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A Study of the Relativistic Euler Equation

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1 Introduction

In this article we study the Cauchy problem to the one-dimensional relativistic Euler equation

\[
\frac{\partial}{\partial t} \frac{\rho + Pu^2/c^4}{1 - u^2/c^2} + \frac{\partial}{\partial x} \frac{(\rho + P/c^2)u}{1 - u^2/c^2} = 0,
\]

\[
\frac{\partial}{\partial t} \frac{(\rho + P/c^2)u}{1 - u^2/c^2} + \frac{\partial}{\partial x} \frac{P + \rho u^2}{1 - u^2/c^2} = 0,
\]

\[\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x).\]  

(1.1)

Here \(c\) is a positive constant, the speed of light, and \(P\) is a given function of \(\rho\). The equation (1.1) governs the one dimensional motion of a perfect gas in the Minkowski space-time. When \(c \to \infty\), (1.1) tends to the usual Euler equation of gas dynamics

\[\rho_t + (\rho u)_x = 0,\]

\[(\rho u)_t + (P + \rho u^2)_x = 0.\]  

(1.3)

Many mathematical investigations for this non-relativistic Euler equation were done. But the first mathematical investigation for the relativistic Euler equation (1.1) was done recently by Smoller and Temple [6]. They assume \(P = \sigma^2 \rho\), where \(\sigma\) is a positive constant < \(c\). Under this assumption, they showed that if the initial data \(\rho_0(x)\) and \(u_0(x)\) satisfy

\[T.V. \log \rho_0 < \infty, \quad T.V. \frac{c + u_0}{c - u_0} < \infty,\]

They showed that

\[T.V. \log \rho_0 < \infty, \quad T.V. \frac{c + u_0}{c - u_0} < \infty,\]
then there exists a global weak solution to the Cauchy problem (1.1)(1.2). The result was obtained by Glimm's scheme and it is the relativistic version of Nishida's result [5] for the non-relativistic problem.

However we would like to consider a more realistic equation of states. We keep in mind the equation of state for a neutron stars, which is given by

\begin{align*}
P &= Kc^5 f(y), \\
\rho &= Kc^3 g(y) \\
f(y) &= \int_0^y \frac{q^4}{\sqrt{1+q^2}} dq, \\
g(y) &= 3 \int_0^y q^2 \sqrt{1+q^2} dq.
\end{align*}

For this equation of state, we have \( P \sim \frac{c^2}{3} \rho \) as \( \rho \to \infty \) but \( P \sim \frac{1}{5} K^{2/3} \rho^{5/3} \) as \( \rho \to 0 \). So we assume the following properties of the function \( P(\rho) \):

(A): 

\[ P(\rho) > 0, \quad 0 < dP/d\rho < c^2, \quad 0 < d^2 P/d\rho^2 \]

for \( \rho > 0 \), and

\[ P = A \rho^\gamma (1 + [\rho^{\gamma-1}/c^2]_1) \]

as \( \rho \to 0 \). Here \( A \) and \( \gamma \) are positive constants and

\[ \gamma = 1 + \frac{2}{2N+1}, \]

\( N \) being a positive integer, and \([X]_1\) denotes a convergent power series of the form \( \sum_{k \geq 1} a_k X^k \).

The result which we want to generalize to the relativistic problem is those by G.-Q. Chen et al [2]. So we assume that the initial data \( \rho_0(x), \ u_0(x) \) satisfy

\[ 0 \leq \rho_0(x) \leq M_0, \quad \left| \frac{c}{2} \log \frac{c+u_0(x)}{c-u_0(x)} \right| \leq M_0. \]

A weak solution of (1.1)(1.2) is defined as follows.

We write

\begin{align*}
E &= \frac{\rho + Pu^2/c^4}{1 - u^2/c^2}, \\
F &= \frac{(\rho + P/c^2)u}{1 - u^2/c^2}, \\
G &= \frac{P + \rho u^2}{1 - u^2/c^2}, \\
U &= (E, F)^T, \quad f(U) = (F, G)^T.
\end{align*}

Then (1.1) can be written as

\[ U_t + f(U)_x = 0. \]
Let us denote by $U_0(x)$ the initial data. Then a weak solution $U(t, x)$ is a bounded measurable function which satisfies
\[
\int \int (U \Phi_t + f(U) \Phi_x)dxdt + \int U_0(x) \Phi(0, x)dx = 0
\]
for any test function $\Phi \in C_0^\infty([0, +\infty) \times R)$.

2 Riemann problems

The Riemann problem is the problem to the special initial data of the form
\[
U_0(x) = \begin{cases} 
    U_L & \text{if } x < 0 \\
    U_R & \text{if } x > 0 
\end{cases}
\]
In order to solve this we introduce the Riemann invariants
\[
w = x + y, \quad z = x - y
\]
where
\[
x = \frac{c}{2} \log \frac{c+u}{c-u}, \quad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho.
\]
Then (1.1) is diagonalized as
\[
w_t + \lambda_2 w_x = 0, \quad z_t + \lambda_1 z_x = 0,
\]
where
\[
\lambda_1 = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}, \quad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \sqrt{P'}u/c^2}.
\]
the possible states $U = U_R$ connected to $U_L$ on the right by rarefaction waves are
\[
R_1: \quad w = w_L, z > z_L
\]
and
\[
R_2: \quad w > w_L, z = z_L.
\]
The Rankine-Hugoniot jump condition
\[
\sigma[U] = [f(U)],
\]
where $[U] = U_R - U_L, [f(U)] = f(U_R) - f(U_L)$, gives the shock curve
\[
\frac{(u_R - u_L)^2}{(1 - u_R^2/c^2)(1 - u_L^2/c^2)} = \frac{(\rho_R - \rho_L)(P_R - P_L)}{(\rho_L + P_L/c^2)(\rho_R + P_R/c^2)}.
\]
Along this curve we have shocks
\[
S_1: \quad \rho_L < \rho_R, u_R < u_L,
\]
\[
S_2: \quad \rho_R < \rho_L, u_R < u_L.
\]
The Riemann problem can be solved uniquely by using these rarefaction waves and shock waves and vacuum state. The detailed discussion can be found in J. Chen [1].

If we look at a region of the form

$$\Sigma_B = \{(w,z) | -B \leq z \leq w \leq B\},$$

we have the following

**Proposition 1** If the initial data $U_L, U_R$ belong to $\Sigma_B$ for some large $B$, then the solution of the Riemann problem is confined to $\Sigma_B$.

Moreover if we consider the image of $\Sigma_B$ in the $(E,F)$-space, we have

**Proposition 2** The region $\Sigma_B$ is convex in the $(E,F)$-plane.

Proof. Let us consider the above hedge $F = F(E)$ which corresponds to $w = B, -B < z < B$. We have to show $d^2 F/dE^2 < 0$. Along the hedge $w = B$, we have

$$u = c \tanh \frac{1}{c} \left( B - \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho \right),$$

from which

$$\frac{du}{d\rho} = -(1 - u^2/c^2) \frac{\sqrt{P'}}{\rho + P/c^2}. $$

By a direct calculation we have

$$\frac{dF}{dE} = \frac{u - \sqrt{P'}}{1 - \sqrt{P}u/c^2} = \lambda_1. $$

Differentiating once more we have

$$\frac{d^2 F}{dE^2} = -\frac{1 - u^2/c^2}{(1 - \sqrt{P'}u/c^2)^3} \left( \frac{P''}{2\sqrt{P'}} + (1 - \frac{P'}{c^2}) \frac{\sqrt{P'}}{\rho + P/c^2} \right) < 0. $$

This was to be seen. QED.

From Proposition 2, we have

**Proposition 3** If $U(s), s \in [a,b]$, is confined to a region $\Sigma_B$, then the average

$$\frac{1}{b-a} \int_a^b U(s) ds$$

belongs to $\Sigma_B$.

Let us look at the shock wave which connects the left state $U_L$ to the right state $U_R$ with the shock speed $\sigma$.

The right state $U_R$ and $\sigma$ are parametrized by $\rho = \rho_R$. Then we have the following fact, which will be used in Section 4.
Proposition 4 Along \( S_1(\rho_L < \rho) \), we have \( d\sigma/d\rho < 0 \), and along \( S_2(\rho < \rho_L) \) we have \( d\sigma/d\rho > 0 \).

Proof. Without loss of generality we can assume \( u_L = 0 \). Then \( u = u_R \) is given by
\[
u = -\sqrt{\frac{[\rho][P]}{(\rho_L + P/c^2)(\rho + P_L/c^2)}},
\]
where \( [\rho] = \rho - \rho_L \), \( [P] = P - P_L \). We have
\[
\sigma = \frac{[F]}{[E]} = \frac{(\rho + P/c^2)u}{\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2)}.
\]
By a direct but tedious computations, we have
\[
\frac{d\sigma}{d\rho} = \frac{(\rho + P/c^2)(\rho_L + P_L/c^2)[\rho]X}{2(\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2))^2u(\rho_L + P/c^2)^2(\rho + P_L/c^2)^2},
\]
where
\[
\]
Since \( P'' > 0 \) we know \( [P] \leq P'[\rho] \). Thus
\[
X \geq (\rho + P_L/c^2)(\rho + P/c^2)[P] + (\rho + P_L/c^2)(- (\rho + P_L/c^2) + [P]/c^2)[P] + (\rho_L + P/c^2)[P]^2/c^2
\]
\[
= [P][(\rho + P_L/c^2)(\rho + [P]/c^2) + ([\rho] - [P]/c^2)[P]/c^2).
\]
But
\[
1 > \frac{[\rho] - [P]/c^2}{[\rho]} = 1 - P'(\rho_L + \theta(\rho - \rho_L))/c^2 > 0.
\]
Using this, it is easy to see \( X > 0 \) both when \( [\rho] > 0 \) and when \( [\rho] < 0 \). Since \( u < 0 \), this completes the proof. QED.

3 Entropies

A pair of functions \( \eta \) and \( q \) is called an entropy-entropy flux if it satisfies the equation
\[
D_U q = D_U \eta . D_U f.
\] (3.1)

Using the Riemann invariants, we can write (3.1) as
\[
\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \quad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.
\]
By eliminating \( q \) from the equation, we get the following second order equation:

\[
\frac{\partial^2 \eta}{\partial w \partial z} + Q \left( J \frac{\partial \eta}{\partial w} - \frac{1}{J} \frac{\partial \eta}{\partial z} \right) = 0,
\]

(3.2)

where

\[
Q = \frac{1}{4\sqrt{P'}} \left( 1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'} \right),
\]

\[
J = \frac{1 - \sqrt{P'}u/c^2}{1 + \sqrt{P'}u/c^2}.
\]

Since this equation tends to the Euler-Poisson-Darboux equation

\[
\frac{\partial^2 \eta}{\partial w \partial z} + \frac{N}{w-z} \left( \frac{\partial \eta}{\partial w} - \frac{\partial \eta}{\partial z} \right) = 0
\]

(3.3)

as \( c \to \infty \), we shall call (3.2) the relativistic Euler-Poisson-Darboux equation.

Among entropies of (3.3) when \( c = \infty \) the kinetic energy

\[
\eta = \frac{1}{2} \rho u^2 + \frac{P}{\gamma - 1}
\]

(3.4)

plays an important role. Therefore we want to find an entropy of (3.2) which tends to (3.4) as \( c \to \infty \). Let us look for an entropy-entropy flux of the form

\[
\eta = H(\rho, u^2), \quad q = Q(\rho, u^2)u.
\]

Inserting this to the equation it is easy to find an entropy-entropy flux

\[
\eta^* = -\frac{\Psi(\rho)}{(1-u^2/c^2)^{1/2}} + c^2 \left( \frac{\rho + Pu^2/c^4}{1-u^2/c^2} \right),
\]

(3.5)

\[
q^* = -\left( \frac{\Psi(\rho)}{(1-u^2/c^2)^{1/2}} + c^2 \frac{\rho + Pu^2/c^4}{1-u^2/c^2} \right)u,
\]

(3.6)

\[
\Psi = \exp \left( \int_1^\rho \frac{d\rho}{\rho + P/c^2} + K_0 \right),
\]

(3.7)

where \( K_0 \) is determined so that \( \eta^* \) tends to the kinetic energy (3.4) as \( c = \infty \). We call the entropy \( \eta^* \) defined by (3.5) the relativistic standard entropy. The important fact is

**Proposition 5**  The Hessian \( D^2_U \eta^* \) is positive definite. For any fixed \( B \) there is a positive constant \( k \) such that

\[
(\xi|D^2_U \eta^*(U)\xi) \geq k|\xi|^2,
\]

for any \( U \in \Sigma_B \) and \( \xi = (\xi_0, \xi_1) \) with \( |\xi|^2 = \xi_0^2 + \xi_1^2 \).
Proof. The proof is due to direct but tedious calculations. We note
\[
\begin{align*}
\frac{\partial \rho}{\partial E} &= \frac{1 + u^2/c^2}{1 - P'u^2/c^4}, \\
\frac{\partial u}{\partial E} &= \frac{(1 + P'/c^2)(1 - u^2/c^2)u}{(\rho + P/c^2)(1 - P'u^2/c^4)} , \\
\frac{\partial \rho}{\partial F} &= -\frac{2u/c^2}{1 - P'u^2/c^4}, \\
\frac{\partial u}{\partial F} &= \frac{(1 - u^2/c^2)(1 + P'u^2/c^4)}{(\rho + P/c^2)(1 - P'u^2/c^4)} .
\end{align*}
\]

