotted torus link, we can obtain a spatial graph by the $n/2$-fold cyclic branched covering, which means that we first take the quotient of the action of the generator of the symmetry group, and then take the n-fold cyclic branched covering of the quotient space over the axis of the action. So the following question naturally arises:

Question 1.1 *For any knotted torus knot or link and natural number $n \geq 3$, does the $n/2$-fold cyclic branched covering arise a hyperbolic spatial graph?*

The purpose of this paper is to answer this question:

Theorem 1.2 *For any knotted torus knot and natural number $n \geq 3$, the $n/2$-fold cyclic branched covering arises a hyperbolic spatial graph. On the other hand, this property does not hold for the knotted torus links.*

We prove this theorem in the next section.

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2 Proof of Theorem 1.2

2.1 Knots case

Take relatively prime integers p and q, and consider a torus knot $T(p,q)$. Let $E\,(T(p,q))$ be the closure of $C\,(T(p,q))$, i.e., the exterior of $T(p,q)$ in $\mathbf{S}^3$. Then it is known (see, for example, [ST, p. 402]) that $E\,(T(p,q))$ is a Seifert fibered manifold with two exceptional fibers. Its base orbifold is a closed disk with two cone points of orders p and q respectively. The Seifert invariant of the exceptional fiber corresponding to the cone point of order p (resp. q) is (p, β_1) (resp. (q, β_2)), where β_1 (resp. β_2) is a natural number uniquely determined from the following two conditions: $0 < \beta_1 < p$ and $q\beta_1 \equiv 1 \bmod p$ (resp. $0 < \beta_2 < q$ and $p\beta_2 \equiv 1 \bmod q$).

Let M be a 3-manifold of trivial $\mathbf{S}^1$ bundle over a closed disk (i.e., M is a trivial solid torus), and L a link in M of two (trivial) fibers. Then $E(T(p,q))$ is a result of Dehn surgery on M along L with surgery coefficients β_1/p and β_2/q. We here note that, since φ acts on $E(T(p,q))$ as involution, the link L must be preserved by the action of φ and its each component must be intersect the axis of φ two times.

We next take the quotient space of $E(T(p,q))$ by φ. Topologically $E(T(p,q))$ is a solid torus and the action φ is the involution. So the quotient space is a closed ball, say M'. Then the fixed point set, say t, of φ comes from two arcs. Actually it is obtained by a tangle addition of two rational tangles β_1/p and β_2/q (for the definition of the rational tangle, see, for example, [Mu, Chapter 9]. See also Figure 3).

Example.

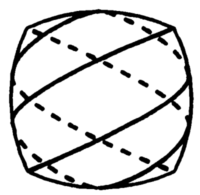
$\frac{2}{3} =$  $\left(\text{slope} = \frac{2}{3}\right)$

Figure 3: The rational tangle 2/3

Now all we have to show is that the n-fold cyclic branched covering of M' over t is a closed hyperbolic manifold with totally geodesic boundary. Let DM' be the double of M', i.e., a closed manifold obtained from M' and its mirror image by gluing them naturally along their boundaries. Since M' is topologically a closed ball, DM' is topologically $\mathbf{S}^3$. Then t arises a link Dt in $\mathbf{S}^3$, usually called Montesinos link $M(0; (p, \beta_1), (q, \beta_2), (q, -\beta_2), (p, -\beta_1))$ (see Figure 4). So what we want to show is equivalent to do the following one: for any $n \geq 3$, the n-fold cyclic branched covering of $\mathbf{S}^3$ over a Montesinos link $M(0; (p, \beta_1), (q, \beta_2), (q, -\beta_2), (p, -\beta_1))$ admits a hyperbolic structure.

These types of Montesinos links are so-called *non-elliptic* Montesinos links (see [BZ]). We here note that the figure eight knot in not contained in this class. Now it is shown in [Oe, Corollary 5] that the exterior $C(M(0; (p, \beta_1), (q, \beta_2), (q, -\beta_2), (p, -\beta_1)))$ has a complete hyperbolic structure. So using the following Theorem 2.1 we can complete the proof of Theorem 1.2 in the knots case.

