数理解析研究所講究録 1225 巻 2001 年 14-33

Asymptotic Profiles of Nonstationary Incompressible Navier-Stokes Flows in $\mathbb{R}^n$ and $\mathbb{R}^n_+$

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1. Introduction

We are interested in the large-time asymptotic behavior of weak and strong solutions of the Navier-Stokes system in the whole-space $\mathbf{R}^n$ and in the upper half-space $\mathbf{R}^n_+$, $n \geq 2$:

$$\begin{aligned} &\partial_t u + u \cdot \nabla u = \Delta u - \nabla p \quad (x \in D^n,\ t > 0) \\ &\nabla \cdot u = 0 \qquad\qquad\qquad\quad (x \in D^n,\ t \geq 0) \\ &u|_{t=0} = a. \end{aligned} \tag{NS}$$

Here, $D^n = \mathbf{R}^n$ or $\mathbf{R}^n_+$; and when $D^n = \mathbf{R}^n_+$, we impose the boundary condition

$$u|_{\partial R^n_+} = 0.$$

$u = (u^1, \cdots, u^n)$ and p denote, respectively, unknown velocity and pressure; a is a given initial velocity; and

$$\partial_t = \partial/\partial t, \qquad \nabla = (\partial_1, \cdots, \partial_n), \quad \partial_j = \partial/\partial x_j \quad (j = 1, \cdots, n),$$
$$\Delta u = \sum_{j=1}^n \partial_j^2 u, \qquad u \cdot \nabla u = \sum_{j=1}^n u^j \partial_j u, \qquad \nabla \cdot u = \sum_{j=1}^n \partial_j u^j.$$

We want to find asymptotic profiles of Navier-Stokes flows under some specific conditions on the initial velocities. In Section 2 we state our main results in $\mathbf{R}^n$. Our first result in $\mathbf{R}^n$ is that, as $t \to \infty$, the strong solution u admits an asymptotic expansion in terms of the space-time derivatives of Gaussian-like functions up to (and including) the order n, the space dimension, provided the initial velocity a satisfies appropriate decay conditions and moment conditions. This result, stated in Section 2, improves that of Carpio [2], in which is deduced the first-order asymptotics of two kinds, one in $\mathbf{R}^3$ and the other in $\mathbf{R}^2$. Our proof shows that one and the same result holds in all dimensions $n \geq 2$. Moreover, our argument utilizes only the Taylor expansion of smooth functions, elementary results on the Fourier transform, and (improvement of) decay results on the L^2-moments of solutions as given in [6], [14].

In Section 3 we state our result in $\mathbb{R}^n_+$. In this case our asymptotic expansion involves only the normal derivatives of Gaussian-like functions in contrast to the case of flows in $\mathbb{R}^n$; but the essential feature is the same. Namely, the functions describing the profiles are of the form $t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$, where K stands for some specific functions which are bounded and L^q-integrable for all $1 < q < \infty$. However, it should be emphasized here that in the case of flows in $\mathbb{R}^n$ the functions K are all in L^1, while this is not always true for flows in the half-space. This suggests that the Stokes semigroup over the half-space would never be bounded in L^1. We then apply our expansion result to the analysis of the modes of energy decay of Navier-Stokes flows in the half-space and prove in Corollary 3.8 a characterization of weak solutions which admit the lower bound of rates of energy decay of the form $\|u(t)\|_2 \geq ct^{-\frac{n+2}{4}}$. This result extends a result of [13] to flows in the half-space.

In Section 4 we sketch the proofs for flows in $\mathbb{R}^n$, and in Section 5 the proofs are outlined for flows in $\mathbb{R}^n_+$. We deduce our asymptotic expansion only for strong solutions. The full proofs are given in [3], [4].

2. Results for flows in $\mathbb{R}^n$

In this section, we consider the Navier-Stokes system in $\mathbb{R}^n$, $n \geq 2$, which will be treated in the form of the integral equation:

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla \cdot (u \otimes u)(s)ds. \tag{2.1}$$

Here, $A = -\Delta$ is the Laplacian, $\{e^{-tA}\}_{t\geq 0}$ is the heat semigroup, and P is the bounded projection [8] onto the spaces of solenoidal vector fields.

In dealing with strong solutions, we will always assume that the initial velocity a is solenoidal, bounded, smooth and satisfies

$$\int (1 + |y|)|a(y)|dy < \infty. \tag{2.2}$$

Assumption (2.2) implies $a \in L^1$; so the divergence-free condition ensures (see [11])

$$\int a(y)dy = 0. \tag{2.3}$$

As for the strong solutions, we know (see [8], [11]) that a unique strong solution u exists in general L^p-spaces, satisfying

$$\|\nabla^k u(t)\|_q \leq C(1+t)^{-\frac{k}{2}-\frac{n}{2}(1+\frac{1}{n}-\frac{1}{q})} \qquad (k = 0, 1, \quad 1 \leq q < \infty), \tag{2.4}$$

if a is bounded, smooth, small in L^n and satisfies (2.2). Hereafter, $\|\cdot\|_r$ denotes the L^r-norm and $\nabla = (\partial_1, \cdots, \partial_n)$, $\partial_j = \partial/\partial x_j$.

We first recall that the kernel function of the projection P onto the solenoidal fields is represented via the Fourier transform in the form $P(x) = (P_{jk}(x))_{j,k=1}^n$, with

$$\hat{P}_{jk}(\xi) \equiv \int e^{-ix\cdot\xi}P_{jk}(x)dx = \delta_{jk} + \frac{i\xi_j i\xi_k}{|\xi|^2} \qquad (i = \sqrt{-1}, \quad x\cdot\xi = \sum_{j=1}^n x_j\xi_j). \tag{2.5}$$

So, the kernel function $F_\ell = (F_{\ell,jk})_{j,k=1}^n$ of the operator $e^{-tA}P\partial_\ell$ has the Fourier transform

$$\widehat{F}_{\ell,jk}(\xi,t) = i\xi_\ell e^{-t|\xi|^2}(\delta_{jk} + i\xi_j i\xi_k/|\xi|^2) \equiv \widehat{F}^1_{\ell,jk}(\xi,t) + \widehat{F}^2_{\ell,jk}(\xi,t).$$

Denoting the heat kernel by

$$E_t(x) = (4\pi t)^{-\frac{n}{2}}\exp(-\tfrac{|x|^2}{4t})$$

and writing $\partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ for any multi-index $\alpha = (\alpha_1,\cdots,\alpha_n)$ of nonnegative integers, we easily see that $F^1_{\ell,jk}(x,t) = (\partial_\ell E_t)(x)\delta_{jk}$, and so

$$\|\partial_t^p\partial_x^\beta F^1_{\ell,jk}(\cdot,t)\|_q \le C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1\le q\le\infty). \tag{2.6}$$

To deduce similar estimates for $F^2_{\ell,jk}$, we invoke the relation $|\xi|^{-2} = \int_0^\infty e^{-s|\xi|^2}ds$, to get

$$\widehat{F}^2_{\ell,jk}(\xi,t) = i\xi_\ell i\xi_j i\xi_k\int_t^\infty e^{-s|\xi|^2}ds, \quad \text{so that} \quad \partial_x^\beta F^2_{\ell,jk}(x,t) = \int_t^\infty \partial_x^\beta\partial_\ell\partial_j\partial_k E_s(x)ds.$$

From this we easily obtain

$$\|\partial_t^p\partial_x^\beta F^2_{\ell,jk}(\cdot,t)\|_q \le C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1\le q\le\infty).$$

Combining this with (2.6) gives

$$\|\partial_t^p\partial_x^\beta F_{\ell,jk}(\cdot,t)\|_q \le C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1\le q\le\infty). \tag{2.7}$$

We now state the results for flows in $\mathbb{R}^n$. Here and in what follows the summation convention will be employed.

