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Regularity of Weak Solutions of the Compressible Navier-Stokes Equations

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Abstract

We prove regularity of weak solutions of the Navier-Stokes equations for compressible, isentropic flow in three space dimension. We allow the presence of vacuum region for the initial data. The pressure law satisfies the general relation $P(\rho) = a\rho^\gamma$, $\gamma \geq 1$. As was found by Hoff[2], Lions[7] and Desjardins[1], the effective viscosity $G$ plays an important role.

keywords: Navier-Stokes equations, isentropic, weak solution, regularity

1 Introduction

The isothermal gases are governed by isentropic compressible Navier-Stokes equations. Although there are many important results, the existence of solutions under general condition remains still open. When the initial velocity has small norm in sufficiently regular space, say $H^3$, and the initial density is near constant, the global existence of classical solution was obtained by Matsumura and Nishida[9]. Then, Hoff[2] extended the global existence of small solutions to more weaker spaces which allow discontinuity of the initial
For the weak solutions, Lions[7] obtained the global existence when the pressure law satisfies $P(\rho) = a\rho^\gamma$, $\gamma \geq 9/5$ for three space dimension, $\gamma \geq 3/2$ for two space dimension and $\gamma > N/2$ for $N$-space dimension with $N \geq 4$. Now, the remaining question will be the extension of the range of the parameter $\gamma$.

On the contrary, Solonnikov[13] showed the local existence of strong solutions if there is no vacuum region for the initial density in the context of classical. Also, Desjardins[1] proved local regularity for the weak solutions when $\gamma \geq 1$ for two space dimension and $\gamma > 3$ for three space dimension.

In this paper, we prove the a priori regularity of weak solutions under the general law $P(\rho) = a\rho^\gamma$, $\gamma \geq 1$ for three space dimension. We allow vacuum region and do not assume any smallness for the initial data. The compactness and local existence of strong solution will be discussed in a forthcoming paper.

First, we consider the isentropic compressible Navier-Stokes equations in periodic domain $\mathbf{T}^3$ with periodicity one to each coordinate direction:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0 \quad \text{in} \quad (0, T) \times \mathbf{T}^3 \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \text{div}(u) + \nabla P(\rho) &= \rho f \quad \text{in} \quad (0, T) \times \mathbf{T}^3,
\end{align*}
\]

where the pressure satisfies for a positive constant $a$

\[P(\rho) = a\rho^\gamma, \quad \gamma \geq 1.
\]

The viscosity constants satisfy $\mu > 0$ and $\lambda + \mu \geq 0$ and the external force $f$ belongs to $L^2((0, T) \times \mathbf{T}^3)$. We need to find the unknown velocity $u \in \mathbf{R}^3$ and the unknown density $\rho \in \mathbf{R}$. The velocity and pressure are to satisfy the initial condition

\[\rho(0, x) = \rho_0(x), u(0, x) = u_0(x).
\]

Although we do not know yet the global existence of weak solution under the general pressure law, we introduce definition of a weak solution. In fact the estimates of local smoothness of the weak solution will lead to the existence of strong solution and we will discuss the existence in different places. $(\rho, u) \in L^1((0, T_0) \times \mathbf{T}^3)$ is a weak solution if it satisfies

\[
\begin{align*}
\int_0^{T_0} \int \rho_0 \psi(0, x) dx + \int_0^{T_0} \int \rho \psi_t + \rho u \cdot \nabla \psi dx dt &= 0 \\
\int_0^{T_0} \int \rho_0 u_0 \psi(x, 0) dx + \int_0^{T_0} \int \rho u \otimes u \nabla u + P \text{div} \psi dx dt &= 0
\end{align*}
\]
\[ = \int_0^{T_0} \int \mu \nabla u \nabla \psi + (\lambda + \mu) \text{div} u \text{div} \psi \, dx \, dt + \int_0^{T_0} \int \rho f \psi \, dx \, dt \]

for all \( \psi \in C_0^\infty[0, T_0 : C^\infty(\mathbb{T}^3)] \) which is periodic. Moreover \((\rho, u)\) satisfies
\[
\sup_{0 \leq t \leq T_0} |\rho|_\gamma(t) + |\sqrt{\rho} u|_2(t) + \int_0^{T_0} |\nabla u|_2 \, dt \leq C.
\]

We denote \( |u|_p = (\int |u|^p \, dx)^{1/p} \) and \( c \) is constant depending only exterior data.