Using these, we have
\[
\begin{align*}
\frac{\partial \eta^*}{\partial E} &= -\frac{\Psi}{(\rho + P/c^2)(1 - u^2/c^2)^{1/2} + c^2}, \\
\frac{\partial \eta^*}{\partial F} &= \frac{\Psi u/c^2}{(\rho + P/c^2)(1 - u^2/c^2)^{1/2}}, \\
\frac{\partial^2 \eta^*}{\partial E^2} &= \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(P' + 2P'u^2/c^2 + u^2)}, \\
\frac{\partial^2 \eta^*}{\partial E\partial F} &= \frac{-\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(P' + 2P'u^2/c^2 + u^2)}, \\
\frac{\partial^2 \eta^*}{\partial F^2} &= \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(1 + 3P'u^2/c^4)}.
\end{align*}
\]

Therefore we get
\[
\begin{align*}
(\xi | D^2_0 \eta^* \xi) &= \eta^*_{EE} \xi_0^2 + 2\eta^*_{EF} \xi_0 \xi_1 + \eta^*_{FF} \xi_1^2 \\
&= \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(P' + 2P'u^2/c^2 + u^2)}Z, \\
Z &= \frac{1}{A + C + \sqrt{(A - C)^2 + 4B^2}}(\xi_0^2 + \xi_1^2), \\
A &= P' + 2P'u^2/c^2 + u^2, \\
B &= (2P'/c^2 + 1 + P'u^2/c^4)u, \\
C &= 1 + 3P'u^2/c^4.
\end{align*}
\]

This completes the proof. QED.

4 Construction of approximate solutions

Let us construct approximate solutions using the Godunov scheme. The construction is similar if we use the Lax-Friedrichs scheme.
Suppose that the initial data $U_0(x)$ is confined to an invariant region $\Sigma_B$. Put $\Lambda_0 = \sup\{ |\lambda_j(U)| | j = 1, 2, U \in \Sigma_B \}$. Fixing $\Lambda_1 > \Lambda_0$, we take mesh lengths $\Delta x, \Delta t$ such that $\Delta x = \Lambda_1 \Delta t$. We denote $\Delta = \Delta x$.

Let us construct the approximate solution $U^\Delta(t, x)$. First we put

$$U_0^\Delta(x) = U_0(x) \chi[-1/\Delta, 1/\Delta].$$

We define

$$U^\Delta(+0, x) = \frac{1}{2\Delta x} \int_{2j\Delta x}^{(2j+2)\Delta x} U_0^\Delta(x) dx$$

for $2j\Delta x < x \leq (2j+2)\Delta x$. Solving the Riemann problem on each interval $[2(j-1)\Delta, 2(j+1)\Delta]$, we define $U^\Delta(t, x)$ for $0 \leq t < \Delta t$. Since the Courant-Friedrichs-Levi condition is satisfied, the wave from the center $2j\Delta$ does not intersect. If $U^\Delta(t, x)$ for $0 \leq t < n\Delta t$ has been defined, then we define

$$U^\Delta(n\Delta t, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U^\Delta(n\Delta t-0, x) dx$$

for $2j\Delta < x \leq (2j+2)\Delta$. Solving the Riemann problem, we define $U^\Delta(t, x)$ for $n\Delta t \leq t < (n+1)\Delta t$.

By Proposition 1 and 3, it is inductively guaranteed that $U^\Delta$ remains in $\Sigma_B$, say,

**Proposition 6** The approximate solution $U^\Delta(t, x)$ satisfies $U^\Delta(t, x) \in \Sigma_B$, therefore,

$$0 \leq \rho^\Delta(t, x) \leq M, \quad \left| \frac{c}{2} \log \frac{c+u^\Delta(t,x)}{c-u^\Delta(t,x)} \right| \leq M.$$ 

Moreover we shall prove

**Proposition 7** For any test function $\Phi$ it holds that

$$\int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0, x) U_0^\Delta(x) dx = O(\Delta^{1/2}).$$

In order to prove Proposition 7, we prepare

**Proposition 8** For any shock wave from $U_L$ to $U_R$ with the shock speed $\sigma$ and for any convex entropy $\eta$, we have

$$\sigma[\eta] - [q] \geq 0,$$

where $[\eta] = \eta(U_R) - \eta(U_L), [q] = q(U_R) - q(U_L)$.
Proof. The right state of shocks can be parametrized by \( \rho = \rho_R \). Putting

\[
Q(\rho) = \sigma[\eta] - [q],
\]

we shall see \( dQ/d\rho \geq 0 \) along \( S_1 : [\rho] > 0 \) and \( dQ/d\rho \leq 0 \) along \( S_2 : [\rho] < 0 \). Using the equation (3.1) and the differentiation of the Rankine-Hugoniot condition, we have

\[
\frac{dQ}{d\rho} = \frac{d\sigma}{d\rho} ([\eta] - D_{U} \eta(U).[U]) = -\frac{d\sigma}{d\rho} \int_0^1 \theta(U - U_L)D_{U}^2 \eta(U + \theta(U - U_L).(U - U_L)) d\theta.
\]

We supposed \( D_{U}^2 \eta \geq 0 \). By Proposition 4, we know \( d\sigma/d\rho < 0 \) on \( S_1 \) and \( d\sigma/d\rho > 0 \) on \( S_2 \). QED.

Proof of Proposition 7.

We fix \( T \) to consider \( U^\Delta \) on \( 0 \leq t \leq T \). First we shall show

\[
\sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, (2j+1)\Delta)|^2 dx \leq C. \quad (4.1)
\]

Let us consider the standard entropy \( \eta^* \). Then we have

\[
0 = \int \eta^*(U(T, x)) dx - \int \eta^*(U(0, x)) dx + L + \Sigma,
\]

\[
L = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\eta^*(U(n\Delta t - 0, x)) - \eta^*(U(n\Delta t + 0, (2j+1)\Delta))) dx,
\]

\[
\Sigma = \int_0^T \sum_{shocks} (\sigma[\eta^*] - [q^*]) dt.
\]

We write \( U_0 = U(n\Delta t + 0, (2j+1)\Delta), U_1 = U(n\Delta t - 0, x) \). Since

\[
U_0 = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U_1 dx,
\]

we see

\[
L = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \int_0^1 (1 - \theta)(U_1 - U_0)D_{U}^2 \eta^*(U_0 + \theta(U_1 - U_0)).(U_1 - U_0)) d\theta dx \\
\geq 0.
\]

On the other hand we have \( \Sigma \geq 0 \) from Proposition 8. Thus \( L \leq C, \Sigma \leq C \). But from Proposition 5, we have \( D_{U}^2 \eta^* \geq k \). Therefore

\[
C \geq L \geq \frac{k}{2} \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U_1 - U_0|^2 dx.
\]
Thus we get (4.1).

Now let us consider a test function $\Phi$. Put

$$J = \int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) \, dx \, dt + \int \Phi(0,x)U_0^\Delta \, dx.$$

Since $U^\Delta$ is a weak solution on each time strip $n\Delta t < t < (n+1)\Delta t$, we have

$$J = \sum_n \int \Phi(n\Delta t, x) (U(n\Delta t-0, x) - U(n\Delta t+0, x)) \, dx$$

and

$$J_1 = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \Phi(n\Delta t, j\Delta) (U(n\Delta t-0, x) - U(n\Delta t+0, x)) \, dx,$$

$$J_2 = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\Phi(t, x) - \Phi(n\Delta t, j\Delta)) (U(n\Delta t-0, x) - U(n\Delta t+0, x)) \, dx.$$ 

Since

$$U(n\Delta t+0, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U(n\Delta t-0, x) \, dx$$

for $2j\Delta < x < (2j+2)\Delta$, we see $J_1 = 0$. It follows from (4.1) that

$$|J_2| \leq C\Delta^{1/2} ||\Phi||_{C^1} (\sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t-0, x) - U(n\Delta t+0, x)|^2 \, dx)^{1/2}$$

$$\leq C'\Delta^{1/2}.$$ 

Here we have used $T/\Delta t = O(1/\Delta)$. QED.

Summing up, we have the following theorem.

**Theorem 1** The approximate solution $U^\Delta(t,x)$ satisfies

$$0 \leq \rho^\Delta(t,x) \leq M, \quad |\frac{c}{2} \log \frac{c + u^\Delta(t,x)}{c - u^\Delta(t,x)}| \leq M$$

and

$$\int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) \, dx \, dt + \int \Phi(0,x)U_0^\Delta(x) = O(\Delta^{1/2})$$

for any test function $\Phi$.

We expect that $U^\Delta$ tends to a weak solution everywhere. For the non-relativistic gas dynamics, this was done by DiPerna [3] and G.Q.Chen et al [2]. In their proof the Darboux formula

$$\eta = \int_z^w ((w-s)(s-z))^N \phi(s) \, ds$$

which gives solutions of the Euler-Poisson-Darboux equation (3.3), $\phi$ being arbitrary, plays an important role. Section 6 will be devoted to find such an integral formula for the relativistic Euler-Poisson-Darboux equation (3.2).
5  Remark

We note that
\[
\lambda_2 - \lambda_1 = \frac{\sqrt{P'}(1 - u^2/c^2)}{1 - u^2 P/c^4} > 0,
\]
\[
\frac{\partial \lambda_1}{\partial z} = \frac{1 - u^2/c^2}{2(1 - \sqrt{P'u/c^2})(1 - \frac{P'}{c^2} + \frac{\rho + P/c^2}{2P} P'') > 0},
\]
\[
\frac{\partial \lambda_2}{\partial w} = \frac{1 - u^2/c^2}{2(1 + \sqrt{P'u/c^2})(1 - \frac{P'}{c^2} + \frac{\rho + P/c^2}{2P} P'') > 0}
\]
for \( \rho > 0 \) and \(|u| < c\).

This says that the system is strictly hyperbolic and genuinely nonlinear on \( \rho > 0 \). Therefore the Glimm's theory can be applied if
\[
\|U_0(x) - U^*\|_{L^\infty} + T.V.U_0
\]
is sufficiently small, where \( U^* \) is a constant state such that \( \rho^* > 0, |u^*| < c \). But the vacuum may not be covered by this application of the general theorem.

6  Generalized Darboux formula

In this section we seek an integration formula for solutions of the relativistic Euler-Poisson-Darboux equation. Let us introduce the variables
\[
x = \frac{c}{2} \log \frac{c + u}{c - u}, \quad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho.
\]
Then the relativistic Euler-Poisson-Darboux equation is
\[
(EPD) \quad \eta_{xx} - \eta_{yy} + A(x, y)\eta_y + B(x, y)\eta_x = 0,
\]
where
\[
A(x, y) = \frac{1}{\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'} P'') \frac{1 + P'u^2/c^4}{1 - P'u^2/c^4},
\]
\[
B(x, y) = -\frac{2u/c^2}{1 - P'u^2/c^4}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'} P'').
\]
The coefficients \( A \) and \( B \) are of the form
\[
A = \frac{2N}{y} + a, \quad a = \frac{y}{c^2}(a_0 + [x^2/c^2, y^2/c^2]),
\]
\[
B = -\frac{4N}{N + 1} \frac{x}{c^2}(1 + [x^2/c^2, y^2/c^2]),
\]
where \([X,Y]_1\) denotes a convergent power series \(\sum_{j+k \geq 1} c_{jk} X^j Y^k\). In order to remove the singularity in \(A\), we use the trick of Weinstein [7]. We introduce the sequence of variables \(\eta_j, j = 0, 1, \ldots, N\) by

\[
\frac{\partial \eta_j}{\partial y} = y \eta_{j+1},
\]

or

\[
\eta_j(x, y) = I \eta_{j+1}(x, y) = \int_0^y Y \eta_{j+1}(x, Y) dY,
\]

where \(\eta_0 = \eta\). The sequence of formal integro-differential operators \(L_j\) is defined by

\[
L_j V = V_{xx} - V_{yy} + \left( \frac{2(N - j)}{y} + a \right) V_y + BV_x + j \tilde{a} V + \sum_{k=1}^{j} H_{jk} I^k V,
\]

where

\[
\tilde{a} = \frac{\partial a}{\partial y} + \frac{a}{y} = \frac{1}{c^2} \left[ \frac{x^2}{c^2}, \frac{y^2}{c^2} \right]_0.
\]

The coefficients \(F_{jk}\) and \(H_{jk}\) are determined inductively by

\[
F_{j+1,k} = \begin{cases} 
F_{j1} + \frac{1}{y} \frac{\partial B}{\partial y} & \text{if } k = 1 \\
F_{jk} + \frac{1}{y} \frac{\partial}{\partial y} F_{j,k-1} & \text{if } k \geq 2
\end{cases}
\]

\[
H_{j+1,k} = \begin{cases} 
H_{j1} + j \frac{1}{y} \frac{\partial \tilde{a}}{\partial y} & \text{if } k = 1 \\
H_{jk} + \frac{1}{y} \frac{\partial}{\partial y} H_{j,k-1} & \text{if } k \geq 2
\end{cases}
\]

It is easy to see that \(F_{jk}\) are of the form \(\frac{x^2}{c^2}[x^2/c^2, y^2/c^2]_0\) and \(H_{jk}\) are of the form \(\frac{1}{c^2}[x^2/c^2, y^2/c^2]_0\). By the definition we have formally

\[
\frac{1}{y} \frac{\partial}{\partial y} (L_j \eta_j) = L_{j+1} \eta_{j+1}.
\]

Now we consider the equation \(L_N V = 0\) for \(V = \eta_N\) with the initial conditions

\[
V = 0, \quad V_y = 2^{N+1} N! \phi(x), \quad \text{at } y = 0.
\]

The problem is

\[
(Q) \quad V_{yy} - V_{xx} = a V_y + BV_x + N \tilde{a} V + \sum_{k=1}^{N} F_k I^k V_x + \sum_{k=1}^{N} H_k I^k V,
\]

\[
V = 0, \quad V_y = 2^{N+1} N! \phi(x) \quad \text{at } y = 0.
\]
Proposition 9 If $\phi \in C^1(R)$, then the problem \((Q)\) admits a unique solution $V$ in $C^2(R \times [0, \infty))$.