Theorem 2.1 ([BP, Corollary 5]) *Let M be a compact orientable irreducible 3-manifold and $L \subset M$ be a hyperbolic link. Then, for $n \geq 3$, any n-fold cyclic covering of M branched over L admits a hyperbolic structure, except when $n = 3$,*

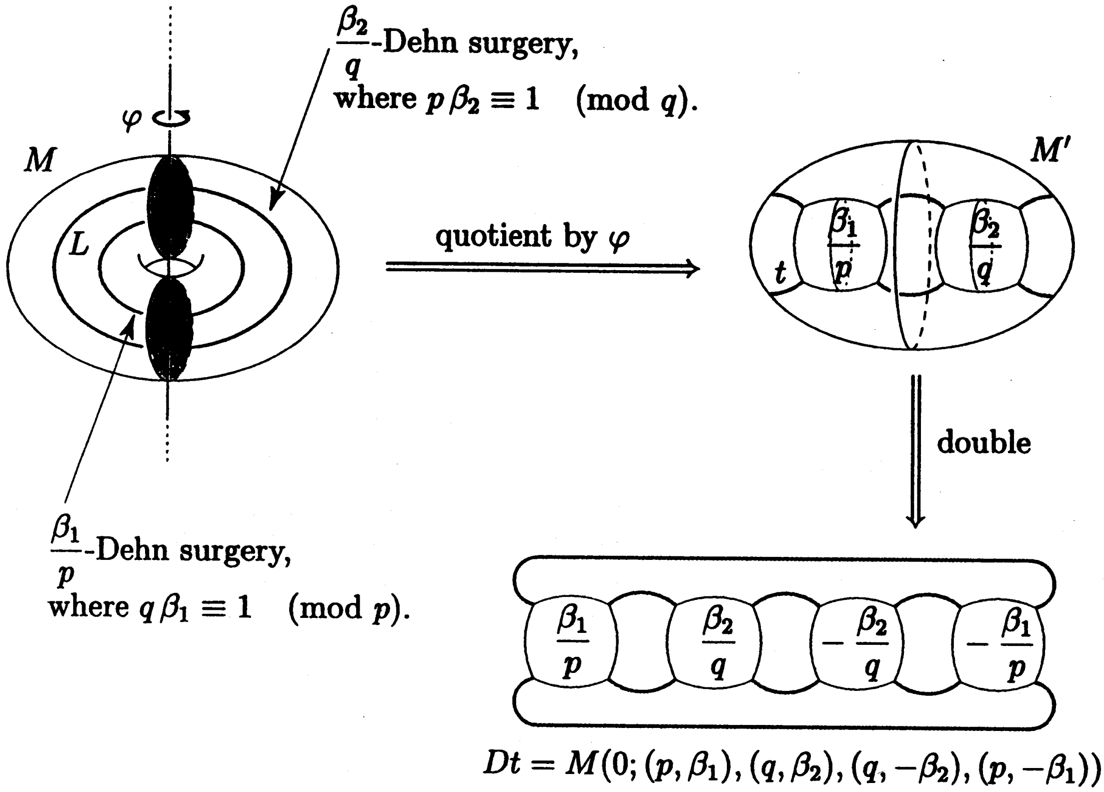


Figure 4: A flow chart of the proof of Theorem 1.2 of knots case

$M = \mathbf{S}^3$ *and* L *is the figure-eight knot.*

2.2 Odd components links case

Let $l \geq 3$ be an odd natural number, and consider a torus link $T(lp, lq)$. Then $E(T(lp, lq))$ has l toric boundary components, and one of them intersects the axis of φ. So the quotient space of $E(T(lp, lq))$ by φ is topologically $\mathbf{B}^3$ minus the interior of $(l-1)/2$ solid tori. Now, as we have done in the previous proof, we take the double of this quotient space with respect to the spherical boundary. Then we obtain the three-dimensional manifold, topologically homeomorphic to $\mathbf{S}^3$ minus the interior of $l-1$ solid tori, with a Montesinos link as the image of the axis of φ (see Figure 5). As in the previous proof, all we have to check is that, for any $n \geq 3$, whether or not the n-fold cyclic branched covering of the manifold over the Montesinos link admits a hyperbolic structure with totally geodesic boundary.