Theorem 2.1. (i) *Let a be bounded, smooth, solenoidal and satisfy* (2.2). *Let $u = (u_1,\cdots,u_n)$ be the corresponding strong solution of* (2.1). *For $1\le q\le\infty$ and $j = 1,\cdots,n$, we have*

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\left\|u_j(t) + (\partial_k E_t)(\cdot)\int y_k a_j(y)dy + F_{\ell,jk}(\cdot,t)\int_0^\infty\!\!\int (u_\ell u_k)(y,s)dyds\right\|_q = 0. \tag{2.8}$$

(ii) *Suppose $a = (a_1,\cdots,a_n)$ satisfies the following additional conditions:*

$$\begin{aligned} &\int |y|^m|a(y)|dy < \infty, && |a(y)|\le C(1+|y|)^{-n-1}, && a_j = \sum_{k=1}^n \partial_k b_{jk},\\ &|b_{jk}(y)|\le C(1+|y|)^{-n}, && b_{jk} \text{ are small in } L^1, && \end{aligned} \tag{2.9}$$

for some integer m with $1\le m\le n$. Then, for $1\le q\le\infty$ and $j = 1,\cdots,n$, we have

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})}\Bigg\|u_j(t) - \sum_{1\le|\alpha|\le m}\frac{(-1)^{|\alpha|}}{\alpha!}(\partial_x^\alpha E_t)(\cdot)\int y^\alpha a_j(y)dy \\ + \sum_{|\beta|+2p\le m-1}\frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(\cdot,t)\int_0^\infty\!\!\int s^p y^\beta(u_\ell u_k)(y,s)dyds\Bigg\|_q = 0.\end{aligned} \tag{2.10}$$

Theorem 2.2. (iii) *For every $a \in \boldsymbol{L}^2$ which is solenoidal and satisfies* (2.2), *there exists a weak solution u which admits the expansion* (2.8) *with $1 \leq q \leq 2$.*

(iv) *Let $n = 3, 4$, and suppose that*

$$\int (1+|y|)^{n-1}|a(y)|dy < \infty, \qquad \int (1+|y|)^{n}|a(y)|^2 dy < \infty. \tag{2.11}$$

Then there exists a weak solution u satisfying (2.10) *for $1 \leq q \leq 2$ and $1 \leq m \leq n-1$.*

Remarks. (i) (2.4) implies $\|u(s)\|_2^2 \leq C(1+s)^{-\frac{n}{2}-1}$, so the last integral in (2.8) is finite.

(ii) Convergence of the integrals in the second sum of (2.10) is ensured by the estimate:

$$\int |y|^m |u(y,s)|^2 dy \leq C(1+s)^{-\frac{n-m}{2}-1} \qquad (0 \leq m \leq n+1), \tag{2.12}$$

which holds if a satisfies (2.9). Estimate (2.12) is deduced in the following way: First, as shown in [11], assumption (2.2) implies $\|u(s)\|_1 \leq C(1+s)^{-\frac{1}{2}}$. Secondly, we know (see [12]) that (2.9) ensures $|y|^{n+1}|u(y,s)| \leq C$ for all $y \in \mathbb{R}^n$ and $s \geq 0$. Therefore,

$$\int |y|^{n+1}|u(y,s)|^2 dy \leq \sup_y(|y|^{n+1}|u(y,s)|) \int |u(y,s)|dy \leq C(1+s)^{-\frac{1}{2}}.$$

Since $\|u(s)\|_2^2 \leq C(1+s)^{-\frac{n}{2}-1}$ by (2.4), we get (2.12) via Hölder's inequality.

(iii) Theorem 2.1 improves an asymptotic result of Carpio [2] in the following sense. First, the result of [2] ignores the vanishing of the average (2.3), and so contains the trivial term $E_t(x)\int a(y)dy \equiv 0$. Secondly, [2] deals only with the case discussed in assertion (i) of Theorem 2.1, but the results given there are incomplete in the two-dimensional case.

(iv) The proof of Theorem 2.2 is almost the same as that of Theorem 2.1 (ii). It differs only in estimating the nonlinear convolution integral of (2.1) in a neighborhood of $s = t$. The restriction $m \leq n-1$ in assertion (iv) is needed since for weak solutions we can prove only that

$$\int |y|^m |u(y,s)|^2 dy \leq C(1+s)^{-(1+\frac{n}{2})(1-\frac{m}{n})} \qquad (0 \leq m \leq n, \quad n = 3,4), \tag{2.13}$$

which is weaker than (2.12). Estimate (2.13) is due to [6], [14].

3. Results for flows in $\mathbb{R}^n_+$

We first prepare a few specific properties of solutions $v = (v', v^n)$, $v' = (v^1, \cdots, v^{n-1})$, of the Stokes system

$$\begin{aligned} &\partial_t v = \Delta v - \nabla p && (x \in \mathbb{R}^n_+,\ t > 0) \\ &\nabla \cdot v = 0 && (x \in \mathbb{R}^n_+,\ t \geq 0) \\ &v|_{\partial R^n_+} = 0, \qquad v|_{t=0} = a. && \end{aligned} \tag{S}$$

Consider the Helmholtz decomposition ([1]):

$$\boldsymbol{L}^r(\mathbb{R}^n_+) \equiv (L^r(\mathbb{R}^n_+))^n = \boldsymbol{L}^r_\sigma \oplus \boldsymbol{L}^r_\pi, \qquad 1 < r < \infty,$$

$$L^r_\sigma = \{u \in L^r(\mathbb{R}^n_+) \,:\, \nabla \cdot u = 0, \quad u^n|_{\partial R^n_+} = 0\},$$
$$L^r_\pi = \{\nabla p \in L^r(\mathbb{R}^n_+) \,:\, p \in L^r_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})\},$$

and let $P = P_r$ be the associated bounded projection onto L^r_σ. Then problem (S) is written in the form

$$\partial_t v + A_r v = 0 \quad (t > 0), \qquad v(0) = a \in L^r_\sigma, \tag{S'}$$

in terms of the Stokes operator

$$A = A_r = -P\Delta, \quad D(A_r) = L^r_\sigma \cap \{u \in W^{2,r}(\mathbb{R}^n_+) \,:\, u|_{\partial R^n_+} = 0\}.$$

We know (see [1]) that $-A_r$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in L^r_σ so that for each $a \in L^r_\sigma$, the function $v(t) = (v', v^n) = e^{-tA}a$ gives a unique solution of (S′) in L^r_σ. Ukai [16] gave the following representation of the solution v:

$$v^n(t) = Ue^{-tB}[a^n - S \cdot a']; \qquad v'(t) = e^{-tB}[a' + Sa^n] - Sv^n. \tag{3.1}$$

Hereafter, $B = -\Delta$ denotes the Dirichlet-Laplacian on $\mathbb{R}^n_+$; $\{e^{-tB}\}_{t\geq 0}$ is the bounded analytic semigroup in L^p-spaces generated by $-B$; $S = (S_1, \cdots, S_{n-1})$ are the Riesz transforms on $\mathbb{R}^{n-1}$; and U is the bounded linear operator from $L^r(\mathbb{R}^n_+)$ to itself, $1 < r < \infty$, which is defined via the Fourier transform on $\mathbb{R}^{n-1}$ as

$$(\widehat{Uf})(\xi', x_n) = |\xi'| \int_0^{x_n} e^{-|\xi'|(x_n - y)} \hat{f}(\xi', y)dy. \tag{3.2}$$

As is well known, we have

$$e^{-tB}f = E_t * f^*\Big|_{R^n_+}, \tag{3.3}$$

for a function f defined on $\mathbb{R}^n_+$, where E_t is the heat kernel on $\mathbb{R}^n$ and f^* is the odd extension of the function f defined on $\mathbb{R}^n_+$:

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & (x_n > 0), \\ -f(x', -x_n) & (x_n < 0). \end{cases} \tag{3.4}$$

Let $\|\cdot\|_q$, $1 \leq q \leq \infty$, denote the norm of $L^q(\mathbb{R}^n_+)$. The following are the standard L^r-L^q estimates for the Stokes semigroup.