**Proof.** Let us denote by $H(x, y, V)$ the right hand side of the equation $L_N = 0$. Then \((Q)\) is transformed to the integral equation

$$V(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2} \int_{x-y+Y}^{x+y-Y} H(X, Y, V) dX dY.$$  

We can solve this integral equation by the iteration

\[ V_0(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi, \]

\[ V^{n+1}(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2} \int_{x-y+Y}^{x+y-Y} H(X, Y, V^n) dX dY. \]

Fixing $L$ arbitrarily, we consider $|x| \leq L$. Then it is easy to get the estimates

$$|V^{n+1}(x, y) - V^n(x, y)| \leq \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$  

Therefore $V^n$ tends to a limit $V$ uniformly on $|x| \leq L, 0 \leq y \leq L$. The limit is the unique solution of \((Q)\). QED.

Now we put

$$\eta_N = V, \quad \eta_{N-k} = I\eta_{N-k+1}. \quad (6.1)$$

Since $\eta_{N-k}$ and its derivatives of order $\leq 2$ all vanish on $y = 0$ for $k \geq 1$, we see $\eta = \eta_0$ gives a solution of the relativistic Euler-Poisson-Darboux equation (EPD).

Next we give an integral formula for the solution $V$ of \((Q)\).

**Proposition 10** There is a $C^{N+2}$-function $G(x, y, \xi)$ of $|x| < \infty, y \geq 0, x - y \leq \xi \leq x + y$ such that the solution $V$ of \((Q)\) satisfies

$$V(x, y) = \int_{x-y}^{x+y} G(x, y, \xi)\phi(\xi)d\xi. \quad (6.1)$$

Moreover

$$G = 2^N N! + O(y/c^2), \quad \partial^p_x \partial^p_\xi \partial^p_y G = O(1/c^2) \quad \text{for} \quad 1 \leq p_1 + p_2 + p_3 \leq N + 2$$

**Proof.** We consider the approximate solution $V^n(x, y)$ which appeared in the iteration of the proof of Proposition 9. By writing $H$ as

$$H = (aV)_y + (BV)_x + bV + \sum (F_k I^k V)_x + \sum H_k I^k V,$$
\[ b = N \tilde{a} - a_y - B_z = \frac{1}{c^2} [x^2/c^2, y^2/c^2]_0, \]
\[ \tilde{H}_k = H_k - (F_k)_x = \frac{1}{c^2} [x^2/c^2, y^2/c^2]_0, \]

It is easy to see inductively that there is a kernel \( G^n(x, y, \xi) \) such that

\[ V^n(x, y) = \int_{x-y}^{x+y} G^n(x, y, \xi) \phi(\xi) d\xi. \]

In fact \( G^0 = 2 \) and \( G^n \) are determined inductively by the formula

\[
G^{n+1} = 2 + \frac{1}{2} (G^n_I + G^n_{II} + G^n_{III} + \sum G^n_{IVk} + \sum G^n_{Vk}),
\]

\[
G_I = \int_{(x+y-\xi)/2}^{y} a(x - y + Y, Y) G(x - y + Y, Y, \xi) dY + \int_{(x+y-\xi)/2}^{y} a(x + y - Y, Y) G(x + y - Y, Y, \xi) dY,
\]

\[
G_{II} = \int_{(x+y-\xi)/2}^{y} B(x + y - Y, Y) G(x + y - Y, Y, \xi) dY - \int_{(-x+y+\xi)/2}^{y} B(x - y + Y, Y) G(x - y + Y, Y, \xi) dY,
\]

\[
G_{III} = \int \int_{D(x, y, \xi)} b(X, Y) G(X, Y, \xi) dX dY,
\]

where

\[
D(x, y, \xi) = \{(X, Y) \mid X - Y \leq \xi \leq X + Y, x - y + Y \leq X \leq x + y - Y, 0 \leq Y \leq y\},
\]

\[
G_{IVk} = \int_{(x+y+\xi)/2}^{y} F_k(x + y - Y, Y) J^k G(x + y - Y, Y, \xi) dY + \int_{(-x+y+\xi)/2}^{y} F_k(x - y + Y, Y) J^k G(x - y + Y, Y, \xi) dY,
\]

where

\[
JG(x, y, \xi) = \int_{|x-\xi|}^{y} Y G(x, Y, \xi) dY,
\]

and

\[
G_{Vk} = \int \int_{D(x, y, \xi)} \tilde{H}_k(X, Y) J^k G(X, Y, \xi) dX dY.
\]

It is easy to see inductively that

\[
|G^{n+1}(x, y, \xi) - G^n(x, y, \xi)| \leq \frac{M^{n+1} y^{n+1}}{(n+1)!}.
\]
therefore $G^n$ converges to a limit $G$ uniformly and (6.1) holds. Moreover, one can differentiate $G^{n+1}$ supposing that $G^n$ is differentiable. In fact

$$G_{I,x} = \frac{1}{2} aG((x-y+\xi)/2, (z+y+\xi)/2, \xi)$$

$$- \frac{1}{2} aG((x+y+\xi)/2, (z+y-\xi)/2, \xi) +$$

$$+ \int_{(-x+y+\xi)/2}^{y} (aG)_{x}(x-y+Y, Y, \xi)dY$$

$$+ \int_{(x+y-\xi)/2}^{y} (aG)_{x}(x-Y+Y, Y, \xi)dY,$$

$$G_{I,\xi} = -\frac{1}{2} aG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) +$$

$$+ \frac{1}{2} aG((x+y+\xi)/2, (z+y-\xi)/2, \xi) +$$

$$+ \int_{(-x+y+\xi)/2}^{y} aG_{\xi}(x-y+Y, Y, \xi)dY +$$

$$+ \int_{(x+y-\xi)/2}^{y} aG_{\xi}(x-Y+Y, Y, \xi)dY,$$

$$G_{I,y} = -\frac{1}{2} aG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) +$$

$$- \frac{1}{2} aG((x+y+\xi)/2, (z+y-\xi)/2, \xi) +$$

$$+ 2aG(x, y, \xi) +$$

$$- \int_{(-x+y+\xi)/2}^{y} (aG)_{x}(x-y+Y, Y, \xi)dY +$$

$$+ \int_{(-x+y+\xi)/2}^{y} (aG)_{x}(x+y-Y, Y, \xi)dY;$$

$$G_{II,x} = -\frac{1}{2} BG((x+y+\xi)/2, (z+y-\xi)/2, \xi) +$$

$$- \frac{1}{2} BG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) +$$

$$+ \int_{(x+y-\xi)/2}^{y} (BG)_{x}(x+y-Y, Y, \xi)dY +$$

$$- \int_{(-x+y+\xi)/2}^{y} (BG)_{x}(x-y+Y, Y, \xi)dY,$$

$$G_{II,\xi} = \frac{1}{2} BG((x+y+\xi)/2, (z+y-\xi)/2, \xi) +$$

$$+ \frac{1}{2} BG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) +$$

$$+ \int_{(x+y-\xi)/2}^{y} BG_{\xi}(x+y-Y, Y, \xi)dY +$$

$$- \int_{(-x+y+\xi)/2}^{y} BG_{\xi}(x-y+Y, Y, \xi)dY,$$
$G_{II,y} = -\frac{1}{2}BG((x+y+\xi)/2,(x+y-\xi)/2,\xi) +$
$+ \frac{1}{2}BG((-x+y+\xi)/2,(-x+y+\xi)/2,\xi) +$
$+ \int_{(x+y-\xi)/2}^{y}(BG)_{x}(x+y-Y,Y,\xi)dY +$
$+ \int_{(-x+y+\xi)/2}^{y}(BG)_{x}(x-y+Y,Y,\xi)dY;$

$G_{III,x} = \int_{(x+y-\xi)/2}^{y}bG(x+y-Y,Y,\xi)dY - \int_{(-x+y+\xi)/2}^{y}bG(x-y+Y,Y,\xi)dY,$

$G_{III,\xi} = \int_{0}^{(x+y-\xi)/2}bG(\xi+Y,Y,\xi)dY + \int_{0}^{(-x+y+\xi)/2}bG(\xi-Y,Y,\xi)dY +$
$+ \int\int_{D(x,y,\xi)}bG(X,Y,\xi)dXdY,$

$G_{III,y} = \int_{(x+y-\xi)/2}^{y}bG(x+y-Y,Y,\xi)dY + \int_{(-x+y+\xi)/2}^{y}bG(x-y+Y,Y,\xi)dY;$

and the derivatives of $G_{IVk}$ are similar to $G_{II}$ and the derivatives of $G_{IVk}$
are similar to $G_{III}$. Then it is easy to see inductively that

$|G_{x}^{n+1} - G_{x}^{n}| + |G_{\xi}^{n+1} - G_{\xi}^{n}| + |G_{y}^{n+1} - G_{y}^{n}| \leq \frac{M^{n}y^{n}}{n!}.$

Thus the limit $G$ is differentiable. In a similar manner we see

$|G_{xx}^{n+1} - G_{xx}^{n}| + |G_{x\xi}^{n+1} - G_{x\xi}^{n}| + |G_{xy}^{n+1} - G_{xy}^{n}| +$
$+ |G_{\xi\xi}^{n+1} - G_{\xi\xi}^{n}| + |G_{\xi y}^{n+1} - G_{\xi y}^{n}| + |G_{yy}^{n+1} - G_{yy}^{n}| \leq$
$\leq \frac{M^{n-1}y^{n-1}}{(n-1)!}.$

Thus $G$ is twice continuously differentiable. In a similar manner we see that
$G$ is $N + 2$-times continuously differentiable. The rough estimates stated in
the propositions is obvious since the coefficients are all of $O(1/c^2)$. QED.

The solution $\eta_{N-k}$ enjoys an integral representation

$\eta_{N-k} = \int_{x-y}^{x+y}K_{N-k}(x,y,\xi)\phi(\xi)d\xi,$

where

$K_{N-k}(x,y,\xi) = JK_{N-k+1}(x,y,\xi) = J^{k}G(x,y,\xi).$

So the solution $\eta$ of the relativistic Euler-Poisson-Darboux equation is given by

$\eta(x,y) = \int_{x-y}^{x+y}K(x,y,\xi)\phi(\xi)d\xi,$
By induction we see

\[ J^k G(x, y, \xi) = \frac{2^N N!}{2^k k!} (y^2 - (x - \xi)^2)^k (1 + O(y/c^2)). \]

Thus we have

**Proposition 11** There is a kernel \( K(x, y, \xi) \) which is of \( C^{N+2} \)-class in \(|x| < \infty, 0 \leq y, x - y \leq \xi \leq x + y \) such that

\[ \eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi \]

gives a solution of the relativistic Euler-Poisson-Darboux equation for any smooth \( \phi \). Moreover

\[ K(x, y, \xi) = (y^2 - (x - \xi)^2)^N (1 + O(y/c^2)). \]

But in order to apply this integration formula, the generalized Darboux formula, to the study of the relativistic Euler equation, more detailed estimates of the remainder are necessary.

**Proposition 12** We have

\[ G_y = O(y/c^2). \]

Proof. Since \( a = O(y/c^2) \), it is clear that \( G_{I,y} = O(y/c^2) \). Next we see

\[ G_{II,y} = -B((x+y+\xi)/2,(x+y-\xi)/2)+B((x-y+\xi)/2,(-x+y+\xi)/2))+O(y/c^2). \]

On the other hand we can write

\[ B = \frac{1}{c^2} B_0(x) + O(y^2/c^2) \]

and

\[ \frac{x+y+\xi}{2} = x + \frac{y+Z}{2}, \quad \frac{x-y+\xi}{2} = x + \frac{-y+Z}{2}, \quad Z = \xi - x. \]

Therefore we see \( G_{II,y} = O(y/c^2) \). It is clear that \( G_{III,y} = O(y/c^2) \) and \( G_{IVk,y}, G_{Vk,y} = O(y^2/c^2) \). QED.