We consider a torus wrapping all toric boundary (see Figure 5 again). Since it wraps several toric boundaries, it is essential in the n-fold cyclic branched covering of the manifold. Thus we have proved Theorem 1.2 in the odd components links case.

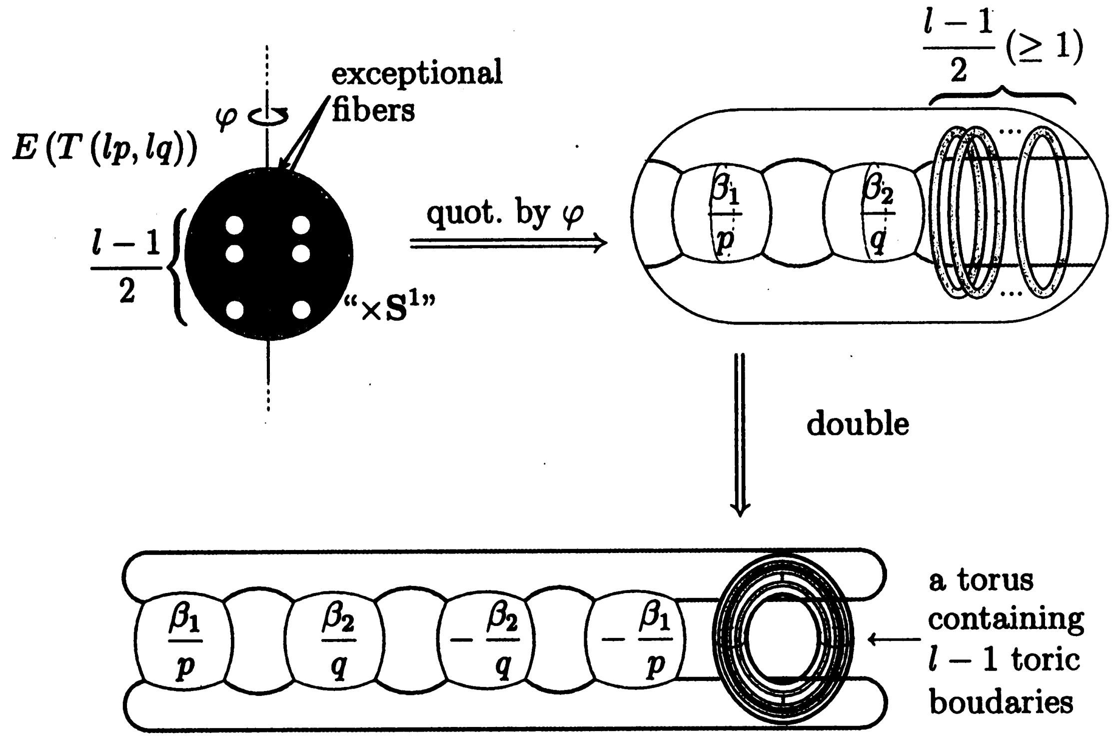


Figure 5: A flow chart of the proof of Theorem 1.2 of odd links case

2.3 Even components links case

Let l be an even natural number, and consider a torus link $T(lp, lq)$. Then $E(T(lp, lq))$ has l toric boundary components, and two of them intersect the axis of φ. So the quotient space of $E(T(lp, lq))$ by φ is topologically $S^2 \times [0,1]$ minus the interior of $(l-2)/2$ solid tori, with two rational tangles as the image of the axis of φ (see Figure 6).