**Theorem 1.1** Suppose that \( \rho_0 \in L^\infty \) and \( u_0 \in H^1 \). Then, there is \( T \) such that the weak solution \((\rho, u)\) satisfies \( \rho \in L^\infty([0, T) \times \mathbb{T}^3) \) and \( u \in L^\infty(0, T : H^1(\mathbb{T}^3)) \). Furthermore we have
\[
\sup_{0 \leq t \leq T} |\rho|_\infty(t) + |\nabla u|_2(t) + \left( \int_0^T |\sqrt{\rho} u_t|_2^2 \, dt \right)^{1/2} \leq c.
\]

For our simplicity of presentation, we assume zero external force.

## 2 Estimate of integral norm of density

We define our objective function \( h \) by
\[
h(t) = |\rho|_\infty(t) + |\nabla u|_2(t).
\]

For computational convenience we introduce two universal Lipschitz function \( \Phi \) and \( \Psi \) which could be different in each appearance. \( \Phi(h(s)) \) depends only on \( h(s) \) and \( \Psi(\int_0^t \Phi \, ds) \) depends only on \( \int_0^t \Phi(h(s)) \, ds \) But, after overall computations, they will be decided in natural way.

First, we estimate the Averages. We denote \( \overline{u} = \int u \, dx \). The initial mass is positive so that
\[
\int \rho_0 \, dx = M > 0,
\]
otherwise the problem is trivial. From mass conservation and momentum conservation,
\[
\overline{\rho}(t) = M \quad \text{and} \quad \overline{\rho u}(t) = \int \rho_0 u_0 \, dx
\]
for all \( t \). From Poincaré inequality we have
\[
|\int \rho(u - \overline{u}) \, dx(t)| \leq |\rho|_\infty \left( \int |u - \overline{u}|^2 \, dx \right)^{1/2}
\]
\[ \leq c|\rho|_\infty|\nabla u|_2(t) \]

and hence we obtain

\[ |\overline{u}|(t) \leq \frac{1}{M} \int \rho u dx(t) + \frac{|\rho|_\infty(t)}{M} |\nabla u|_2(t) \leq \Phi(h(t)) \]

for some Lipschitz function \( \Phi \). We also have

\[ \int \rho|u|^2 dx(t) - M\overline{|u|^2}(t) \leq \int \rho||u|^2 - \overline{|u|^2}| dx(t) \]

\[ \leq |\rho|_\infty(t) \int |u||\nabla u| dx(t) \leq \frac{1}{4} M\overline{|u|^2}(t) + 4 \left( \frac{|\rho|_\infty(t)}{M} \right)^2 |\nabla u|^2_2(t) \]

and hence it follows that

\[ \overline{|u|^2}(t) \leq \frac{2}{M} \int \rho|u|^2 + 6 \left( \frac{|\rho|_\infty(t)}{M} \right)^2 |\nabla u|^2_2(t) \leq \Phi(h(t)). \]

Now we estimate the integral norms of density. We apply \( \rho^{k-1} \) as a test function to mass conservation. Then, we obtain

\[ (\rho^k)_t + \text{div}(\rho^k u) + (k - 1)\rho^k \text{div}(u) = 0 \]

for any positive constant \( k \) and hence integrating in time and space

\[ \int \rho^k dx(t) = \int \rho^k_0 dx - (k - 1) \int_0^t \int \rho^k(s, x) \text{div}(u(s, x)) dx ds \]

\[ \leq \int \rho^k_0 dx + \int_0^t |\nabla u|^2_2(s) ds + c(k)|\rho|^2_\infty(s) ds \]

\[ \leq c + \int_0^t \Phi(h(s)) ds. \]

Therefore we conclude

\[ |\rho|_k(t) \leq \Psi(\int_0^t \Phi(h(s)) ds) \]

for all fixed positive constant and for some Lipschitz functions \( \Phi \) and \( \Psi \). We decide appropriate \( k \) later.
3 Estimate of velocity