Proposition 3.1. *There hold the estimates*

$$\|\nabla^k e^{-tA}a\|_q \leq Ct^{-\frac{k}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})}\|a\|_r \tag{3.5}$$

with $k = 0, 1, 2 \ldots$, *provided either* $1 \leq r < q \leq \infty$, *or* $1 < r \leq q < \infty$.

Furthermore,

$$\|\nabla e^{-tA}a\|_r \leq Ct^{-\frac{1}{2}}\|a\|_r \qquad (r = 1, \infty). \tag{3.6}$$

Note that in (3.5) the exponents r and q may take on values 1 and ∞, respectively, although the Stokes semigroup would not be bounded in L^1, nor in L^∞. Estimates (3.5) are proved in [1]; and estimates (3.6) are proved in [5] for $r = 1$ and in [15] for $r = \infty$, respectively.

We further need the following estimates:

Proposition 3.2. *Let $a \in L^q_\sigma$ for some $1 < q < \infty$ and*

$$\int_{R^n_+} (1 + y_n)|a(y)|dy < \infty. \tag{3.7}$$

Then,

$$\begin{aligned} &\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \left(\|a\|_q + \int_{R^n_+} y_n |a(y)|dy \right) \\ &\|\nabla^j e^{-tA} a\|_r \le C t^{-\frac{1+j}{2}-\frac{n}{2}(1-\frac{1}{r})} \int_{R^n_+} y_n|a(y)|dy \qquad (1 < r \le \infty, \quad j = 0, 1). \end{aligned} \tag{3.8}$$

Proof. We use representation (3.1) for $e^{-tA}a$. It is easy to see that

$$e^{-tB} S \cdot a' = e^{t\Delta} S \cdot (a')^*,$$

where $e^{t\Delta}$ means convolution with the heat kernel on $\mathbb{R}^n$. The Fourier image of the kernel function of the convolution operator $e^{t\Delta}S$ is $e^{-t|\xi|^2} i\xi'/|\xi'|$, $\xi' = (\xi_1, \dots, \xi_{n-1})$. Inserting $|\xi'|^{-1} = \pi^{-1/2} \int_0^\infty \eta^{-\frac{1}{2}} e^{-\eta|\xi'|^2} d\eta$ gives

$$e^{-t|\xi|^2} i\xi'/|\xi'| = \pi^{-\frac{1}{2}} i\xi' e^{-t|\xi|^2} \int_0^\infty \eta^{-\frac{1}{2}} e^{-\eta|\xi'|^2} d\eta = \pi^{-\frac{1}{2}} e^{-t\xi_n^2} \int_0^\infty \eta^{-\frac{1}{2}} i\xi' e^{-(\eta+t)|\xi'|^2} d\eta.$$

Thus, the kernel function $F_t = (F_t^1, \cdots, F_t^{n-1})$ of $e^{t\Delta}S$ is

$$F_t(x) \equiv (e^{t\Delta}S)(x) = \pi^{-\frac{1}{2}} E_t(x_n) \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(x') d\eta, \tag{3.9}$$

where $\nabla' = (\partial_1, \cdots, \partial_{n-1})$ and $\Delta' = \sum_{j=1}^{n-1} \partial_j^2$. It is easy to see that

$$\|\partial_t^\ell \nabla^m F_t\|_p \le C t^{-\frac{m+2\ell}{2}-\frac{n}{2}(1-\frac{1}{p})} \qquad \text{for } 1 < p \le \infty \text{ and } \ell,\ m = 0, 1, \dots. \tag{3.10}$$

We now prove (3.8). In what follows integration with respect to the space variables will be performed on the whole space $\mathbb{R}^n$ unless otherwise specified. Suppose (3.7) holds. Since $\int_{-\infty}^\infty f^*(y', y_n) dy_n = 0$ for a.e. $y' \in \mathbb{R}^{n-1}$ whenever $f \in L^1(\mathbb{R}^n_+)$, direct calculation gives

$$\begin{aligned} e^{t\Delta}(a^n)^* &= \int E_t(x'-y')[E_t(x_n-y_n) - E_t(x_n)](a^n)^*(y)dy \\ &= -\int_0^1 \int y_n E_t(x'-y')(\partial_n E_t)(x_n - y_n\theta)(a^n)^*(y) dy d\theta. \end{aligned} \tag{3.11}$$

So, application of Minkowski's inequality for integrals yields

$$\|e^{-tB}a^n\|_q \le C\|E_t\|_q \|\partial_n E_t\|_q \int |y_n| \cdot |(a^n)^*(y)| dy \le C t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \int_{R^n_+} y_n |a(y)|dy.$$

Similarly, we get

$$\|e^{-tB} S \cdot a'\|_q \ \le \ C t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \int_{R^n_+} y_n|a(y)|dy.$$

Since U and S are bounded in $L^r(\mathbb{R}^n_+)$ for $1 < r < \infty$, these calculations imply that

$$\|e^{-tA}a\|_q \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_{R^n_+} y_n|a(y)|dy \qquad \text{for all } 1 < q < \infty.$$

On the other hand, we have $\|e^{-tA}a\|_q \leq C\|a\|_q$ by Proposition 3.1; so we obtain

$$\|e^{-tA}a\|_q \leq C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\left(\|a\|_q + \int_{R^n_+} y_n|a(y)|dy\right).$$

The above argument and Proposition 3.1 together yield

$$\|\nabla e^{-tA}a\|_r \leq Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_r \leq Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_{R^n_+} y_n|a(y)|dy \qquad \text{for all } 1 < r < \infty.$$

This proves (3.8) in case $1 < r < \infty$. When $r = \infty$, we apply Proposition 3.1 to get

$$\|e^{-tA}a\|_\infty \leq Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_n \leq Ct^{-\frac{1+n}{2}}\int_{R^n_+} y_n|a(y)|dy,$$
$$\|\nabla e^{-tA}a\|_\infty \leq Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_\infty \leq Ct^{-1-\frac{n}{2}}\int_{R^n_+} y_n|a(y)|dy.$$

This proves Proposition 3.2.

We write problem (NS) in the form of the integral equation

$$\begin{aligned} u(t) &= e^{-tA}a - \int_0^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds \\ &= e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds \end{aligned} \tag{IE}$$

with $u\otimes u = (u^ju^k)^n_{j,k=1}$, and discuss the existence of weak and strong solutions with specific decay properties that are needed in proving our main result.

We first deal with the weak solutions, which are known (see [1], [10]) to exist globally in time for all $a \in L^2_\sigma$, satisfying the identity:

$$\langle u(t), \varphi\rangle = \langle e^{-tA}a, \varphi\rangle + \int_0^t \langle u\otimes u, \nabla e^{-(t-s)A}\varphi\rangle ds \tag{3.12}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n_+)$ with $\nabla\cdot\varphi = 0$ and the energy inequality:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \leq \|a\|_2^2 \qquad \text{for all } t \geq 0. \tag{E}$$

Theorem 3.3. *Suppose $a \in L^2_\sigma$ satisfies*

$$\int_{R^n_+}(1+y_n)|a(y)|dy < \infty. \tag{3.13}$$

(i) *There exists a weak solution u, which is unique in case $n = 2$, such that*

$$\|u(t)\|_2 \leq C(1+t)^{-\frac{n+2}{4}}. \tag{3.14}$$

Furthermore, this weak solution satisfies

$$\|u(t)\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad \text{for all } 1 < q \le 2. \tag{3.15}$$

(ii) *When* $n = 3, 4$, *the weak solution* u *given in* (i) *is constructed via approximate solutions* $\{u_N\}$ *as given in* [1], [7], [13], *which satisfy*

$$\lim_{N\to\infty} \int_0^\infty \|u_N(t) - u(t)\|_2^2 dt = 0.$$

Proposition 3.2 implies $\|e^{-tA}a\|_2 \le C(1+t)^{-\frac{n+2}{4}}$, so the existence of a weak solution with decay property (3.14) is deduced in exactly the same way as in [1]. The assumption implies $a \in \boldsymbol{L}^q_\sigma$ for all $1 < q \le 2$; so Proposition 3.2 and (3.13) together imply

$$\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad \text{for all } 1 < q \le 2. \tag{3.16}$$

By using this, we can deduce assertion (3.15) in a standard manner.