**Proposition 13** We have

\[ G = 2^N N! + \frac{1}{c^2} C_0(x, c)(\xi - x) + O(y^2/c^2), \]

where \( C_0(x, c) \) is a function of the form

\[ [x^2/c^2]_0 + \frac{x}{c^2} [x^2/c^2]_0. \]
Proof. It is clear that \(G_I = O(y^2/c^2)\) since \(a = O(y/c^2)\). Next we see

\[
G_{II} = 2^N N! \int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY + O(y^2/c^2),
\]

since \(G = 2^N N! + O(y/c^2)\). If we write

\[
B = \frac{1}{c^2} B_0(x) + O(y^2/c^2), \quad Z = \xi - x
\]

then we see

\[
\int_{(x+y-\xi)/2}^{y} B(x + y - Y, Y)dY - \int_{(-x+y+\xi)/2}^{y} B(x - y + Y, Y)dY = \\
= \frac{1}{c^2} \left( \int_{x}^{x+\frac{-y+Z}{2}} B_0(s)ds - \int_{x+\frac{-y+Z}{2}}^{x} B_0(s)ds \right) + O(y^2/c^2)
\]

\[
= \frac{1}{c^2} B_0(x) Z + O(y^2/c^2).
\]

Note \(|Z| \leq y\). It is clear that \(G_{III}, G_{IV, k}, G_{Vk} = O(y^2/c^2)\). QED.

**Proposition 14** We have

\[
G_x + G_\xi = O(y/c^2).
\]

Proof. First we see

\[
G_{I,x} + G_{I,\xi} = \int_{(-x+y+\xi)/2}^{y} ((aG)_x + aG_\xi)(x - y + Y, Y, \xi)dY + \\
+ \int_{(x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)(x + y - Y, Y, \xi)dY
\]

\[
= O(y^2/c^2),
\]

since \(a, a_x = O(y/c^2)\). Next we see

\[
G_{II,x} + G_{II,\xi} = \int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_\xi)(x + y - Y, Y, \xi)dY + \\
- \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x - y + Y, Y, \xi)dY
\]

\[
= O(y/c^2).
\]

It is clear that \(G_{III,x}, G_{III,\xi}, G_{Vk,x}, G_{Vk,\xi} = O(y/c^2)\). \(G_{IV, k,x} + G_{IV, k,\xi}\) is estimated in a similar manner as \(G_{II,x} + G_{II,\xi}\). QED.

**Proposition 15** We have

\[
(G_x + G_\xi)_y = O(y/c^2).
\]
Proof. First we see

\[
(G_{I,x} + G_{I,\xi})_y = 2((aG)_x + aG_\xi)(x, y, \xi) + \\
- \frac{1}{2}((aG)_x + aG_\xi)((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \\
- \frac{1}{2}((aG)_x + aG_\xi)((x + y + \xi)/2, (x + y - \xi)/2, \xi) + \\
- \int_{(-x+y+\xi)/2}^{y} ((aG)_x + aG_\xi)(x - y + Y, Y, \xi) dY + \\
+ \int_{((x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)(x + y - Y, Y, \xi) dY
\]

\[= O(y/c^2),\]

since \(a, a_x = O(y/c^2)\). Next we see

\[
(G_{II,x} + G_{II,\xi})_y = -\frac{1}{2}((BG)_x + BG_\xi)((x + y + \xi)/2, (x + y - \xi)/2, \xi) + \\
+ \frac{1}{2}((BG)_x + BG_\xi)((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \\
+ \int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_\xi)(x + y - Y, Y, \xi) dY + \\
+ \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x - y + Y, Y, \xi) dY
\]

\[= 2^{N-1}N!B_x((x - y + \xi)/2, (-x + y + \xi)/2) \\
- 2^{N-1}N!B_x((x + y + \xi)/2, (x + y - \xi)/2) + \\
+ O(y/c^2),\]

since \(G = 2^N N! + O(y/c^2)\) and \(G_x + G_\xi = O(y/c^2)\). But

\[
B_x = \frac{1}{c^2} B_0'(x) + O(y^2/c^2)
\]

and

\[
B_x((x - y + \xi)/2, (-x + y + \xi)/2) - B_x((x + y + \xi)/2, (x + y - \xi)/2) = \\
= \frac{1}{c^2} B_0''(x)(-y) + O(y^2/c^2) \\
= O(y/c^2).
\]

It is clear that

\[
(G_{III,x} + G_{III,\xi})_y = \int_{(x+y-\xi)/2}^{y} ((bG)_x + bG_\xi)(x + y - Y, Y, \xi) dY + \\
+ \int_{(-x+y+\xi)/2}^{y} ((bG)_x + bG_\xi)(x - y + Y, Y, \xi) dY
\]

\[= O(y/c^2).
\]
Similarly we can estimate \((G_{IVK,x} + G_{IVK,\xi})_y, (G_{VK,x} + G_{VK,\xi})_y\) bearing in mind that \((JG)_x + (JG)_\xi = J(G_x + G_\xi)\). QED.

**Proposition 16** We have

\[
G_x + G_\xi = \frac{1}{c^2} C_1(x, c)(\xi - x) + O(y^2 / c^2),
\]

where \(C_1(x, c)\) is a function of the form

\[
[x^2 / c^2]_0 + \frac{x}{c^2} [x^2 / c^2]_0.
\]

Proof. We already observed that \(G_{Ix} + G_{I\xi} = O(y^2 / c^2)\). Next we look at

\[
G_{II,x} + G_{II,\xi} = \int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_\xi)(x+y-Y,Y,\xi)dY + \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x-y+Y,Y,\xi)dY
\]

\[
= 2^N N! \int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY + O(y^2 / c^2),
\]

since \(G = 2 + O(y / c^2)\) and \(G_x + G_\xi = O(y / c^2)\). Bearing in mind that \(B_y = O(y / c^2)\), we see

\[
\int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY - \int_{(-x+y+\xi)/2}^{y} B_x(x-y+Y,Y)dY = -2B(x, y) + B((x+y+\xi)/2, (x+y-\xi)/2) +
\]

\[
+ B((x+y+\xi)/2, (x+y-\xi)/2) + O(y^2 / c^2)
\]

\[
= \frac{1}{c^2} (-2B_0(x) + B_0(x + \frac{y+Z}{2}) + B_0(x + \frac{-y+Z}{2}) + O(y^2 / c^2)
\]

Next we look at

\[
G_{III,x} + G_{III,\xi} = \int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY - \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y,Y,\xi)dY + \int_{0}^{(x+y-\xi)/2} bG(\xi+Y,Y,\xi)dY - \int_{0}^{(-x+y+\xi)/2} bG(\xi-Y,Y,\xi)dY + \int \int_{D(x,y,\xi)} bG(X,Y,\xi)dXdY.
\]
\[ b(x, y) = \frac{1}{c^2} b_0(x) + O(y^2/c^2), \]

we see

\[ G_{III,x} + G_{III,\xi} = 2^N N! \left( \int_{x}^{x+\frac{y+Z}{2}} b_0(s) ds - \int_{x+\frac{y-Z}{2}}^{x} b_0(s) ds + \int_{x+Z}^{x+\frac{y-Z}{2}} b_0(s) ds - \int_{x+\frac{y+Z}{2}}^{x+Z} b_0(s) ds \right) + O(y^2/c^2) \]

\[ = \frac{2^N N!}{c^2} b_0(x) (\frac{y+Z}{2} - \frac{y-Z}{2} + \frac{y-Z}{2} - \frac{y+Z}{2}) + O(y^2/c^2) \]

\[ = O(y^2/c^2). \]

\[ G_{IVk,x} + G_{IVk,\xi} \text{ can be estimated in a similar manner as } G_{II,x} + G_{II,\xi}. \]

Finally \( G_{Vk,x}, G_{Vk,\xi} = O(y^3/c^2) \) since \( J^k G = O(y^2/c^2) \) for \( k \geq 1 \). QED.

**Proposition 17** We have

\( (G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2). \)

**Proof.** First we see

\( (G_{I,x} + G_{I,\xi})_x + (G_{I,x} + G_{I,\xi})_\xi = \)

\[ \int_{(-x+y+\xi)/2}^{y} ((aG)_{xx} + 2(aG_{\xi})_{x} + aG_{\xi\xi})(x-y, Y, \xi) dY + \]

\[ + \int_{(x+y-\xi)/2}^{y} ((aG)_{xx} + 2(aG_{\xi})_{x} + aG_{\xi\xi})(x+y-Y, Y, \xi) dY = O(y^2/c^2), \]

since \( a, a_x, a_{xx} = O(y/c^2) \). Next

\( (G_{II,x} + G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi = \)

\[ \int_{(x+y-\xi)/2}^{y} (((BG)_{x} + BG_{\xi})_x + (((BG)_{x} + BG_{\xi})_\xi)(x+y-Y, Y, \xi) dY + \]

\[ + \int_{(-x+y+\xi)/2}^{y} (((BG)_{x} + BG_{\xi})_x + (((BG)_{x} + BG_{\xi})_\xi)(x+y-Y, Y, \xi) dY = O(y/c^2). \]

It is easy to see

\( (G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi = O(y/c^2). \)

The estimates of \( G_{IVk} \) and \( G_{V_k} \) can be seen similarly. QED.
Proposition 18 We have 

$$(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = \frac{1}{c^2} C_2(x, c)(\xi - x) + O(y^2/c^2),$$

where $C_2(x, c)$ is a function of the form 

$$[x^2/c^2]_0 + \frac{x}{c^2}[x^2/c^2]_0.$$

Proof. We already observed that 

$$(G_{I,x} + G_{I,\xi})_x + (G_{I,x} + G_{I,\xi})_\xi = O(y^2/c^2).$$

Next, bearing in mind that $G_x + G_\xi = O(y/c^2)$ and $(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2)$, we see 

$$(G_{II,x} + G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi =$$

$$= \int_{(x+y-\xi)/2}^{y} (B_{xx}G + 2B_x(G_x + G_\xi) + B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi))(x+y-Y, Y, \xi)dY +$$

$$- \int_{(-x+y+\xi)/2}^{y} (B_{xx}G + 2B_x(G_x + G_\xi) + B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi))(x-y+Y, Y, \xi)dY$$

$$= 2^N N! \int_{(x+y-\xi)/2}^{y} B_{xx}(x+y-Y,Y)dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B_{xx}(x-y+Y,Y)dY +$$

$$+ O(y^2/c^2).$$

The same discussion to that of the proof of Proposition 16 can be applied by replacing $B$ by $B_x$. Let us look at $(G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi$. Note that 

$$(bG)_x + bG_\xi = b_x G + b(G_x + G_\xi)$$

$$= 2^N N! b_x + O(y/c^2),$$

$$bG = 2^N N! b + O(y/c^2).$$

Applying the discussion of the proof of Proposition 16 by replacing $b$ by $b_x$, we see 

$$(G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi =$$

$$= 2^N N!(\int_{x+\xi/2}^{x+Z} b_0(s)ds - \int_{x+\xi/2}^{x+Z} b_0(s)ds) +$$

$$+ O(y^2/c^2)$$

$$= -2^N N! b_0(x)Z + O(y^2/c^2).$$

The estimates of $G_{IV_k}, G_{V_k}$ are parallel. QED.
**Proposition 19** We have

\[ G_{\xi} = \frac{1}{c^2} C_3(x, c) + O(y/c^2). \]

**Proof.** It is sufficient to note that

\[ G_{II,\xi} = 2^{N-1} N! \left( B((x + y + \xi)/2, (x + y - \xi)/2) + B((x - y + \xi)/2, (-x + y + \xi)/2) + O(y/c^2) \right) \]

\[ = \frac{2^{N-1} N!}{c^2} (B_0(x + \frac{y+Z}{2}) + B_0(x + \frac{-y+Z}{2})) + O(y/c^2) \]

\[ = \frac{2^N N!}{c^2} B_0(x) + O(y/c^2). \]

QED.

**Proposition 20** We have

\[ (G_x + G_{\xi})_{\xi} = \frac{1}{c^2} C_3(x, c) + O(y/c^2). \]

**Proof.** We see

\[ (G_{I,x} + G_{I,\xi})_{\xi} = O(y/c^2) \]

by \( a, a_x = O(y/c^2) \). Next we see

\[ (G_{II,x} + G_{II,\xi})_{\xi} = 2^{N-1} N! \left( B_x((x + y + \xi)/2, (x + y - \xi)/2) + B_x((x - y + \xi)/2, (-x + y + \xi)/2) + O(y/c^2) \right) \]

\[ = \frac{2^N N!}{c^2} B_0'(x) + O(y/c^2). \]

And we see

\[ (G_{III,x} + G_{III,\xi})_{\xi} = \]

\[ = 2^N N! b((x - y + \xi)/2, (-x + y + \xi)/2) + O(y/c^2) \]

\[ = \frac{2^N N!}{c^2} b_0(x) + O(y/c^2). \]

Other terms can be estimated similarly. QED.