To handle the nonlinear convection term $\rho u \cdot \nabla u$, we first estimate

$$\sup_{0 \leq s \leq t} \int \rho |u|^4 dx(s) + \int_0^t \int |u|^2 |\nabla u|^2 dx ds.$$  

For our convenience we define effective pressure $Q$ and effective viscosity flux $G$ by

$$Q = - (\lambda + \mu) \text{div}(u) + P(\rho),$$

$$G = (\lambda + 2\mu) \text{div}(u) - P(\rho) = \mu \text{div}(u) - Q.$$

Taking $|u|^2 u$ as test function for momentum conservation equation, we have

$$\frac{1}{4} \int \rho (|u|^4)_t dx + \frac{1}{4} \int \rho u \cdot \nabla (|u|^4) dx + \mu \int |u|^2 |\nabla u|^2 dx + \frac{\mu}{8} \int |\nabla (|u|^2)|^2 dx = \int Q \text{div}(|u|^2 u) dx.$$  

We note that

$$\int \rho (|u|^4)_t dx + \int \rho u \cdot \nabla (|u|^4) dx = \frac{d}{dt} \int \rho |u|^4 dx.$$  

Hence, integrating in time, we have

$$\int \rho |u|^4 dx(t) + \mu \int_0^t \int |u|^2 |\nabla u|^2 dx ds$$

$$\leq \int \rho_0 |u_0|^4 dx + c \int_0^t \int |Q||u||u \nabla u| dx ds.$$  

It is important to find right exponent to derive closed estimates. From Hölder inequality and Sobolev inequality, we have

$$\int_0^t \int |Q||u||u \nabla u| dx ds \leq \int_0^t \left[ \int |Q|^{12/5} dx \right]^{5/12} \left[ \int (|u|^2)^6 dx \right]^{1/12} \left[ \int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds$$

$$\leq \int_0^t \left[ \int |Q|^{12/5} dx \right]^{5/12} \left[ \int (|u|^2 - |\overline{u}|^2(s))^6 dx \right]^{1/12} \left[ \int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds$$

$$+ \int_0^t \left[ \int |Q|^{12/5} dx \right]^{5/12} \left[ \int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds.$$
\[
\leq c \int_0^t \left[ \int |Q|^{12/5} dx \right]^{5/12} \left[ \int |u|^2 |\nabla u|^2 dx \right]^{3/4} ds
+ \int_0^t (\overline{|u|^2})^{1/2} \left[ \int |Q|^{12/5} dx \right]^{5/12} \left[ \int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds
\leq c \int_0^t \left[ \int |Q|^{12/5} dx \right]^{5/3} + \overline{|u|^2} \left[ \int |Q|^{12/5} dx \right]^{5/6} ds
+ \frac{\mu}{4} \int_0^t \int |u|^2 |\nabla u|^2 dx ds.
\]

The estimates for generalized pressure \( Q \) can be replaced by effective viscosity flux \( G \) so that

\[
\int |Q(s, x)|^{12/5} dx \leq c \int |G(s, x)|^{12/5} dx + \Phi(h(s)).
\]

From the definition of control variable \( h \) and \( G \), we also have

\[
|\overline{G}(s)| \leq \Phi(h(s)).
\]

Thus from Sobolev inequality and, we find that

\[
\left( \int |G(s, x)|^{12/5} dx \right)^{5/3} \leq \left( \int |G(s, x) - \overline{G}(s)|^{12/5} dx \right)^{5/3} + \Phi(h(s))
\leq \left( \int |G(s, x) - \overline{G}(s)|^2 dx \right)^{5/4} \left( \int |G(s, x) - \overline{G}(s)|^{18/5} dx \right)^{5/12} + \Phi(h(s))
\leq \varepsilon_0 \left( \int |G(s, x) - \overline{G}(s)|^{18/5} dx \right)^{1/2} + c \left( \int |G(s, x) - \overline{G}(s)|^2 dx \right)^{15/2} + \Phi(h(s)).
\]