To deal with strong solutions, note first that (IE) can be rewritten as

$$\begin{aligned} u(t) &= e^{-tA}a - \int_0^t e^{-(t-s)A} P\nabla\cdot(u\otimes u)(s)ds \\ &= e^{-tA/2}u(t/2) - \int_{t/2}^t e^{-(t-s)A} P\nabla\cdot(u\otimes u)(s)ds. \end{aligned}$$

We need this representation to prove the following

Theorem 3.4. *Let* $a \in \boldsymbol{L}^q_\sigma$ *for all* $1 < q < \infty$. *Given* $1 < p \le 2$, *there is a number* $\eta_p > 0$ *so that if* $\|a\|_n \le \eta_p$, *a unique strong solution* u *exists for all* $t \ge 0$, *satisfying* $u \in BC([0,\infty) : \boldsymbol{L}^q_\sigma)$ *for all* $p \le q < \infty$, *and*

$$\left.\begin{aligned} &\|u(t)\|_q \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \\ &\|\nabla u(t)\|_q \le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \end{aligned}\right. \qquad \text{for all } p \le q < \infty. \tag{3.17}$$

Theorem 3.4 is proved by following the argument given in [8], [9], [11]. Applying Proposition 3.2 and Theorem 3.4, we can deduce

Theorem 3.5. *Let* $a \in \boldsymbol{L}^1(\mathbb{R}^n_+) \cap \boldsymbol{L}^q_\sigma$ *for all* $1 < q < \infty$ *and satisfy* (3.13). *If* a *is small in* $\boldsymbol{L}^n_\sigma$, *the strong solution* u *given in Theorem* 3.4 *satisfies*

$$\left.\begin{aligned} &\|u(t)\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \\ &\|\nabla u(t)\|_q \le Ct^{-1-\frac{n}{2}(1-\frac{1}{q})} \end{aligned}\right. \qquad \text{for all } 1 < q < \infty. \tag{3.18}$$

To deduce our main result, we need some specific functions of (x,t) which will describe the profiles of general solutions as $t \to \infty$. We use one and the same notation E_t to denote simultaneously the heat kernel of one variable and several variables.

The following is the complete list of necessary functions.

$$
\begin{aligned}
&F_t(x) = \pi^{-\frac{1}{2}} E_t(x_n) \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(x')d\eta. \\
&E_{jk}(x,t) = \int_0^\infty \partial_j\partial_k\partial_n E_{\tau+t}(x)d\tau, \qquad j,k=1,\dots,n. \\
&F_{jk}(x,t) = \int_0^\infty \partial_j\partial_k\partial_n F_{\tau+t}(x)d\tau, \qquad j,k=1,\dots,n. \\
&G_{jk}(x,t) = \int_0^\infty\int_{-\infty}^\infty (\partial_j\partial_k\nabla' E_{\tau+t})(x')\,\mathrm{sgn}\,(z_n)\times \\
&\qquad\qquad \times E_t(x_n-z_n)E_\tau(z_n)dz_nd\tau \qquad (j,k\le n-1). \\
&G_{nn}(x,t) = \int_0^\infty\int_{-\infty}^\infty (\nabla' E_{\tau+t})(x')\,\mathrm{sgn}\,(z_n)\times \qquad\qquad (3.19)\\
&\qquad\qquad \times (\partial_n E_t)(x_n-z_n)(\partial_n E_\tau)(z_n)dz_nd\tau. \\
&H_{jk}(x,t) = -\pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty\int_{-\infty}^\infty \eta^{-\frac{1}{2}}(\partial_j\partial_k E_{\eta+\tau+t})(x')\,\mathrm{sgn}\,(z_n)\times \\
&\qquad\qquad \times (\partial_n E_t)(x_n-z_n)(\partial_n E_\tau)(z_n)dz_nd\eta d\tau \qquad (j,k\le n-1). \\
&H_{nn}(x,t) = \pi^{-\frac{1}{2}} \int_0^\infty\int_0^\infty\int_{-\infty}^\infty \eta^{-\frac{1}{2}}\Delta' E_{\eta+\tau+t}(x')\,\mathrm{sgn}\,(z_n)\times \\
&\qquad\qquad \times (\partial_n E_t)(x_n-z_n)(\partial_n E_\tau)(z_n)dz_nd\eta d\tau.
\end{aligned}
$$

Here $\mathrm{sgn}\,(z_n) = z_n/|z_n|$ for $z_n \neq 0$ and $\mathrm{sgn}\,(0) = 0$. The important fact is that all of the above functions except F_t are written in the form $K_t(x) \equiv t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$ in terms of some functions K which are bounded and L^p-integrable on $\mathbb{R}^n$ for all $1 < p < \infty$ together with their derivatives. So each K_t satisfies

$$\|\partial_t^\ell \nabla^m K_t\|_q = C_{q\ell} t^{-\frac{1+m+2\ell}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1 < q \le \infty,\ \ \ell,\ m = 0,1,2,\dots).$$

By using the functions listed above, we can prove

Proposition 3.6. (i) *Let $u = (u', u^n)$ be the strong solution given in Theorem 3.5. For all $1 < q < \infty$,*

$$
\begin{aligned}
\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| & u^n(t) + 2U\left[(\partial_n E_t)(\cdot)\int_{R_+^n} y_n a^n(y)dy - (\partial_n F_t)(\cdot)\cdot\int_{R_+^n} y_n a'(y)dy\right] \\
&+2U\left[(\partial_n E_t)(\cdot)\int_0^\infty\int_{R_+^n}|u^n|^2dyds - (\partial_n F_t)(\cdot)\cdot\int_0^\infty\int_{R_+^n}u^n u' dyds\right] \\
&+2U\left[\sum_{j,k=1}^{n-1} E_{jk}(\cdot,t)\int_0^\infty\int_{R_+^n}u^j u^k dyds + E_{nn}(\cdot,t)\int_0^\infty\int_{R_+^n}|u^n|^2dyds\right] \qquad (3.20)\\
&-2U\left[\sum_{j,k=1}^{n-1} H_{jk}(\cdot,t)\int_0^\infty\int_{R_+^n}u^j u^k dyds + H_{nn}(\cdot,t)\int_0^\infty\int_{R_+^n}|u^n|^2dyds\right]\Bigg\|_q = 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u'(t) &+ 2\left[(\partial_n F_t)(\cdot)\int_{R^n_+} y_n a^n(y)dy + (\partial_n E_t)(\cdot)\int_{R^n_+} y_n a'(y)dy\right] \\
&-2SU\left[(\partial_n E_t)(\cdot)\int_{R^n_+} y_n a^n(y)dy - (\partial_n F_t)(\cdot)\cdot\int_{R^n_+} y_n a'(y)dy\right] \\
&+2\left[(\partial_n E_t)(\cdot)\int_0^\infty\int_{R^n_+} u^n u' dyds + (\partial_n F_t)(\cdot)\int_0^\infty\int_{R^n_+} |u^n|^2 dyds\right] \\
&-2SU\left[(\partial_n E_t)(\cdot)\int_0^\infty\int_{R^n_+} |u^n|^2 dyds - (\partial_n F_t)(\cdot)\cdot\int_0^\infty\int_{R^n_+} u^n u' dyds\right] \\
&+2\left[\sum_{j,k=1}^{n-1} G_{jk}(\cdot,t)\int_0^\infty\int_{R^n_+} u^j u^k dyds + G_{nn}(\cdot,t)\int_0^\infty\int_{R^n_+} |u^n|^2 dyds\right] \\
&+2\left[\sum_{j,k=1}^{n-1} F_{jk}(\cdot,t)\int_0^\infty\int_{R^n_+} u^j u^k dyds + F_{nn}(\cdot,t)\int_0^\infty\int_{R^n_+} |u^n|^2 dyds\right] \\
&-2SU\left[\sum_{j,k=1}^{n-1} E_{jk}(\cdot,t)\int_0^\infty\int_{R^n_+} u^j u^k dyds + E_{nn}(\cdot,t)\int_0^\infty\int_{R^n_+} |u^n|^2 dyds\right] \\
&+2SU\left[\sum_{j,k=1}^{n-1} H_{jk}(\cdot,t)\int_0^\infty\int_{R^n_+} u^j u^k dyds + H_{nn}(\cdot,t)\int_0^\infty\int_{R^n_+} |u^n|^2 dyds\right]\Bigg\|_q = 0.
\end{aligned}
\tag{3.21}
$$

(ii) *The weak solutions* u *given in Theorem* 3.3 (ii) *satisfy* (3.20) *and* (3.21) *for* $1 < q \leq 2$.