We consider an annulus naturally connecting two spherical boundaries (see Figure 6 again). Then we can easily see that it is essential in the n-fold cyclic branched covering of the quotient space branched over the rational tangles. Thus we have proved Theorem 1.2 in the even components links case.

We have thus completely finished the proof of Theorem 1.2. □

3 Remarks

1. A torus knot exterior is the one obtained from S^3 as Seifert fibered manifold fibered by the torus knot by removing the interior of a tubular neighborhood of a regular fiber. Since S^3 is a special case of lens spaces, we can generalize the statement of Theorem 1.2 to lens spaces as Seifert fibered manifold with two (nontrivial) exceptional fibers.

2. As we mentioned in the introduction, Theorem 1.2 has already proved when the knot is $T(2,3)$. The way of the previous proof is to determine the fundamental polyhedra of hyperbolic manifolds (see [PZ, Us]). One

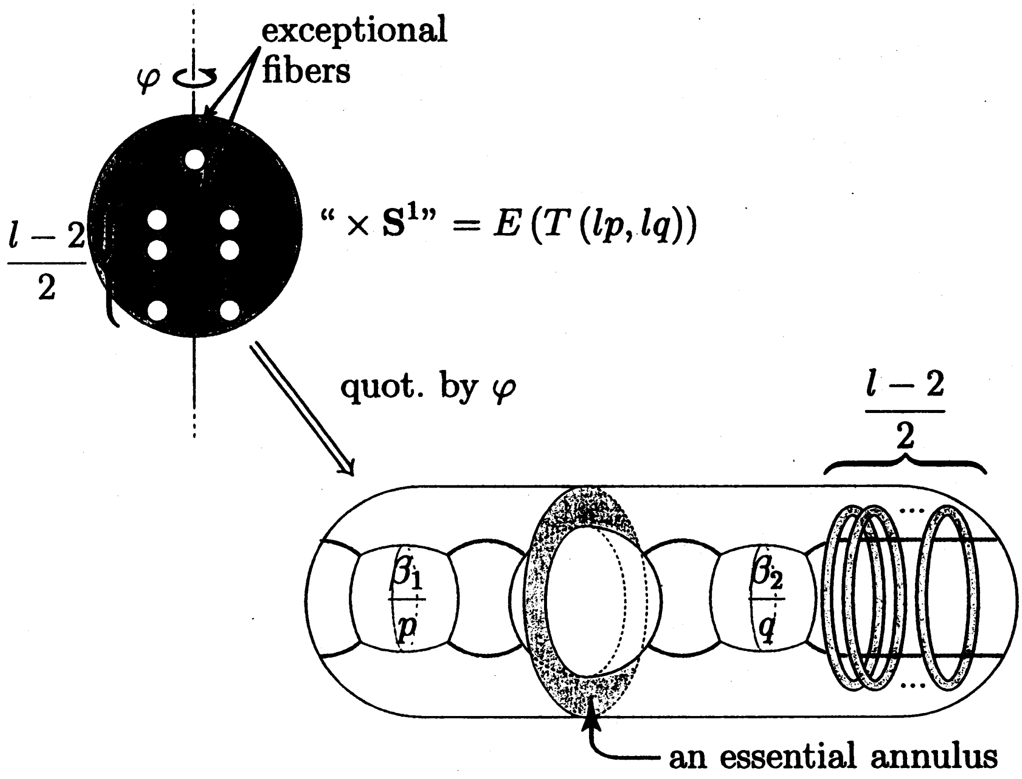


Figure 6: A flow chart of the proof of Theorem 1.2 of even links case

of the advantages of such proof is to obtain various different manifolds not coming from cyclic branched coverings (by changing the identification rules of the faces of the polyhedra). So there still be the problem of determining fundamental polyhedra of the hyperbolic manifolds obtained by $n/3$-fold cyclic branched coverings of $E\,(T(p,q))$.

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