We note that

\[
c \left( \int |G(s, x) - \overline{G}(s)|^2 dx \right)^{15/2} \leq c |\nabla u|_2^{15}(s) + |P(\rho)|_2^{15} \leq \Phi(h(s))
\]

and

\[
\left( \int |G(s, x) - \overline{G}(s)|^{18/5} dx \right)^{1/2} \leq c |\nabla G|_{15/8}^{9/5}.
\]

Here important fact is the exponent 9/5 is less than 2 and 15/8 is less also less than 2. Therefore combining all the previous estimates, we conclude

\[
\sup_{0 \leq s \leq t} \int \rho|u|^4 dx(s) + \int_0^t \int |u|^2 |\nabla u|^2 dx ds
\]
\[ \leq \left( \int_0^t |\nabla G|_{15/8}^2(s) ds \right)^{9/10} + \int_0^t \Phi(h(s)) ds. \]

We let \( \mathbf{P} \) be the projection operator to divergence free vector space. Then, from the definition of \( G \) and \( \mathbf{P}u \), we have

\[ \Delta G = \text{div}(\rho u_t) + \text{div}(\rho u \cdot \nabla u) \]
\[ \Delta \mathbf{P}u = \mathbf{P}(\rho u_t + \rho u \cdot \nabla u). \]

For a given nonnegative constant \( \delta \in [0, 1) \), we have

\[ |\nabla G|_{2-\delta}^2 + |\Delta \mathbf{P}u|_{2-\delta}^2 \leq c \left( |\rho u_t|_{2-\delta}^2 + |\rho u \cdot \nabla u|_{2-\delta}^2 \right) \]
\[ \leq c(|\rho|_m^2 + 1) \left( |\sqrt{\rho} u_t|_{2-\delta}^2 + |u \nabla u|_{2-\delta}^2 \right) \]

for some \( m \) depends only on \( \delta \) and integrating with respect to time we obtain

\[ \int_0^t |\nabla G|_{2-\delta}^2 + |\Delta \mathbf{P}u|_{2-\delta}^2 ds \]
\[ \leq c \sup_{0 \leq s \leq t}(|\rho|_m^2 + 1) \int_0^t \rho |u_t|^2 + |u \nabla u|^2 dx ds \]
\[ \leq \Psi \left( \int_0^t \Phi(h(s)) ds \right) \int_0^t \rho |u_t|^2 + |u \nabla u|^2 dx ds. \]

Moreover, the Sobolev inequality implies that

\[ |\nabla u|_5 \leq c|\nabla G|_{15/8} + c|\Delta \mathbf{P}u|_{15/8} + \Phi(h(s)). \]

Finally we estimate \( |\nabla u|_2(t) \). We multiply \( u_t \) to our momentum conservation equation and integrate. Consequently, we have

\[ \int_0^t \int \rho |u_t|^2 dx ds + \mu \int |\nabla u|^2 dx(t) + (\lambda + \mu) \int |\text{div}u|^2 dx(t) \]
\[ \leq \mu \int |\nabla u_0|^2 dx + (\lambda + \mu) \int |\text{div}u_0|^2 dx + \int_0^t \int \rho |u \nabla u|^2 dx ds - \int_0^t \int \nabla p \cdot u_t dx ds. \]

Again from Hölder inequality, for a given constant \( 0 < \epsilon < 1 \), we have

\[ \int_0^t \int \rho |u \nabla u|^2 dx ds \leq \left( \int_0^t \int |u \nabla u|^2 dx ds \right)^{1-\epsilon} \left( \int_0^t \int \rho^{1/\epsilon} |u \nabla u|^2 dx ds \right)^{\epsilon} \]
\[
\int_0^t \int \rho^{1/\varepsilon} |u \nabla u|^2 dx ds \leq \int_0^t \left( \int \rho^{10/\varepsilon - 5} dx \right)^{1/10} \left( \int \rho |u|^4 dx \right)^{1/2} \left( \int |\nabla u|^5 dx \right)^{2/5}
\]

\[
\leq \sup_{0 \leq s \leq t} |\rho|^{1/\varepsilon - 1/2} (s) \left( \sup_{0 \leq s \leq t} \int \rho |u|^4 dx(s) \right)^{1/2} \int_0^t |\nabla u|^2(s) ds
\]