### 7 Estimates of the derivatives of entropies

Let us consider the entropy \( \eta \) generated by \( \phi \) of \( C^3 \)-class, that is,

\[ \eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) \, d\xi. \]
In this section we will find estimates of the derivatives of $\eta$ with respect to $E, F$. As auxiliary variables we introduce

$$R = y^{2N+1}, \quad M = xy^{2N+1}.$$  \hspace{1cm} (7.1)

We are going to prove the following

**Proposition 21** We have

\[
\frac{\partial \eta}{\partial M} = 2^{2N+1} \int_{0}^{1} (s-s^2)^N D\phi(x+(2s-1)y)ds + O(y^2/c^2), \hspace{1cm} (7.2)
\]

\[
\frac{\partial \eta}{\partial R} = 2^{2N+1} \int_{0}^{1} (s-s^2)^N \phi ds + 2^{2N+1} \int_{0}^{1} (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^2/c^2), \hspace{1cm} (7.3)
\]

\[
\frac{\partial^2 \eta}{\partial M^2} = 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N D^2 \phi ds + O(y^{-2N+1}/c^2), \hspace{1cm} (7.4)
\]

\[
\frac{\partial^2 \eta}{\partial R \partial M} = 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D^2 \phi ds + O(y^{-2N+1}/c^2), \hspace{1cm} (7.5)
\]

\[
\frac{\partial^2 \eta}{\partial R^2} = 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N ((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2} s(1-s)y^2) D^2 \phi(x + (2s-1)y)ds + O(y^{-1}/c^2). \hspace{1cm} (7.6)
\]

**Proof.** We write

$$\eta = 2R^{\frac{1}{2N+1}} \int_{0}^{1} K(M/R, R^{\frac{1}{2N+1}}, M/R + (2s-1)R^{\frac{1}{2N+1}}) \phi(M/R + (2s-1)R^{\frac{1}{2N+1}})ds.$$  \hspace{1cm} (7.6)

Differentiating $\eta$ with respect to $M$, we have

\[
\frac{\partial \eta}{\partial M} = (1) + (2),
\]

\[
(1) = 2R^{\frac{2N}{2N+1}} \int_{0}^{1} (K_x + K_\xi)(x, y, x + (2s-1)y)\phi(x + (2s-1)y)ds,
\]

\[
(2) = 2R^{\frac{2N}{2N+1}} \int_{0}^{1} K(x, y, x + (2s-1)y) D\phi(x + (2s-1)y)ds.
\]

Since $K(x, y, \xi) = J^NG(x, y, \xi)$, i.e.

\[
K(x, y, \xi) = \int_{|x-\xi|}^{Y_N} \int_{|x-\xi|}^{Y_{N-1}} \cdots \int_{|x-\xi|}^{Y_1} Y_1 G(x, Y_1, \xi) dY_1 \cdots Y_N,
\]
by Proposition 16 we see

\[(K_x + K_{\xi})(x, y, x + (2s - 1)y)\]
\[= \int_{|2s-1|y}^{y} Y_N \int_{|2s-1|y}^{Y_N} Y_{N-1} \cdots \int_{|2s-1|y}^{Y_2} Y_1 (G_x + G_{\xi})(x, Y_1, x + (2s - 1)y) dY_1 \cdots\]
\[= \frac{C_1(x, c)}{2^{N-1}N!c^2} y^{2N+1} (2s-1)(1 - (2s-1)^2)^N + O(y^{2N+2}/c^2)\]
\[= \frac{2^N C_1(x, c)}{(N+1)!c^2} y^{2N+1} \frac{d}{ds} (s - s^2)^{N+1} + O(y^{2N+2}/c^2).\]

Therefore by integration by parts we get

\[(1) = R^{\frac{2N}{2N+1}} y^{2N+2} \frac{2^{N+1} C_1(x, c)}{(N+1)!c^2} \int_0^1 (s - s^2)^{N+1} D\phi ds + O(y^2/c^2)\]
\[= O(y^2/c^2).\]

By Proposition 13 we see

\[K(x, y, \xi) = \int_{|x-\xi|}^{y} Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1} \cdots \int_{|x-\xi|}^{Y_2} Y_1 G(x, Y_1, \xi) dY_1 \cdots Y_N,\]
\[= 2^N (s - s^2)^N y^{2N} + \frac{2^N C_0(x, c)}{N!c^2} (2s-1) (s-s^2)^N y^{2N+1} + O(y^{2N+2}/c^2).\]

Therefore by integration by parts we get

\[(2) = 2^{N+1} R^{\frac{2N}{2N+1}} y^{2N} \int_0^1 (s(1-s))^N D\phi(x + (2s-1)y) ds\]
\[+ R^{\frac{2N}{2N+1}} O(y^{2N+2}/c^2).\]

Thus we have (7.2). Next we show (7.3). We have

\[
\frac{\partial \eta}{\partial R} = (3) + (4) + (5),
\]

(3) \[= \frac{2}{2N+1} R^{\frac{2N}{2N+1}} \int_0^1 K(x, y, x + (2s-1)y)\phi(x + (2s-1)y) ds,\]

(4) \[= 2R^{\frac{2N}{2N+1}} \int_0^1 (-x(K_x + K_{\xi}) + \frac{1}{2N+1} y(K_y + (2s-1)K_{\xi})) \times \phi(x + (2s-1)y) ds,\]

(5) \[= 2R^{\frac{2N}{2N+1}} \int_0^1 K(x, y, x + (2s-1)y)(-x + \frac{y}{2N+1}(2s-1)) D\phi(...) ds.\]

By Proposition 13 we get

\[(3) = \frac{2^{2N+1}}{2N+1} \int_0^1 (s - s^2)^N \phi(...) ds + O(y^2/c^2).\]

As for (4) we use Proposition 16 and

\[K_y + (2s-1)K_{\xi} = yJ^{N-1}G - (2s-1)(\xi - x)G(x, |\xi - x|, \xi)J^{N-1} + (2s-1)J^NG_{\xi}\]
\[2^{2N+1}N(s-s^2)^ny^{2N-1} + \frac{2^{N-1}C_0(x,c)}{(N-1)!c^2} (2s-1)(s-s^2)^Ny^{2N} + \]
\[\frac{2NC_3(x,c)}{N!c^2} (2s-1)(s-s^2)^Ny^{2N} + O(y^{2N+1}/c^2)\]

(See Proposition 19). Then by integration by parts we have

\[\int_{0}^{1} (s-s^2)^N \phi(...)ds + O(y^2/c^2).\]

As (2) we get

\[2^{2N+1} \int_{0}^{1} (s-s^2)(-x + \frac{y}{2N+1}(2s-1))D\phi(...)ds + O(y^2/c^2).\]

Thus we get (7.3).

Next we show (7.4). We have

\[\frac{\partial^2 \eta}{\partial M \partial H} = (9) + (10) + (11) + (12) + (13) + (14),\]

By Proposition 18 we have

\[((K_x + K_\xi) - (K_x + K_\xi))(x, y, x + (2s-1)y) =\]
\[= \frac{2NC_2(x,c)}{N!c^2} (s-s^2)^N(2s-1)y^{2N+1} + O(y^{2N+2}/c^2).\]

Thus by integration by parts we get

\[(6) = O(y^{-2N+1}/c^2).\]

By the same discussion as (1) we see (7) = \(O(y^{-2N+1}/c^2)\). By the same discussion as (2) we see

\[(8) = 2^{2N+1}y^{-2N-1} \int_{0}^{1} (s-s^2)^N D^2\phi(...)ds + O(y^{-2N+1}/c^2).\]

Thus we get (7.4).

Next we show (7.5). We see
\[ (9) = -\frac{4N}{2N+1} R^{\frac{-4N+1}{2N+1}} \int_0^1 (K_x + K_\xi)(x, y, \ldots) \phi(\ldots) ds, \]

\[ (10) = 2R^{\frac{-4N+1}{2N+1}} \int_0^1 (-x(K_x + K_\xi)x + (K_x + K_\xi)_x + \frac{y}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi)) \phi(\ldots) ds, \]

\[ (11) = 2R^{\frac{4N}{2N+1}} \int_0^1 (K_x + K_\xi)(-x + \frac{y}{2N+1}(2s-1)) D\phi ds, \]

\[ (12) = -\frac{4N}{2N+1} R^{\frac{-4N+1}{2N+1}} \int_0^1 K D\phi ds, \]

\[ (13) = 2R^{\frac{-4N+1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) D\phi ds \]

\[ (14) = 2R^{\frac{-4N+1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1)) D^2\phi ds. \]

We already know that \((9) = O(y^{-2N+1}/c^2)\). (Recall (1).) Next we look at (10). The first term is \(O(y^{-2N+1}/c^2)\). (Recall (6)). By Proposition 16 and 20 we see

\[(K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi = \]

\[= \frac{2^{N-1}C_1(x,c)}{(N-1)!c^2} y^{2N}(2s-1)(s-s^2)^N \]

\[+ \frac{2^N C_4(x,c)}{N!c^2} y^{2N}(2s-1)(s-s^2)^N \]

\[+ \frac{2^{N-1}C_1(x,c)}{(N-1)!c^2} y^{2N}(2s-1)^3(s-s^2)^{N-1} + O(y^{2N+1}/c^2). \]

By integration by parts we see \((10) = O(y^{-2N+1}/c^2)\). We already know \((11) = O(y^{-2N+1}/c^2)\). Clearly

\[ (12) = -\frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N D\phi ds + O(y^{-2N+1}/c^2). \]

We see

\[ (13) = O(y^{-2N+1}/c^2) + \frac{2}{2N+1} R^{\frac{-4N+1}{2N+1}} \int_0^1 (K_y + (2s-1)K_\xi) D\phi ds. \]

As (4) we have

\[ (13) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N D\phi ds + O(y^{-2N+1}/c^2). \]

Finally we see

\[ (14) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x+(2s-1)\frac{y}{2N+1}) D^2\phi ds + O(y^{-2N+1}/c^2) \]
(Recall (5)). Summing up we get (7.5).

Next we show (7.6).

\[ \frac{\partial^2 \eta}{\partial R^2} = \frac{\partial}{\partial R} (3) + \frac{\partial}{\partial R} (4) + \frac{\partial}{\partial R} (5), \]

\[ \frac{\partial}{\partial R} (3) = (15) + (16) + (17), \]

\[ (15) = -\frac{4N}{(2N + 1)^2} R^{-\frac{4N-1}{2N+1}} \int_0^1 K \phi ds, \]

\[ (16) = \frac{2}{2N + 1} R^{-\frac{4N-1}{2N+1}} \int_0^1 (x(K_x + K_\xi) + \frac{y}{2N + 1}(K_y + (2s - 1)K_\xi)) \phi ds, \]

\[ (17) = \frac{2}{2N + 1} R^{-\frac{4N-1}{2N+1}} \int_0^1 K (-x + \frac{y}{2N + 1}(2s - 1)) D\phi ds, \]

\[ \frac{\partial}{\partial R} (4) = (18) + (19) + (20), \]

\[ (18) = \frac{4N}{2N + 1} R^{-\frac{4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N + 1}(K_y + (2s - 1)K_\xi)) \phi ds, \]

\[ (19) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 K'' \phi ds, \]

where

\[ K'' = x(K_x + K_\xi) + x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) + \]

\[ + \frac{y}{2N + 1}((K_x + K_\xi)_y + (2s - 1)(K_x + K_\xi)_\xi) + \]

\[ - \frac{xy}{2N + 1}((K_x + K_\xi)_y + (2s - 1)(K_x + K_\xi)_\xi) + \]

\[ + \frac{y^2}{2N + 1}((K_y + (2s - 1)K_\xi)_y + (2s - 1)(K_y + (2s - 1)K_\xi)_\xi) + \]

\[ + \frac{y}{(2N + 1)^2}(K_y + (2s - 1)K_\xi), \]

\[ (20) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N + 1}(K_y + (2s - 1)K_\xi)) \]

\[ \times (-x + \frac{y}{2N + 1}(2s - 1)) D\phi ds, \]

\[ \frac{\partial}{\partial R} (5) = (21) + (22) + (23) + (24), \]

\[ (21) = -\frac{4N}{2N + 1} R^{-\frac{4N-1}{2N+1}} \int_0^1 K (-x + \frac{y}{2N + 1}(2s - 1)) D\phi ds, \]

\[ (22) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N + 1}(K_y + (2s - 1)K_\xi)) \]

\[ \times (-x + \frac{y}{2N + 1}(2s - 1)) D\phi ds, \]

\[ (23) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 K (x + \frac{y}{(2N + 1)^2}(2s - 1)) D\phi ds, \]
\[
(24) = 2R^{-\frac{4N+1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))^2 D^2 \phi ds.
\]