Expansions (3.20) and (3.21) are simplified into the following form:

Theorem 3.7. (i) *For all* $1 < q < \infty$, *the strong solution* u *given in Theorem* 3.5 *satisfies*

$$
\begin{aligned}
\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u^n(t) &+ 2U\partial_n E_t(\cdot)\int_{R^n_+} y_n a^n(y)dy \\
&-2U\partial_n F_t(\cdot)\cdot\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty\int_{R^n_+} u^n u'(y,s)dyds\right)\Bigg\|_q = 0
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u'(t) &+ 2(\partial_n F_t(\cdot) - SU\partial_n E_t(\cdot))\int_{R^n_+} y_n a^n(y)dy \\
&+2\partial_n E_t(\cdot)\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty\int_{R^n_+} u^n u'(y,s)dyds\right) \\
&+2SU\partial_n F_t(\cdot)\cdot\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty\int_{R^n_+} u^n u'(y,s)dyds\right)\Bigg\|_q = 0.
\end{aligned}
\tag{3.23}
$$

(ii) *The weak solution* u *given in Theorem* 3.3 (ii) *satisfies* (3.22) *and* (3.23) *for* $1 < q \leq 2$.

Theorem 3.7 can be applied to characterizing flows with the lower bound of rates of energy decay. The result below extends a result of [13] to flows in the half-space.

Corollary 3.8. *The weak solution* u *given in Theorem* 3.3 (ii) *satisfies*

$$\|u(t)\|_2 \geq ct^{-\frac{n+2}{4}} \qquad \textit{for large } t > 0 \tag{3.24}$$

if and only if

$$\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty \int_{R^n_+} (u^n u')(y,s)dyds, \ \int_{R^n_+} y_n a^n(y)dy\right) \neq (0,0). \tag{3.25}$$

The characterization by [13] for flows in $\mathbb{R}^n$ involves all of the quantities $\int y_j a^k(y)dy$ and $\int_0^\infty \int (u^j u^k)(y,s)dyds$, while Corollary 3.8 shows that in characterizing flows in the half-space a distinguished role is played by the normal components a^n and u^n and the normal derivatives $\partial_n E_t$ and $\partial_n F_t$. Moreover, the integrals $\int_0^\infty \int_{R^n_+} (u^j u^k)(y,s)dyds$, $j,k = 1,\ldots,n-1$, and $\int_0^\infty \int_{R^n_+} (u^n u^n)(y,s)dyds$ do not appear in Corollary 3.8.

4. Proof of Theorem 2.1

We prove Theorem 2.1 by estimating linear and nonlinear terms separately.

Theorem 4.1. *Suppose* a *is solenoidal and satisfies*

$$\int (1+|y|)^m |a(y)|dy < \infty \tag{4.1}$$

for an integer $m \geq 1$. *Then for* $1 \leq q \leq \infty$,

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| e^{-tA}a - \sum_{1\leq|\alpha|\leq m} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha E_t)(\cdot) \int y^\alpha a(y)dy \right\|_q = 0. \tag{4.2}$$

Outline of Proof. Since a is solenoidal and integrable, it satisfies (2.3). So,

$$(e^{-tA}a)(x) = \int [E_t(x-y) - E_t(x)]a(y)dy.$$

Applying Taylor's theorem to $E_t(x-y)$ and estimating the remainder, we can prove (4.2).

Consider next the nonlinear term

$$w(t) = (w_1(t), \cdots, w_n(t)) = -\int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s)ds. \tag{4.3}$$

Theorem 4.2. *Under the assumption of Theorem* 2.1 (ii), *we have*

$$\begin{aligned} &\left\| w_j(t) + \sum_{|\beta|+2p\leq m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!} (\partial_t^p \partial_x^\beta F_{\ell,jk})(\cdot,t) \int_0^\infty \int s^p y^\beta (u_\ell u_k)(y,s)dyds \right\|_q \\ &= o(t^{-\frac{m}{2}-\frac{n}{2}(1-\frac{1}{q})}) \qquad \textit{as } t\to\infty, \end{aligned} \tag{4.4}$$

for all $1 \le q \le \infty$, $m = 1, \cdots, n$ *and* $j = 1, \cdots, n$.

Proof. By (4.3) and the definition of $F_{\ell,jk}$ given in Section 2, we have

$$w_j(t) = -\left(\int_0^{t/2} + \int_{t/2}^t\right)\int F_{\ell,jk}(x-y,t-s)(u_\ell u_k)(y,s)dyds \equiv J_1 + J_2$$

and

$$\|J_2\|_q \le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{2q}^2 ds = o(t^{-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{q})}) \qquad \text{as } t \to \infty.$$

To estimate J_1, we write

$$\begin{aligned} J_1 &= -\int_0^{t/2}\int F_{\ell,jk}(x-y,t-s)(u_\ell u_k)(y,s)dyds \\ &= -\sum_{|\beta|+2p\le m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(x,t)\int_0^{t/2}\int s^p y^\beta (u_\ell u_k)(y,s)dyds \\ &\qquad -\int_0^{t/2}\int R^m_{\ell,jk}(x,y,t,s)(u_\ell u_k)(y,s)dyds, \end{aligned}$$

obtaining

$$\begin{aligned} J_1 + &\sum_{|\beta|+2p\le m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(x,t)\int_0^{\infty}\int s^p y^\beta (u_\ell u_k)(y,s)dyds \\ &= \sum_{|\beta|+2p\le m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(x,t)\int_{t/2}^{\infty}\int s^p y^\beta (u_\ell u_k)(y,s)dyds \\ &\qquad\qquad -\int_0^{t/2}\int R^m_{\ell,jk}(x,y,t,s)(u_\ell u_k)(y,s)dyds \\ &\equiv J_{11} + J_{12}. \end{aligned} \tag{4.5}$$

Recall (2.6), (2.7) and (2.12), *i.e.*, that

$$\begin{aligned} &\|(\partial_t^p\partial_x^\beta F_{\ell,jk})(\cdot,t)\|_q \le C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})}, \\ &\int s^p|y|^{|\beta|}|u(y,s)|^2 dy \le C(1+s)^{-\frac{n-(|\beta|+2p)}{2}-1}. \end{aligned} \tag{4.6}$$

If $|\beta| + 2p \le m-1$, each term of J_{11} behaves in L^q like $t^{-\frac{n+1}{2}-\frac{n}{2}(1-\frac{1}{q})}$ as $t \to \infty$; hence

$$t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})}\|J_{11}\|_q \le Ct^{-\frac{n-m+1}{2}} \le Ct^{-\frac{1}{2}} \to 0 \qquad \text{as } t\to\infty.$$