\[
\leq \Psi_{\varepsilon}(\int_0^t \Phi(h(s)) ds) \left( \left( \int_0^t |\nabla G|_{15/8}^2 ds \right)^{9/20} + \Psi \right)
\]

\[
\left( \int_0^t |\nabla G|_{15/8}^2 + |\Delta P u|_{15/8}^2 ds + \int_0^t \Phi(h(s)) ds \right),
\]

where \( \Psi_{\varepsilon} \) is a Lipschitz function depending on \( \varepsilon \) and we choose \( \varepsilon = \frac{1}{11} \). From the estimate for \( \int_0^t \int |u \nabla u|^2 dx ds \), we have

\[
\int_0^t \int \rho |u \nabla u|^2 \leq \Psi_{\varepsilon}(\int_0^t \Phi(h(s)) ds) \left( \int_0^t |\nabla G|_{15/8}^2 + |\Delta P u|_{15/8}^2 ds \right)^{\frac{9}{10} + \frac{11\varepsilon}{20}} + \Psi(\int_0^t \Phi(h(s)) ds).
\]

Now if we choose \( \varepsilon = \frac{1}{11} \), then

\[
\frac{9}{10} + \frac{11\varepsilon}{20} = \frac{19}{20} < 1
\]

and

\[
\int_0^t \int \rho |u \nabla u|^2 \leq \Psi(\int_0^t \Phi(h(s)) ds) \left( \int_0^t |\nabla G|_{15/8}^2 + |\Delta P u|_{15/8}^2 ds \right)^{19/20} + \Psi(\int_0^t \Phi(h(s)) ds).
\]

From integration by parts,

\[
- \int \nabla P \cdot u_s dx = \int P \text{div} u_s dx = \frac{d}{ds} \int P \text{div} u dx + \int P_s \text{div} u dx.
\]

Integrating in time, we have

\[
- \int_0^t \int \nabla P \cdot u_s ds = - \int \int P \text{div} u dx(t) + \int P_0 \text{div} u_0 dx.
\]
$$+ \int_0^t \int P' \rho_s \text{div}u dx ds.$$ 

We find that

$$\left| \int P(\rho) \text{div}u dx(t) \right| \leq \frac{\mu}{4} \int |\nabla u|^2 dx(t) + \frac{4}{\mu} \int P^2 dx$$

$$\leq \frac{\mu}{4} \int |\nabla u|^2 dx(t) + \Psi(\int_0^t \Phi(h(s)) ds).$$

Since \( \rho_t = -\text{div}(\rho u) \), we find that

$$\int_0^t \int P' \rho_s \text{div}u dx ds = - \int_0^t \int P' \text{div}(\rho \text{div}u + u \cdot \nabla \rho) dx ds$$

$$- \int_0^t \int P' \rho |\text{div}u|^2 dx ds - \int_0^t \int \nabla P \cdot u \text{div}u dx ds$$

$$\int_0^t \int (P - P') |\text{div}u|^2 dx ds + \int_0^t \int Pu \cdot \nabla \text{div}u dx ds.$$ 

Clearly we have

$$\left| \int_0^t \int (P - P') |\text{div}u|^2 dx ds \right|$$

$$\leq \int_0^t |P - P'|_{\infty}(s) |\nabla u|_{2}(s) ds$$

$$\leq \int_0^t \Phi(h) ds.$$

Since

$$\text{div}u = \frac{1}{\lambda + \mu} (G + P),$$

we have

$$\left| \int_0^t \int \nabla P \cdot u \text{div}u dx ds \right| = \frac{1}{\lambda + \mu} \left| \int_0^t \int Pu \nabla(G + P) dx ds \right|$$

$$\leq c \int_0^t \int P^2 |\text{div}u| dx ds + c \int_0^t \int P |u||\nabla G| dx ds$$

$$\leq \left( \int_0^t |\nabla G|_{15/8}^2 ds \right)^{19/20} + \Psi(\int_0^t \Phi ds).$$

Therefore combining all the estimates, we have

$$\int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t)$$
\[
\leq \Psi \left( \int_0^t |\nabla G|_{15/8}^2 + |\Delta P u|_{15/8}^2 ds \right)^{19/20} + \Psi \\
\leq \Psi \left( \int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) \right)^{19/20} + \Psi \\
\leq \frac{1}{2} \int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) + \Psi
\]
and we conclude that
\[
\int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) \leq \Psi.
\]