First we see
\[
(15) = -\frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),
\]
\[
(16) = \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),
\]
\[
(17) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).
\]

Thus we have
\[
\frac{\partial}{\partial R}(3) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).
\]

Since (18) is similar to (16), we have
\[
(18) = -\frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).
\]

Next let us look at (19). We already know
\[
2^{\frac{4N+1}{2N+1}} \int_0^1 x(K_x + K_\xi) \phi ds = O(y^{-2N+1}/c^2),
\]
\[
2^{\frac{4N+1}{2N+1}} \int_0^1 x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2).
\]

Recalling (10), we see
\[
\frac{2}{2N+1} R^{-\frac{4N+1}{2N+1}} y \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2),
\]
\[
\frac{2}{2N+1} R^{-\frac{4N+1}{2N+1}} x y \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2).
\]

When \(N = 1\), we have
\[
(K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi = 8(s-s^2) + \frac{C_3}{c^2}(2s-1)y - \frac{C_0}{c^2}(2s-1)^3y - \frac{2C_3}{c^2}(2s-1)^3y + O(y^2/c^2).
\]

When \(N \geq 2\), there are bounded functions \(F_j(x, c)\) such that
\[
(K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi =
\]
\[
= 2^{2N+1}N(2N-1)(s-s^2)^N y^{2N-2} + \frac{F_1(x, c)}{c^2}(2s-1)(s-s^2)^{N-1} y^{2N-1} +
\]
\[
+ \frac{F_2(x, c)}{c^2}(2s-1)(s-s^2)^{N-2} y^{2N-1} + \frac{F_3(x, c)}{c^2}(2s-1)^3(s-s^2)^{N-2} y^{2N-1} +
\]
\[
+ \frac{F_4(x, c)}{c^2}(2s-1)^3(s-s^2)^{N-1} y^{2N-1} + \frac{F_5(x, c)}{c^2}(2s-1)^5(s-s^2)^{N-2} y^{2N-1} +
\]
\[
+ O(y^{2N}/c^2).
\]
Thus we see
\[
2R^{\frac{-4N+1}{2(N+1)}} \frac{y^2}{(2N+1)^2} \int_0^1 ((K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_y) \phi ds
= \frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds - \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds
+ O(y^{-2N+1}/c^2).
\]

We have
\[
\frac{2}{(2N+1)^2} R^{\frac{-4N+1}{2}} y \int_0^1 (K_y + (2s-1)K_\xi) \phi ds
= \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).
\]

Therefore
\[
(19) = \frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).
\]

We see
\[
(20) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).
\]

Therefore
\[
\frac{\partial}{\partial R}(4) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).
\]

Next we see
\[
(21) = -\frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),
\]
\[
(22) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),
\]
\[
(23) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),
\]
\[
(24) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1))^2 D^2\phi ds + O(y^{-2N+1}/c^2).
\]

Therefore we get
\[
\frac{\partial}{\partial R}(5) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{(2N+1)^2}(2s-1)) D\phi ds + \\
+ 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1))^2 D^2\phi ds + O(y^{-2N+1}/c^2)
\]

Summing up, we have
\[
\frac{\partial^2 \eta}{\partial R^2} = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds
\]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N (x + \frac{y}{(2N+1)^2} (2s-1)) D\phi ds \]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N (-x + \frac{y}{2N+1} (2s-1))^2 D^2\phi ds \]
\[ = \frac{2^{2N+2} (N+1)}{(2N+1)^2} y^{-2N-1} \int_{0}^{1} (s-s^2)^N (2s-1) y D\phi ds + \]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N (-x + \frac{y}{2N+1} (2s-1))^2 D^2\phi ds \]
\[ = \frac{2^{2N+3}}{(2N+1)^2} y^{-2N-1} \int_{0}^{1} (s-s^2)^{N+1} y^2 D^2\phi ds + \]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N (-x + \frac{y}{2N+1} (2s-1))^2 D^2\phi ds. \]

Thus we get (7.6). QED.

Let us recall the standard entropy \( \eta^* \). This is generated by

\[ \phi^*(x) = A' c^2 \left( \frac{1}{1-u^2/c^2} - \frac{1}{\sqrt{1-u^2/c^2}} \right), \]

where

\[ A' = (2N+1)^{-2N} ((2N+1)/(2N+3)A)^{\frac{2N+1}{2}} (2N-1)!!/2^{N+1}N!. \]

We note that

\[ D^2\phi^*(x) = A'(1 + \frac{u^2/c^2}{1-u^2/c^2})(2 - \sqrt{1-u^2/c^2}) \geq A'. \]

We are going to show that the Hessian \( D_U^2 \eta^* \) dominates any \( D_U^2 \eta \).

**Proposition 22** For each \( \phi \) fixed in \( C^3 \) we have on each compact subset of \( \{\rho \geq 0\} \)

\[ |\langle \xi | D_U^2 \eta \xi \rangle| \leq C(\xi | D_U^2 \eta^* \xi), \]

provided that \( c \) is sufficiently large.

By the assumption we have

\[ R = y^{2N+1} = K \rho (1 + [\rho^{3N+1}/c^2]_1), \]
\[ \frac{dR}{d\rho} = K + [\rho^{3N+1}/c^2]_1, \]
\[ \frac{d^2R}{d\rho^2} = \frac{\rho^{3N+1}}{c^2} [\rho^{3N+1}/c^2]_0, \]

where \( K = ((2N+3)(2N+1)A)^{\frac{2N+1}{2}} \). Using these, we have

\[ \frac{\partial R}{\partial E} = \frac{dR}{d\rho} \frac{1+u^2/c^2}{1-P'u^2/c^4} \]
\[
\frac{\partial R}{\partial F} = -\frac{dR}{d\rho} \frac{2u/c^2}{1 - Pu^2/c^4},
\]

\[
\frac{\partial R}{\partial E} = \frac{R}{\rho + P/c^2} \frac{1 + Pu^2/c^4}{1 - Pu^2/c^4} - \frac{dR}{d\rho} 2xu/c^2 \frac{1}{1 - Pu^2/c^4},
\]

\[
\frac{\partial R}{\partial F} = K(1 - 2xu/c^2) + O(y^2/c^2).
\]

Differentiating once more, we see

\[
\frac{\partial^2 R}{\partial E^2} = -\frac{2}{c^2} \frac{K^2}{y^{2N+1}} \frac{2u}{c^2} (1 - u^2/c^2) + O(y^{-2N+1}/c^2),
\]

\[
\frac{\partial^2 M}{\partial E^2} = -\frac{K^2}{y^{2N+1}} \frac{2u}{c^2} (1 - u^2/c^2) + O(y^{-2N+1}/c^2),
\]

\[
\frac{\partial^2 R}{\partial F^2} = -\frac{2}{c^2} \frac{K^2}{y^{2N+1}} (1 - u^2/c^2) + O(y^{-2N+1}/c^2),
\]

\[
\frac{\partial^2 M}{\partial F^2} = -\frac{K^2}{y^{2N+1}} 2(u + x(1 - u^2/c^2))/c^2 + O(y^{-2N+1}/c^2).
\]

The chain rule gives

\[
\frac{\partial^2 \eta}{\partial E^2} = \left(\frac{\partial R}{\partial E}\right)^2 \frac{\partial^2 \eta}{\partial R^2} + 2 \frac{\partial R}{\partial E} \frac{\partial M}{\partial E} \frac{\partial^2 \eta}{\partial R \partial M} + \left(\frac{\partial M}{\partial E}\right)^2 \frac{\partial^2 \eta}{\partial M^2} + \frac{\partial^2 R}{\partial E \partial F} \frac{\partial \eta}{\partial R} + \frac{\partial^2 M}{\partial E \partial F} \frac{\partial \eta}{\partial M},
\]

and so on. Inserting (7.7) and (7.8) into (7.9), and using Proposition 21, we have

\[
(\xi|D^2 \eta|\xi) = \frac{2^{2N+1}K^2}{y^{2N+1}} \int_0^1 (s - s^2)^N Z[\xi] D^2 \phi ds +
\]

\[
- \frac{2K^2}{y^{2N+1} c^2} (1 - u^2/c^2)(u\xi_0 - \xi_1)^2 \frac{\partial \eta}{\partial R} +
\]

\[
- \frac{2K^2}{y^{2N+1} c^2} (u + x(1 - u^2/c^2))(u\xi_0 - \xi_1)^2 \frac{\partial \eta}{\partial M} +
\]

\[
O(y^{-2N+1}/c^2),
\]

where

\[
Z[\xi] = Z_{00}\xi_0^2 + 2Z_{01}\xi_0\xi_1 + Z_{11}\xi_1^2,
\]
\[ Z_{00} = (1 + u^2/c^2)^2((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) + \\
+ 2(1 + u^2/c^2)(-u + x(1 + u^2/c^2))(-x + \frac{y}{2N+1}(2s-1)) \\
+ (-u + x(1 + u^2/c^2))^2, \\
Z_{01} = -2(1 + u^2/c^2)u/c^2((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) + \\
+ (1 + 3u^2/c^2 - 4x(1 + u^2/c^2)u/c^2)(-x + \frac{y}{2N+1}(2s-1)) + \\
+ (-u + x(1 + u^2/c^2))(1 - 2xu/c^2), \\
Z_{11} = \frac{4u^2}{c^4}((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) + \\
- \frac{4u}{c^2}(1 - 2xu/c^2)(-x + \frac{y}{2N+1}(2s-1)) + \\
+ (1 - 2xu/c^2)^2. \\
\]

It can be shown that
\[ Z[\xi] \geq \kappa s(1-s)y^2, \]
where \( \kappa \) is a positive constant depending on the compact subset of \( \{ \rho \geq 0 \} \).

In fact we see
\[ Z_{00}Z_{11} - Z_{01}^2 = (1 - u^2/c^2) \frac{4}{(2N+1)^2}s(1-s)y^2. \]

On the other hand, we can estimate
\[ |\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1 - u^2/c^2)\frac{\partial \eta}{\partial R}| \leq \frac{\epsilon}{y^{2N+1}}, \]
\[ |\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u + x(1 - u^2/c^2))\frac{\partial \eta}{\partial M}| \leq \frac{\epsilon}{y^{2N+1}}, \]
where \( \epsilon = K'/c^2 \). Let us introduce the parameters
\[ \zeta_0 = \xi_0, \quad \zeta_1 = \xi_1 - u\xi_0. \]

Then we have
\[ Z[\xi] = Q_{00}\zeta_0^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{11}\zeta_1^2, \]
and
\[ Q_{00} = Q_{00}^{(1)}(x)(2s-1)y + Q_{00}^{(2)}(x, s)y^2, \]
\[ Q_{01} = Q_{01}^{(1)}(x)(2s-1)y + Q_{01}^{(2)}(x, s)y^2, \]
\[ Q_{11} = Z_{11} = 1 + O(1/c^2) > 0. \]

Therefore if \( |D^2\phi| \leq C \), we see
\[ |(\xi|D^2\eta|\xi)| \leq \frac{2^{2N+1}K^2C}{y^{2N+1}} \int_0^1 (s - s^2)^N Z[\xi]ds \]
\[ + \frac{12\epsilon}{y^{2N+1}} \int_0^1 (s-s^2)^N \zeta^2 ds + O(y^{-2N+1}/c^2) \]
\[ \leq \frac{2^{2N+1}K^2C}{y^{2N+1}} \int_0^1 (s-s^2)^N (Q_{11}(1+\epsilon')\zeta_1^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{00} + O(y^{-2N+1}/c^2). \]

But since \( Q_{00}^{(0)} = Q_{01}^{(0)} = 0 \), \( \int_0^1 (s-s^2)^N (2s-1)ds = 0 \), we see
\[ \int_0^1 (s-s^2)^N (-2\epsilon'Q_{01}\zeta_0\zeta_1 - \epsilon'Q_{00}\zeta_0^2)ds = O(y^{-2N+1}/c^2). \]

Therefore we get
\[ |(\xi|D_U^2\eta\xi)| \leq \frac{2^{2N+1}K^2C(1+\epsilon')}{y^{2N+1}} \int_0^1 (s-s^2)^N \xi[Z\xi]ds + O(y^{-2N+1}/c^2). \]

Similarly, if \( D^2\phi^* \geq \mu \), we have
\[ (\xi|D_U^2\eta^*\xi) \geq \frac{2^{2N+1}K^2\mu(1-\epsilon')}{y^{2N+1}} \int_0^1 (s-s^2)^N \xi[Z\xi]ds + O(y^{-2N+1}/c^2). \]

Thus we get
\[ |(\xi|D_U^2\eta\xi)| \leq \frac{C(1+\epsilon')}{\mu(1-\epsilon')}(\xi|D_U^2\eta^*\xi) + O(y^{-2N+1}/c^2). \]

But we know
\[ (\xi|D_U^2\eta^*\xi) \geq \kappa|\xi|^2y^{-2N+1}. \]

Hence if \( c \) is sufficiently large we get the required estimate. QED.