So we need only estimate J_{12}, which is a finite linear combination of terms of the form

$$\int_0^1\int_0^{t/2}\int (1-\theta)^{|\beta|+p-1}(\partial_t^p\partial_x^\beta F_{\ell,jk})(x-y\theta,t-s\theta)s^p y^\beta (u_\ell u_k)(y,s)dydsd\theta,$$

with $|\beta| + 2p = m$, and this term is written as

$$
\begin{aligned}
&= (\partial_t^p \partial_x^\beta F_{\ell,jk})(x,t) \int_0^1 \int_0^{t/2} \int (1-\theta)^{|\beta|+p-1} s^p y^\beta (u_\ell u_k)(y,s)\,dy\,ds\,d\theta \\
&\quad + \int_0^1 \int_0^{t/2} \int (1-\theta)^{|\beta|+p-1} [(\partial_t^p \partial_x^\beta F_{\ell,jk})(x-y\theta, t-s\theta) - (\partial_t^p \partial_x^\beta F_{\ell,jk})(x, t-s\theta)] \times \\
&\qquad\qquad \times s^p y^\beta (u_\ell u_k)(y,s)\,dy\,ds\,d\theta \\
&\quad + \int_0^1 \int_0^{t/2} \int (1-\theta)^{|\beta|+p-1} [(\partial_t^p \partial_x^\beta F_{\ell,jk})(x, t-s\theta) - (\partial_t^p \partial_x^\beta F_{\ell,jk})(x,t)] \times \\
&\qquad\qquad \times s^p y^\beta (u_\ell u_k)(y,s)\,dy\,ds\,d\theta \\
&\equiv R_1 + R_2 + R_3.
\end{aligned}
$$

So the proof will be complete if we show that

$$
\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \|R_k\|_q = 0 \qquad (k = 1,2,3, \quad 1 \le q \le \infty). \tag{4.7}
$$

Since $m \le n$, for R_1 we have

$$
t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \|R_1\|_q \le C_q t^{-\frac{1}{2}} \int_0^{t/2} (1+s)^{-\frac{n-m}{2}-1} ds \le C_q t^{-\frac{1}{2}} \int_0^t (1+s)^{-1} ds \to 0
$$

as $t \to \infty$. To estimate R_2, note that we can write $\partial_t^p \partial_x^\beta F_{\ell,jk} = t^{-\frac{n+1+m}{2}} K(xt^{-\frac{1}{2}})$ in terms of a function K which is bounded, integrable, and uniformly continuous over $\mathbb{R}^n$. Hence, denoting

$$
\varphi_t(y,s,\theta) = \|K(\cdot - y\theta(t-s\theta)^{-\frac{1}{2}}) - K(\cdot)\|_q
$$

and invoking the boundedness of φ_t, we get

$$
t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \|R_2\|_q \le C_q t^{-\frac{1}{2}} \int_0^{t/2} \int s^p |y|^{|\beta|} |u(y,s)|^2 dy\,ds \le C_q t^{-\frac{1}{2}} \int_0^t (1+s)^{-\frac{n-m}{2}-1} ds,
$$

so that, since $m \le n$,

$$
t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \|R_2\|_q \le C t^{-\frac{1}{2}} \int_0^t (1+s)^{-1} ds \to 0 \qquad \text{as } t \to \infty.
$$

Finally, we write R_3 in the form

$$
R_3 = \int_0^1 \int_0^1 \int_0^{t/2} \int \theta (\partial_t^{p+1} \partial_x^\beta F_{\ell,jk})(x, t-s\theta\tau) s^{p+1} y^\beta (u_\ell u_k)(y,s)\,dy\,ds\,d\theta\,d\tau.
$$

Since $|\beta| + 2p = m \le n$, we have

$$
t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \|R_3\|_q \le C t^{-\frac{3}{2}} \int_0^{t/2} (1+s)^{-\frac{n-m}{2}} ds \le C t^{-\frac{1}{2}} \to 0
$$

as $t \to \infty$. This completes the proof of (4.7) and so Theorem 4.2 is proved.

5. Proof of Proposition 3.6, Theorem 3.7 and Corollary 3.8

We first deduce Corollary 3.8 from Theorem 3.7, and then Theorem 3.7 from Proposition 3.6, and finally, we give an outline of the proof of Proposition 3.6.

Proof of Corollary 3.8. In view of (3.2), the functions $U\partial_n E_t$ and $U\partial_n F_t^j$, $j = 1, \cdots, n-1$, have the form $t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$, so

$$\|U\partial_n E_t\|_2^2 = C_1 t^{-\frac{n+2}{2}} > 0, \qquad \|U\partial_n F_t^j\|_2^2 = C_2 t^{-\frac{n+2}{2}} > 0.$$

We easily see that $U\partial_n E_t$ is an even function of x', and $U\partial_n F_t^j$ are odd functions of x'. Furthermore, let $j \leq n-1$, $k \leq n-1$ and $j \neq k$. Then $U\partial_n F_t^j$ is odd in x_j and even in x_k, while $U\partial_n F_t^k$ is odd in x_k and even in x_j. Therefore,

$$\begin{aligned} &(U\partial_n E_t, U\partial_n F_t^j) = 0, \qquad j = 1, \ldots, n-1, \\ &\|U\partial_n F_t^1\|_2^2 = \ldots = \|U\partial_n F_t^{n-1}\|_2^2, \\ &(U\partial_n F_t^j, U\partial_n F_t^k) = \delta_{jk}\|U\partial_n F_t^j\|_2^2, \end{aligned} \tag{5.1}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{R}_+^n)$. Using (5.1) we see that if we set

$$\alpha = 2\int_{R_+^n} y_n a^n(y)dy, \quad \beta = 2\int_{R_+^n} y_n a'(y)dy, \quad \gamma = 2\int_0^\infty\int_{R_+^n} (u^n u')(y,s)dyds,$$

then

$$\begin{aligned} \|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2^2 &= \|U\partial_n E_t\|_2^2\alpha^2 + \|U\partial_n F_t^1\|_2^2|\beta+\gamma|^2 \\ &= Ct^{-\frac{n+2}{2}}. \end{aligned} \tag{5.2}$$

We shall apply (5.2) to the proof of Corollary 3.8. Firstly, suppose that $(\beta+\gamma,\alpha) \neq (0,0)$. Then (5.2) implies

$$\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 = Ct^{-\frac{n+2}{4}} > 0 \qquad \text{for all } t > 0;$$

so (3.22) yields, for large $t > 0$,

$$\begin{aligned} \|u^n(t)\|_2 &\geq \|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\quad - \|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &= Ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) \geq ct^{-\frac{n+2}{4}}. \end{aligned}$$

Secondly, suppose that $\|u^n(t)\|_2 \geq ct^{-\frac{n+2}{4}}$ for large $t > 0$. Then (3.22) implies

$$\begin{aligned} &\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\quad \geq \|u^n(t)\|_2 - \|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\quad \geq ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) > 0 \end{aligned}$$

for large $t > 0$, and so we conclude that $(\beta + \gamma, \alpha) \neq (0,0)$. Suppose finally that

$$\liminf_{t\to\infty} t^{\frac{n+2}{4}} \|u^n(t)\|_2 = 0 \qquad \text{and} \qquad \|u(t)\|_2 \geq ct^{-\frac{n+2}{4}}. \tag{5.3}$$

In this case we invoke

$$\begin{aligned} C &= t^{\frac{n+2}{4}} \|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\leq t^{\frac{n+2}{4}} \|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 + t^{\frac{n+2}{4}}\|u^n(t)\|_2. \end{aligned}$$

Passing to the limit as $t \to \infty$ and applying (3.22) and (5.3) gives $C = 0$, since

$$\liminf_{t\to\infty}[f(t) + g(t)] = \lim_{t\to\infty} f(t) + \liminf_{t\to\infty} g(t).$$

This implies that $(\beta + \gamma, \alpha) = (0,0)$, so (3.22) and (3.23) together yield

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \|u(t)\|_2 = 0,$$

contradicting the assumption (5.3). Hence $\|u(t)\|_2 \geq ct^{-\frac{n+2}{4}}$ implies $\|u^n(t)\|_2 \geq ct^{-\frac{n+2}{4}}$, and so we get $(\beta+\gamma,\alpha) \neq (0,0)$. This completes the proof of Corollary 3.8.