4 \textit{ } \textbf{}} \text{ } L^\infty \text{-bound of density}

From the mass conservation law, we have
\[
(\log \rho)_t + u \cdot \nabla (\log \rho) + \text{div} = 0
\]
and from momentum conservation law,
\[
(\Delta^{-1} \text{div}(\rho u)) + u \cdot \nabla (\Delta^{-1} \text{div}(\rho u)) + [u_j, R_i R_j](\rho u_i) - (\lambda + 2\mu) \text{div} + P = 0.
\]
Thus, if we define \( F = (\lambda + 2\mu) \log \rho + \Delta^{-1} \text{div}(\rho u) \), \( F \) satisfies
\[
F_t + u \cdot \nabla F + P = [u, R_i R_j](\rho u_i).
\]
Next we define the Lagrange flow \( X \) of \( u \) so that
\[
(X(t, s, x))_t = u(t, X(t, s, x)), \quad X(s, s, x) = x
\]
and derive
\[
F(t, X(t, 0, x)) = F_0 - \int_0^t P(\rho(s, X(s, 0, x))) ds \\
+ \int_0^t [u, R_i R_j](\rho u_i)(s, X(s, 0, x)) ds.
\]
Using the fact that \( \rho_0 \) is nonnegative, we have
\[
F(t, X(t, 0, x)) \leq F_0 + \int_0^t [u, R_i R_j](\rho u_i)(s, X(s, 0, x)).
\]
\[
\log \rho(t, x) \leq \log(|\rho_0|_\infty) + c|\Delta^{-1}\text{div}(\rho_0 u_0)|_\infty \\
+ c|\Delta^{-1}\text{div}(\rho u)|_\infty(t) + c \int_0^t |[u, R_i R_j](\rho u_i)|_\infty(s) ds.
\]

In view of Sobolev embedding, we have
\[
|\Delta^{-1}\text{div}(\rho_0 u_0)|_\infty \leq |\rho_0 u_0|_{7/2}
\]
and
\[
|\Delta^{-1}\text{div}(\rho u)|_\infty(t) \leq c|\rho u|_{7/2} \leq \rho_{21}^{6/7}(t) + \left(\int \rho|u|^4 dx(t)\right)^{2/7} \leq \Psi.
\]

Again, from Sobolev embedding, we obtain
\[
|[u, R_i R_j](\rho u_i)|_\infty(s) \leq |[u, R_i R_j](\rho u_i)|_{W^{1,7/2}} \\
\leq c|\nabla u|_5|\rho u|_{20} \leq c|\nabla u|_5|\rho|^{39/40}_{39} |u|^{9/10}_\infty \left(\int \rho|u|^4 dx\right)^{1/40}.
\]

we know that
\[
|u|_\infty(s) \leq |u - \overline{u}|_\infty(s) + |\overline{u}|_\infty(s) \leq |\nabla u|_5(s) + \Phi(h(s)) \\
|\rho|_{39}(s) \leq \Psi(\int_0^t \Phi(h(s)) ds)
\]
\[
\sup_{0 \leq s \leq t} \int \rho|u|^4 dx(s) \leq \Psi(\int_0^t \Phi(h(s)) ds).
\]

Hence, we get
\[
\int_0^t |[u, R_i R_j](\rho u_i)|_\infty(s) ds \leq \int_0^t |\nabla u|_5^2(s) ds + \Psi \\
\leq c \int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P}u|_{15/8}^2 ds + \Psi \leq \Psi.
\]

and this implies
\[
\rho(t, x) \leq \Psi.
\]

With the estimate of $|\nabla u|_2(t)$, we conclude that
\[
h(t) \leq \Psi(\int_0^t \Phi(h(s)) ds)
\]
for some Lipschitz functions $\Psi$ and $\Phi$. Since $\Psi$ and $\Phi$ are Lipschitz, there is $T_0$ such that
\[
h(t) \leq C \quad \text{for all} \quad 0 \leq t \leq T_0.
\]

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