As for the first derivatives, the following conclusion is now clear.

**Proposition 23** On each compact subset of \( \{\rho \geq 0\} \), we have
\[ \left|\frac{\partial \eta}{\partial E}\right| + \left|\frac{\partial \eta}{\partial F}\right| \leq C. \]

8 **Usefull entropies**

Let us consider an entropy \( \eta \) generated by \( \phi \), that is,
\[ \eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)d\xi. \]

The corresponding entropy flux \( q \) is given by integrating the different equations
\[ \frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \quad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}. \]
We can solve these equations as
\[ q = \lambda_2 \eta - \int_z^w \frac{\partial \lambda_2}{\partial w} \eta dw = \lambda_1 \eta + \int_z^w \frac{\partial \lambda_1}{\partial z} \eta dz. \]

Thus we get the formula
\[ q(x, y) = \int_{x-y}^{x+y} L(x, y, \xi) \phi(\xi) d\xi, \quad (8.2) \]

where
\[ L(x, y, \xi) = \lambda_1 K(x, y, \xi) + L_1(x, y, \xi) \]
\[ = \lambda_2 K(x, y, \xi) + L_2(x, y, \xi), \]
\[ L_1(x, y, \xi) = 2 \int_{(x+y-\xi)/2}^{y} \mu_1(x + y - Y, Y) K(x + y - Y, Y, \xi) dY, \]
\[ L_2(x, y, \xi) = -2 \int_{(-x+y+\xi)/2}^{y} \mu_2(x - y + Y, Y) K(x - y + Y, Y, \xi) dY, \]
\[ \mu_1(x, y) = \frac{\partial \lambda_1}{\partial z} = \frac{1 - u^2/c^2}{2(1 - \sqrt{P}u/c^2)} \left(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P}\right) = \frac{N}{2N + 1} + O(1/c^2), \]
\[ \mu_2(x, y) = \frac{\partial \lambda_2}{\partial w} = \frac{1 - u^2/c^2}{2(1 + \sqrt{P}u/c^2)} \left(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P}\right) = \frac{N}{2N + 1} + O(1/c^2). \]

In this section we will construct various kinds of useful entropies.

1) Let us put
\[ \eta_k^1(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) k^{N+1} e^{k\xi} d\xi, \]
\[ \eta_k^2(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) k^{N+1} e^{-k\xi} d\xi. \]

**Proposition 24** If $1/c^2$ is sufficiently small, we have
\[ \eta_k^1 > 0, \quad \eta_k^2 > 0 \quad \text{for } y > 0, \quad (8.3) \]
\[ \eta_k^1 = 2^N N! y^N (1 + O(y/c^2)) e^{k(x+y)} (1 + O(1/k)), \]
\[ \eta_k^2 = 2^N N! y^N (1 + O(y/c^2)) e^{-k(x-y)} (1 + O(1/k)) \quad (8.4) \]
uniformly on each compact subset of \( \{y > 0\} \). Moreover

\[
q_k^1 = \eta_k^1(\lambda_2 + O(1/k)), \\
q_k^2 = \eta_k^2(\lambda_1 + O(1/k))
\]  

(8.5)

uniformly on each compact subset of \( \{y \geq 0\} \) and

\[
\eta_k^2 q_k^1 - \eta_k^1 q_k^2 = (2^N N!)^2 y^{2(N-1)} \left( \frac{1}{2N+1} + O(1/c^2) \right) e^{2ky}(y+O(1/k))^3. 
\]  

(8.6)

Proof. Since \( K = (1 + O(y/c^2))(y^2 - (x - \xi)^2)^N \), we see

\[
\eta_k^1 = (1 + O(y/c^2)) \int_{x-y}^{x+y} (y^2 - (x - \xi)^2)^N k^{N+1} e^{k\xi} d\xi \\
\eta_k^2 = (1 + O(y/c^2)) 2^{2N+1} y^N e^{k\xi} f(ky)
\]

where

\[
f(r) = r^{N+1} e^{-r} \int_0^1 (s(1-s))^N e^{2rs} ds \\
= e^r \int_0^r \left( \sigma - \frac{\sigma^2}{r} \right)^N e^{-2\sigma} d\sigma.
\]

It is easy to see

\[
e^{-r} f(r) = 2^{-(N+1)} N! + O(1/r)
\]

This implies (8.4). We note

\[
\eta^1 = (1 + O(1/c^2)) 2^N N! y^{N-1} e^{k(x+y)} (y + O(1/k)) \\
\eta^2 = (1 + O(1/c^2)) 2^N N! y^{N-1} e^{-k(x-y)} (y + O(1/k))
\]

uniformly on \( \{y \geq 0\} \). Let us consider the flux. We have

\[
L_2(x, y, \xi) = -2 \int_{(-x+y+\xi)/2}^y \mu_2(x - y + Y, Y) K(x - y + Y, Y, \xi) dY \\
= -2 \left( \frac{N}{2N+1} + O(1/c^2) \right) \int_{(-x+y+\xi)/2}^y (Y^2 - (x - y + Y - \xi)^2)^N dY \\
= -\left( \frac{N}{(2N+1)(N+1)} + O(1/c^2) \right) (y - x + \xi)^N (y + x - \xi)^{N+1},
\]

\[
q^1 - \lambda_2 \eta^1 = -\left( \frac{N}{(2N+1)(N+1)} + O(1/c^2) \right) \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi.
\]

But

\[
0 \leq \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi \\
= (N+1) k^N \int_{x-y}^{x+y} (y^2 - (x - \xi)^2)^N e^{k\xi} d\xi
\]
\[- Nk^N \int_{x-y}^{x+y} (y - x + \xi)^{N-1}(y + x - \xi)^{N+1}e^{k\xi}d\xi \leq (N + 1) \frac{1}{k} \int_{x-y}^{x+y} (y^2 - (x - \xi)^2)^N k^{N+1}e^{k\xi}d\xi. \]

Thus
\[ q^1 - \lambda_2 \eta^1 = O(1/k)\eta^1. \]

Since
\[ \lambda_2 - \lambda_1 = \frac{\sqrt{P'}(1 - u^2/c^2)}{1 - P'u^2/c^4} = \left(\frac{1}{2N+1} + O(1/c^2)\right)y, \]
we have
\[ \eta^2 q^1 - \eta^1 q^2 = \eta^1 \eta^2 \left(\frac{1}{2N+1} + O(1/c^2)\right)y + O(1/k). \]

This implies (8.6). QED.

2) Let \( \psi \) be a function in \( C_0^\infty(-1,1) \) such that \( \psi \geq 0, \int \psi = 1 \). We put
\[
\begin{align*}
\phi_n^3(x) &= \psi_n(x) = n\psi(n(x-a)), \\
\phi_n^4(x) &= -D\psi_n(x), \\
\eta_n^3(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)\phi_n^3(\xi)d\xi, \\
\eta_n^4(x,y) &= \int_{x-y}^{x+y} K(x,y,\xi)\phi_n^4(\xi)d\xi, \\
\eta^3(x,y) &= K(x,y,a)X, \\
\eta^4(x,y) &= K_\xi(x,y,a)X, \\
q^3(x,y) &= L(x,y,a)X, \\
q^4(x,y) &= L_\xi(x,y,a)X, \\
X &= 1 \quad (x - y < a < x + y) \\
    &= \frac{1}{2} \quad (|x - a| = y) \\
    &= 0 \quad (|x - a| > y).
\end{align*}
\]

**Proposition 25** As \( n \to \infty \), we have
\[ \eta_n^3 \to \eta^3, \quad q_n^3 \to q^3, \quad \eta_n^4 \to \eta^4, \quad q_n^4 \to q^4. \]

Moreover
\[
\begin{align*}
|\eta_n^3| &\leq My^{2N}, & |q_n^3| &\leq My^{2N}(|x| + y), & (8.7) \\
|\eta_n^4| &\leq My^{2N-1}, & |q_n^4| &\leq My^{2N-1}(|x| + y), & (8.8) \\
\eta^3 q^4 - \eta^4 q^3 &= \frac{N}{(2N+1)(N+1)}(1 + O(1/c^2))(y^2 - (x - a)^2)^{2N}. & (8.9)
\end{align*}
\]
Proof. We note

\[ K_\xi = -(\xi - x)G(x, |\xi - x|, \xi) \frac{1}{2^{N-1}(N-1)!} (y^2 - (x - \xi)^2)^{N-1} + J^N G_\xi \]

\[ = (2N(x - \xi) + O(1/c^2)(\xi - x)^2)(y^2 - (x - \xi)^2)^{N-1} + O(1/c^2)(y^2 - (x - \xi)^2)^N, \]

\[ L_{1,\xi} = 2 \int_{(x+y-\xi)/2}^{y} \mu_1(x+y-Y, Y)K_\xi(x+y-Y,Y,\xi)dY. \]

The estimates (8.7), (8.8) can be seen easily. Let us consider

\[ \eta^3 q^4 - \eta^4 q^3 = (KL_\xi - LK_\xi)(x,y,a). \]

Suppose \( x - a \geq 0 \). Then

\[ \frac{1}{2}(KL_\xi - LK_\xi) = K \int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y, Y, a)dY - K_\xi \int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y, Y, a)dY. \]

We note

\[ 0 \leq \frac{x+y-a}{2} \leq x-y+Y-a \leq x-a \leq y. \]

Hence we have

\[ \int_{(x+y-a)/2}^{y} \mu_1 K_\xi(x+y-Y, Y, a)dY \]

\[ = \left( \frac{N}{2N+1} + O(1/c^2) \right) 2N \int_{(x+y-a)/2}^{y} (x+y-Y-a) (Y^2 - (x+y-Y-a)^2)^{N-1} dY + \]

\[ + O(1/c^2) \int_{(x+y-a)/2}^{y} (Y^2 - (x+y-Y-a)^2)^N dY \]

\[ = \left( \frac{N^2}{2(2N+1)} + O(1/c^2) \right) (x+y-a)^{N-1} (-x+y+a)^N \frac{1}{N(N+1)} (y+(2N+1)(x-a)) + O(1/c^2)(y^2 - (x-a)^2)^N. \]

Thus

\[ K \int_{(x+y-a)/2}^{y} \mu_1 K dY \]

\[ = \left( \frac{N}{2(2N+1)(N+1)} + O(1/c^2) \right) (y^2 - (x-a)^2)^{2N-1} (-x+y+a)(y+(2N+1)(x-a)) + O(1/c^2)(y^2 - (x-a)^2)^{2N}. \]

Also we have

\[ K_\xi \int_{(x+y-a)/2}^{y} \mu_1 dY \]

\[ = \left( \frac{N^2}{(2N+1)(N+1)} + O(1/c^2) \right) (x-a)(-x+y+a)(y^2 - (x-a)^2)^{2N-1} + O(1/c^2)(-x+y+a)(y^2 - (x-a)^2)^{2N}. \]
\[
\frac{1}{2}(KL_{\xi} - LK_{\xi}) = \left( \frac{N}{2(2N+1)(N+1)} + O(1/c^{2}) \right)(y^{2} - (x-a)^{2})^{2N}.
\]

Here we have used

\[
0 \leq (x-a)(y-(x-a)) \leq y^{2} - (x-a)^{2},
\]

\[
0 \leq (y-x+a)(y+(2N+1)(x-a)) \leq (2N+1)(y^{2} - (x-a)^{2})
\]

provided that \(0 \leq x-a \leq y\). When \(x-a \leq 0\), we can discuss in a similar manner by using \(L_{2}\). QED.