Proof of Theorem 3.7. Let

$$c^{jk} = \int_0^\infty \int_{R^n_+} (u^j u^k)(y,s)\,dyds,$$

where $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$, or $j = k = n$. By (3.20) and (3.21), it suffices to show that

$$\partial_n E_t c^{nn} + E_{jk}c^{jk} + E_{nn}c^{nn} - (H_{jk}c^{jk} + H_{nn}c^{nn}) = 0 \tag{5.4}$$

and

$$\partial_n F_t c^{nn} + G_{jk}c^{jk} + G_{nn}c^{nn} + F_{jk}c^{jk} + F_{nn}c^{nn} = 0. \tag{5.5}$$

Here, and in what follows, we will employ the summation convention for repeated indices with respect to $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$. We apply the Fourier transform with respect to $x' \in \mathbb{R}^{n-1}$ to the left-hand side of (5.4); then multiply the resulting function by $|\xi'|e^{t|\xi'|^2}$; and use $\partial_\tau E_\tau(x_n) = \partial_n^2 E_\tau(x_n)$, to get

$$\begin{aligned} &|\xi'|(c^{nn}|\xi'|^2 - c^{jk}\xi_j\xi_k)\int_0^\infty e^{-\tau|\xi'|^2}\partial_n E_{\tau+t}(x_n)d\tau \\ &\quad + (c^{nn}|\xi'|^2 - c^{jk}\xi_j\xi_k)\int_0^\infty e^{-\tau|\xi'|^2}\int_{-\infty}^\infty \operatorname{sgn}(z_n)\partial_n E_t(x_n - z_n)\partial_n E_\tau(z_n)dz_n d\tau. \end{aligned} \tag{5.6}$$

We regard the above function as an odd function of $x_n \in \mathbb{R}$ and apply the Fourier transform with respect to x_n, to conclude by direct calculation that the function in (5.6) vanishes identically. This proves (5.4). The proof of (5.5) is almost the same as that of (5,4).

We next prove Proposition 3.6. We begin by establishing

Theorem 5.1. *Let $a \in L^q_\sigma(\mathbb{R}^n_+)$ satisfy* (3.7). *Then, for $v = (v', v^n) = e^{-tA}a$ we get*

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\left\| v^n(t) + 2U\left[(\partial_n E_t)(\cdot)\int_{R^n_+} y_n a^n(y)dy - (\partial_n F_t)(\cdot)\cdot\int_{R^n_+} y_n a'(y)dy\right]\right\|_q \to 0$$

$$\begin{aligned} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Bigg\| v'(t) &+ 2\left[(\partial_n E_t)(\cdot)\int_{R^n_+} y_n a'(y)dy + (\partial_n F_t)(\cdot)\int_{R^n_+} y_n a^n(y)dy\right] \\ &- 2SU\left[(\partial_n E_t)(\cdot)\int_{R^n_+} y_n a^n(y)dy - (\partial_n F_t)(\cdot)\cdot\int_{R^n_+} y_n a'(y)dy\right]\Bigg\|_q \to 0 \end{aligned}$$

as $t \to \infty$. Here, F_t is the function given in (3.9).

Proof. We can rewrite (3.11) in the form

$$\begin{aligned} e^{t\Delta}(a^n)^* &= -2(\partial_n E_t)(x)\int_{R^n_+} y_n a^n(t)dy \\ &\quad - \int_0^1\!\!\int y_n E_t(x'-y')[(\partial_n E_t)(x_n - y_n\theta) - (\partial_n E_t)(x_n)](a^n)^*(y)dyd\theta \\ &\quad - (\partial_n E_t)(x_n)\int y_n[E_t(x'-y') - E_t(x')](a^n)^*(y)dy, \end{aligned}$$

since $\int_{-\infty}^{\infty} y_n(a^n)^*(y)dy_n = 2\int_0^\infty y_n a^n(y)dy_n$. Thus, for $1 < q < \infty$, we obtain

$$\begin{aligned} &\left\| Ue^{-tB}a^n + 2U(\partial_n E_t)(\cdot)\int_{R^n_+} y_n a^n(y)dy\right\|_q \\ &\le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_0^1\!\!\int \|(\partial_n E_1)(\cdot - y_n t^{-\frac{1}{2}}\theta) - (\partial_n E_1)(\cdot)\|_q |y_n|\cdot|(a^n)^*(y)|dyd\theta \\ &\quad + Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int \|E_1(\cdot - y't^{-\frac{1}{2}}) - E_1(\cdot)\|_q |y_n|\cdot|(a^n)^*(y)|dy. \end{aligned}$$

But, $\|(\partial_n E_1)(\cdot - y_n t^{-\frac{1}{2}}\theta) - (\partial_n E_1)(\cdot)\|_q$ and $\|E_1(\cdot - y't^{-\frac{1}{2}}) - E_1(\cdot)\|_q$ are bounded and

$$\lim_{t\to\infty}\|E_1(\cdot - y't^{-\frac{1}{2}}) - E_1(\cdot)\|_q = \lim_{t\to\infty}\|(\partial_n E_1)(\cdot - y_n t^{-\frac{1}{2}}\theta) - (\partial_n E_1)(\cdot)\|_q = 0$$

for any fixed y and θ. So the dominated convergence theorem gives

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\left\| Ue^{-tB}a^n + 2U(\partial_n E_t)(\cdot)\int_{R^n_+} y_n a^n(y)dy\right\|_q = 0.$$

Similarly, we rewrite $e^{t\Delta}(S\cdot a')^*$ as

$$\begin{aligned} e^{t\Delta}(S\cdot a')^* &= -2(\partial_n F_t)(x)\cdot\int_{R^n_+} y_n a'(y)dy \\ &\quad - \pi^{-\frac{1}{2}}\int_0^1\!\!\int\!\!\int_0^\infty y_n\eta^{-\frac{1}{2}}[(\partial_n E_t)(x_n - y_n\theta) - (\partial_n E_t)(x_n)]\times \\ &\qquad\qquad \times\nabla' E_{\eta+t}(x'-y')\cdot(a')^*(y)d\eta dyd\theta \\ &\quad - \pi^{-\frac{1}{2}}(\partial_n E_t)(x_n)\int\!\!\int_0^\infty y_n\eta^{-\frac{1}{2}}\nabla'[E_{\eta+t}(x'-y') - E_{\eta+t}(x')]\cdot(a')^*(y)d\eta dy \end{aligned}$$

and apply

$$\lim_{t\to\infty} \|(\partial_n E_1)(\cdot - y_n t^{-\frac{1}{2}}\theta) - (\partial_n E_1)(\cdot)\|_q = 0$$

$$\lim_{t\to\infty} \|(\nabla' E_1)(\cdot - y'(\eta + t)^{-\frac{1}{2}}) - (\nabla' E_1)(\cdot)\|_q = 0,$$

to obtain

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| Ue^{-tB}S\cdot a' + 2U(\partial_n F_t)(\cdot)\cdot \int_{R^n_+} y_n a'(y)dy \right\|_q = 0.$$

The other terms of formula (3.1) are similarly estimated, and we obtain the desired result. This proves Theorem 5.1.

Let u be the strong solution given in Theorem 3.5. We write the nonlinear term of (IE) as

$$-\left(\int_0^{t/2} + \int_{t/2}^t\right) e^{-(t-s)A} P\nabla\cdot(u\otimes u)(s)ds.$$

By (3.18) and the boundedness of $A^{-\frac{1}{2}}P\nabla\cdot$, the second term is estimated in L^q_σ as

$$\le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u(s)\|^2_{2q}ds \le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}(1+s)^{-1-n(1-\frac{1}{2q})}ds = o(t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})})$$

as $t\to\infty$. Therefore, in view of Theorem 5.1, we need only estimate the function

$$\begin{aligned} w(t) &= -\int_0^{t/2} e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds \\ &= w_1(t) + w_2(t) = (w_1'(t), w_1^n(t)) + (w_2'(t), w_2^n(t)) \end{aligned}$$

where $u\otimes u = (u^j u^k)^n_{j,k=1}$ and

$$\begin{aligned} w_1^n(t) &= -\int_0^{t/2} Ue^{-(t-s)B}[\nabla\cdot(uu^n) - S\cdot(\nabla\cdot(uu'))](s)ds \\ w_1'(t) &= -\int_0^{t/2} e^{-(t-s)B}[(\nabla\cdot(uu') + S\nabla\cdot(uu^n))](s)ds - Sw_1^n(t) \\ w_2^n(y) &= -\int_0^{t/2} Ue^{-(t-s)B}[\partial_n N(\partial_j\partial_k(u^ju^k)) - S\cdot(\nabla' N(\partial_j\partial_k(u^ju^k)))](s)ds \\ w_2'(t) &= -\int_0^{t/2} e^{-(t-s)B}[(\nabla' N(\partial_j\partial_k(u^ju^k)) + S\partial_n N(\partial_j\partial_k(u^ju^k))](s)ds - Sw_2^n(t). \end{aligned} \tag{5.8}$$

Here, $g = Nf$ denotes the solution of the Neumann problem

$$-\Delta g = f \quad \text{in } \mathbb{R}^n_+; \qquad \partial_n g|_{\partial R^n_+} = 0.$$

Since $u = 0$ on $\partial\mathbb{R}^n_+$, by using the summation convention we have

$$P\nabla\cdot(u\otimes u) = \partial_j(u^j u) + \nabla N(\partial_j\partial_k(u^ju^k)). \tag{5.9}$$

Let Q_n be the fundamental solution of $-\Delta$. We easily see that

$$Nf = Q_n f_*|_{R^n_+} \equiv Q_n * f_*|_{R^n_+}, \qquad f_*(x', x_n) = \begin{cases} f(x', x_n) & (x_n > 0) \\ f(x', -x_n) & (x_n < 0). \end{cases}$$

The integrals in (5.8) are estimated by applying

Lemma 5.2. *Let $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n$, $t > 0$, and consider the function*

$$K(x,y,t) = t^{-\frac{n+1}{2}} K^0(xt^{-\frac{1}{2}}, yt^{-\frac{1}{2}}),$$

where $K^0(\xi,\eta)$ is smooth and satisfies $\|\nabla^m_{\xi,\eta} K^0(\cdot,\eta)\|_q \le C_{q,m}$ for all $m = 0,1,2,\ldots$, all $\eta \in \mathbb{R}^n$ and some $1 < q \le \infty$. Then

$$\|\partial_t^\ell \nabla^m_{x,y} K(\cdot,y,t)\|_q \le C_{q,\ell,m} t^{-\frac{1+2\ell+m}{2} - \frac{n}{2}(1-\frac{1}{q})}, \qquad \ell, m = 0,1,2,\ldots, \tag{5.10}$$

for all $y \in \mathbb{R}^n$. Moreover, if we set

$$I_*(x,t) = \int_0^{t/2}\!\!\int K(x,y,t-s)(u\otimes u)_*(y,s)dyds,$$

$$I^*(x,t) = \int_0^{t/2}\!\!\int K(x,y,t-s)(u\otimes u)^*(y,s)dyds,$$

with u the strong solutions given in Theorem 3.5, then

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| I_*(t) - 2K(\cdot,0,t)\int_0^\infty\!\!\int_{R^n_+} (u\otimes u)(y,s)dyds \right\|_q = 0 \tag{5.11}$$

and

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \|I^*(t)\|_q = 0. \tag{5.12}$$

Via complicated calculation we can show that the kernel function of each term of (5.8) has the properties of function $K(x,y,t)$ treated above. So we can prove (3.20) and (3.21) by applying Lemma 5.2. The details are given in [4].

Proof of Lemma 5.2. We here prove only (5.11), since (5.12) is proved similarly and (5.10) is directly verified. We can write

$$\begin{aligned} I_*(x,t) &= 2K(x,0,t)\int_0^\infty\!\!\int_{R^n_+} (u\otimes u)(y,s)dyds - K(x,0,t)\int_{t/2}^\infty\!\!\int (u\otimes u)_*(y,s)dyds \\ &\quad + \int_0^{t/2}\!\!\int [K(x,y,t-s) - K(x,0,t-s)](u\otimes u)_*(y,s)dyds \\ &\quad + \int_0^{t/2}\!\!\int [K(x,0,t-s) - K(x,0,t)](u\otimes u)_*(y,s)dyds \\ &\equiv 2K(x,0,t)\int_0^\infty\!\!\int_{R^n_+} (u\otimes u)(y,s)dyds + I_1 + I_2 + I_3. \end{aligned}$$

We easily see that $\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I_1\|_q = 0$. Since $t - s \ge t/2$ if $0 \le s \le t/2$, application of Minkowski's inequality for integrals and a change of variables gives

$$\begin{aligned} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I_2\|_q &\le C\int_0^{t/2}\!\!\int \|K^0(\cdot, y(t-s)^{-\frac{1}{2}}) - K^0(\cdot,0)\|_q |u_*(y,s)|^2 dyds \\ &\equiv C\int_0^{t/2}\!\!\int \varphi_t(y,s)dyds \equiv C\int_0^{t/2} \psi_t(s)ds. \end{aligned}$$

By the assumption on K^0, the function $\varphi_t(y,s) = \|K^0(\cdot, y(t-s)^{-\frac{1}{2}}) - K^0(\cdot,0)\|_q |u_*(y,s)|^2$ satisfies $0 \le \varphi_t(y,s) \le C|u_*(y,s)|^2$. Furthermore, $\lim_{t\to\infty} \varphi_t(y,s) = 0$. Indeed, from the relation $K^0(x, y(t-s)^{-\frac{1}{2}}) - K^0(x,0) = y(t-s)^{-\frac{1}{2}} \cdot \int_0^1 (\nabla_y K^0)(x, y(t-s)^{-\frac{1}{2}}\theta)d\theta$ and the assumption on K^0, we obtain the pointwise convergence

$$\varphi_t(y,s) \le |y|(t-s)^{-\frac{1}{2}} \sup_z \|\nabla_z K^0(\cdot,z)\|_q |u_*(y,s)|^2 \le C|y|(t-s)^{-\frac{1}{2}}|u_*(y,s)|^2 \to 0$$

as $t \to \infty$. Since $|u_*(y,s)|^2$ is integrable in $y \in \mathbb{R}^n$ for fixed s, the dominated convergence theorem gives $\lim_{t\to\infty} \psi_t(s) = \lim_{t\to\infty} \int \varphi_t(y,s)dy = 0$. Since $\psi_t(s) \le C\|u(s)\|_2^2$, we get

$$\lim_{t\to\infty} \int_0^T \psi_t(s)ds = 0 \qquad \text{for each fixed } T > 0.$$

Now, given an $\varepsilon > 0$, choose $T > 0$ so that $\int_T^\infty \|u(s)\|_2^2 ds < \varepsilon$. For $t > 2T$, we have

$$\int_0^{t/2} \psi_t(s)ds \le \int_0^T \psi_t(s)ds + C\int_T^\infty \|u(s)\|_2^2 ds \le \int_0^T \psi_t(s)ds + C\varepsilon.$$

Hence, $\limsup_{t\to\infty} \int_0^{t/2} \psi_t(s)ds \le C\varepsilon$, and this proves $\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I_2\|_q = 0$. To estimate I_3, note that $K(x,0,t) - K(x,0,t-s) = -\int_0^1 s(\partial_t K)(x,0,t-s\theta)d\theta$; so for $0 \le s \le t/2$,

$$\|K(\cdot,0,t) - K(\cdot,0,t-s)\|_q \le C\int_0^1 s(t-s\theta)^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{q})}d\theta \le Cst^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{q})}.$$

Since $\|u(s)\|_2^2 \le C(1+s)^{-1-\frac{n}{2}}$ and $n \ge 2$, it follows that

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|I_3\|_q \le Ct^{-1}\int_0^{t/2} s\|u(s)\|_2^2 ds \le Ct^{-1}\int_0^t (1+s)^{-1}ds \to 0$$

as $t \to \infty$. This proves Lemma 5.2.

Acknowledgments. FUJIGAKI is partially supported by Japan Scholarship Foundation. MIYAKAWA is partially supported by Grant-in-Aid for Scientific Research, No. 10440045, No. 12874027, from the Japan Society for the Promotion of Science.

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