3) Let \(\Phi\) be a function in \(C_{0}^{\infty}(-1,1)\) such that \(\int \Phi = 0\) and the support \(\text{supp} \Phi\) is \([-1+\alpha, 1+\alpha]\), where \(\alpha\) is a small positive number. We put

\[
\psi_{n}(x) = n\Phi(n(x-a)),
\]

\[
\eta_{n}^{5}(x,y) = \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1} \psi_{n}(\xi) d\xi,
\]

\[
\eta_{n}^{5}(x,y) = \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1} \psi_{n}(\xi) d\xi;
\]

\[
\hat{\Phi}(x) = \frac{d}{dx}(x \int_{-1}^{x} \Phi),
\]

\[
\hat{\psi}_{n}(x) = n\hat{\Phi}(n(x-a)),
\]

\[
\eta_{n}^{6}(x,y) = \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1} \hat{\psi}_{n}(\xi) d\xi,
\]

\[
\eta_{n}^{6}(x,y) = \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1} \hat{\psi}_{n}(\xi) d\xi;
\]

\[
B_{n}^{3} = \eta_{n}^{5}q_{n}^{5} - \eta_{n}^{5}q_{n}^{5},
\]

\[
B_{n}^{4} = \eta_{n}^{5}q_{n}^{5} - \eta_{n}^{5}q_{n}^{5},
\]

\[
B_{n} = \eta_{n}^{5}q_{n}^{6} - \eta_{n}^{6}q_{n}^{6}.
\]

Let us divide the domain \(\Sigma = \{-B \leq x-y \leq x+y \leq B\}\) into the following 5 parts.

\[
S_{0} = \left\{ -\frac{1}{n} < x+y-a \leq \frac{1}{n}, -\frac{1}{n} \leq x-y-a < \frac{1}{n} \right\} \cap \Sigma,
\]

\[
S_{1} = \left\{ \frac{1}{n} < x+y-a, x-y-a < -\frac{1}{n} \right\} \cap \Sigma,
\]

\[
S_{L} = \left\{ -\frac{1}{n} < x+y-a \leq \frac{1}{n}, x-y-a < -\frac{1}{n} \right\} \cap \Sigma,
\]

\[
S_{R} = \left\{ \frac{1}{n} < x+y-a, -\frac{1}{n} \leq x-y-a < \frac{1}{n} \right\} \cap \Sigma,
\]

\[
S = \Sigma - (S_{0} \cup S_{1} \cup S_{L} \cup S_{R} \cup S_{R}).
\]
Proposition 26 We have

\[ |B_n^3| \leq M/n, \quad |B_n^4| \leq M \]  \hspace{1cm} (8.10)

on \( \Sigma \), and

\[ |B_n| \leq M/n \]  \hspace{1cm} (8.11)

on \( S_0 \cup S_1 \cup S \). Moreover, on \( S_L \), we have

\[ B_n = ny^{2N}A_1 + y^N A_2 + A_3, \]  \hspace{1cm} (8.12)

where

\[
A_1 = \left( \frac{N(2^N N!)^2}{2N+1} + O(1/\epsilon^2) \right) \left( \int_{-1}^{n(x+y-a)} \Phi \right)^2, \\
|A_2| \leq M \left( \left| \int_{-1}^{n(x+y-a)} \Phi \right| + |\Phi(n(x+y-a))| \right), \\
|A_3| \leq \frac{M}{n}.
\]
on \( S_R \), we have

\[ B_n = ny^{2N}C_1 + y^N C_2 + C_3, \]

\[
C_1 = \left( \frac{N(2^N N!)^2}{2N+1} + O(1/\epsilon^2) \right) \left( \int_{-1}^{n(x-y-a)} \Phi \right)^2, \\
|C_2| \leq M \left( \left| \int_{-1}^{n(x-y-a)} \Phi \right| + |\Phi(n(x-y-a))| \right), \\
|C_3| \leq \frac{M}{n}.
\]

Proof. For the simplicity, we write \( \eta_n = \eta_n^5, q_n = q_n^5, \hat{\eta}_n = \eta_n^6, \hat{q}_n = q_n^6 \).

It is easy to see inductively that, for \( G_j = J^j G = K_{N-j} \), we have

\[ \partial_{\xi}^p G_j = J \partial_{\xi}^p G_{j-1} \]

for \( j \geq p + 1 \) and

\[ \partial_{\xi}^p G_p = (-1)^p (\xi - x)^p G(x, |\xi - x|, \xi) + J \partial_{\xi}^p G_{p-1}. \]

Therefore

\[ \partial_{\xi}^p K = \partial_{\xi}^p G_N(x, y, \xi) = 0 \]

for \( p \leq N-1 \) and \( y = |x - \xi| \). Thus by integration by parts we have

\[
\eta_n = (-1)^N \delta^N_{\xi} K(x, y, x + y) \psi_n(x + y) + \\
- (-1)^N \delta^N_{\xi} K(x, y, x - y) \psi_n(x - y) + \\
+ F^1_n(x, y), \\
F^1_n(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \delta^{N+1}_{\xi} K(x, y, \xi) \psi_n(\xi) d\xi.
\]
We see
\[ \partial_{\xi}^{p} L_{2}(x, y, \xi) = -2 \int_{(-x+y+\xi)/2}^{y} \mu_{2} \partial_{\xi}^{p} K(x-y+Y, Y, \xi) dY \]
for \( p \leq N-1 \). Therefore
\[ \partial_{\xi}^{p} L_{2}(x, y, x+y) = \partial_{\xi}^{p} L_{2}(x, y, x-y) = 0 \]
for \( p \leq N-1 \). Moreover we see
\[ \partial_{\xi}^{N} L_{2}(x, y, x+y) = 0. \]

Therefore by integration by parts we have
\[ \sigma_{n}(x, y) = q_{n}(x, y) - \lambda_{2} \eta_{n}(x, y) \]
\[ = (-1)^{N} \partial_{\xi}^{N} L_{2}(x, y, x-y) \psi_{n}(x-y) + F_{n}^{2}(x, y), \]
\[ F_{n}^{2}(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_{\xi}^{N+1} L_{2}(x, y, \xi) \psi_{n}(\xi) d\xi. \]

Similarly
\[ \bar{\sigma}_{n}(x, y) = q_{n}(x, y) - \lambda_{1} \eta_{n}(x, y) = \]
\[ = (-1)^{N} \partial_{\xi}^{N} L_{1}(x, y, x+y) \psi_{n}(x+y) + \bar{F}_{n}^{2}(x, y), \]
\[ \bar{F}_{n}^{2}(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_{\xi}^{N+1} L_{1}(x, y, \xi) \psi_{n}(\xi) d\xi. \]

We note
\[ \partial_{\xi}^{N} K(x, y, \xi) = (-1)^{N} (\xi-x)^{N} G(x, |x-\xi|, \xi) + J \partial_{\xi}^{N} G_{n-1}. \]

It is easy to see inductively that
\[ \partial_{\xi}^{p+1} G_{p}(x, y, \xi) = (-1)^{p} \frac{p(p+1)}{2} (\xi-x)^{p-1} G(x, |x-\xi|, \xi) + \]
\[ + (\xi-x)^{p} H_{p}(x, \xi) + J \partial_{\xi}^{p} G_{p-1}, \]
where \( H_{p} = O(1/c^{2}) \). Therefore
\[ \partial_{\xi}^{N+1} K(x, y, \xi) = (-1)^{N} \frac{N(N+1)}{2} (\xi-x)^{N-1} G(x, |x-\xi|, \xi) + \]
\[ + (\xi-x)^{N} H_{N}(x, \xi) + J \partial_{\xi}^{N} G_{N-1}. \]

1) Suppose \((x, y) \in S\). Then it is clear that \( \eta^{3}, \eta^{4}, q^{3}, q^{4}, \eta_{n}, q_{n}, \)
\( B_{n}^{3}, B_{n}^{4}, B_{n} \) all vanish.
2) Suppose \((x, y) \in S_0\). Then we see

\[
\eta^3 = K(x, y, a) = O(((y^2 - (x - a)^2)^N)
\]
\[
= O(n^{-2N}),
\]
\[
\eta^4 = K_\xi(x, y, a) = O(|x - a|(y^2 - (x - a)^2)^{N-1}) + O((y^2 - (x - a)^2)^N)
\]
\[
= O(n^{-2N+1}),
\]
\[
\sigma^3 = L_2(x, y, a)
\]
\[
= -2 \int_{(-x+y+a)/2}^{y} \mu_2 K(x, Y, Y, a) dY
\]
\[
= O(n^{-2N-1}),
\]
\[
\sigma^4 = L_{2,\xi}(x, y, a)
\]
\[
= -2 \int_{(-x+y+a)/2}^{y} \mu_2 K_\xi(x, Y, Y, a) dY
\]
\[
= O(n^{-2N}).
\]

Since \(y = O(1/n)\) and \(\psi_n = O(n)\), we see

\[
(-1)^N \partial_\xi^N K(x, y, x+y) \psi_n(x+y) +
\]
\[
- (-1)^N \partial_\xi^N K(x, y, x-y) \psi_n(x-y) =
\]
\[
= O(n^{-N}+1).
\]

Since \(P_n^1 = O(1)\), we have \(\eta_n = O(1)\). We see

\[
\partial_\xi^N L_2(x, y, x-y) = -2 \int_0^y \mu_2 \partial_\xi^N K(x-y+Y, Y, x-y) dY = O(n^{-N-1}).
\]

Therefore

\[
-(-1)^N \partial_\xi^N L_2(x, y, x-y) \psi_n(x-y) = O(n^{-N}).
\]

Since

\[
\partial_\xi^{N+1} L_2(x, y, \xi) = \mu_2 \partial_\xi^N K((x-y+\xi)/2, (-x+y+\xi)/2, \xi) +
\]
\[
- 2 \int_{(-x+y+\xi)/2}^{y} \partial_\xi^{N+1} K(x-y+Y, Y, \xi) dY
\]
\[
= O((x+y+\xi)^N) + O(x+y-\xi),
\]

we see

\[
F_n^2(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x, y, \xi) \psi_n(\xi) d\xi
\]
\[
= O(n^{-1}).
\]
Hence $\sigma_n = O(n^{-1})$. Therefore

$$
B_n^3 = \eta^3 \sigma_n - \eta_n \sigma^3 = O(n^{-2N-1}),
B_n^4 = \eta^4 \sigma_n - \eta_n \sigma^4 = O(n^{-2N}),
B_n = \eta_n \tilde{\sigma}_n - \tilde{\eta}_n \sigma_n = O(n^{-1}).
$$

3) Suppose $(x, y) \in S_1$, where $x + y > a + \frac{1}{n}$ and $x - y < a - \frac{1}{n}$. Then $\psi_n(x+y) = \psi_n(x-y) = \hat{\psi}_n(x+y) = \hat{\psi}_n(x-y) = 0$. So, $\eta_n = F_n^1, \sigma_n = F_n^2$, and so on. But

$$
F_n^1(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_{\xi}^{N+1} K(x, y, \xi) \psi_n(\xi) d\xi
$$

since $\int \Phi = 0$ and $\partial_{\xi}^{N+1} K$ is Lipschitz continuous. Same estimates hold for $F_n^2, \hat{F}_n^1, \hat{F}_n^2$. Thus

$$
B_n^3 = \eta^3 \sigma_n - \eta_n \sigma^3 = O(1/n),
B_n^4 = \eta^4 \sigma_n - \eta_n \sigma^4 = O(1/n),
B_n = F_n^1 \hat{F}_n^2 - \hat{F}_n^1 F_n^2 = O(1/n^2).
$$

4) Suppose $(x, y) \in S_L$, where $|x + y - a| \leq 1/n$. It is easy to see $\eta^3 = O(n^{-N}), \eta^4 = O(n^{-N+1}), \sigma^3 = O(n^{-N-1}), \sigma^4 = O(n^{-N})$. Since $n(x-y-a) < -1$, we have $\psi_n(x-y) = 0$. Thus $\eta_n = O(n), \sigma_n = F_n^2 = O(1)$. Therefore

$$
B_n^3 = \eta^3 \sigma_n - \eta_n \sigma^3 = O(n^{-N}),
B_n^4 = \eta^4 \sigma_n - \eta_n \sigma^4 = O(n^{-1-N}).
$$

Let us estimate $B_n = \eta_n \tilde{\sigma}_n - \tilde{\eta}_n \sigma_N$. Since

$$
\partial_{\xi}^{N+1} K = (-1)^{N} \frac{N(N+1)}{2} (\xi - x)^{N-1} G(x, |x-\xi|, \xi) +
+ (\xi - x)^{N} H_N(x, \xi) + J \partial_{\xi}^{N} G_{N-1},
$$

we have

$$
F_n^1 = (-1)^{N+1} \int_{x-y}^{x+y} \partial_{\xi}^{N+1} K(x, y, \xi) \psi_n(\xi) d\xi =
$$

$$
= (-1)^{N+1}((-1)^{N} \frac{N(N+1)}{2} 2^N N!(a-x)^{N-1} + F'(x, a)) \int_{-1}^{n(x+y-a)} \Phi
+ O(1/n) =
= \frac{N(N+1)}{2} 2^N N!y^{N-1}(1 + F''(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi +
+ O(1/n),
$$
where $F' = O(1/c^2)|x - a|^N$, $F'' = O(1/c^2)$. On the other hand
\[ \partial_x^N K(x, y, x + y) = (-1)^N y^N G(x, y, x + y). \]

Hence
\[
\eta_n = ny^NG(x, y, x + y)\Phi(n(x + y - a)) + \frac{N(N + 1)}{2}2^N N!y^{N-1}(1 + F''(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi + O(1/n).
\]

Since
\[
\partial_x^{N+1} L_2(x, y, \xi) = \mu_2 \partial_x^N K((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + 2 \int_{(-x+y+\xi)/2}^{y} \mu_2 \partial_x^{N+1} K(x-y+Y, Y, \xi) dY = \left(\frac{N}{2N+1} + O(1/c^2)\right)(-1)^N \left(\frac{-x+y+\xi}{2}\right)^N \times G((x+y+\xi)/2, (-x+y+\xi)/2, \xi) + O(x+y-\xi),
\]
we see
\[
\sigma_n = \frac{F_n^2}{F_n} = (-1)^{N+1} \int_{x-y}^{x+y} \partial_x^{N+1} L_2(x, y, \xi) \psi_n(\xi) d\xi = -\frac{N}{2N+1}2^N N!y^N (1 + L'(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi + O(1/n),
\]
where $L' = O(1/c^2)$. Here we have used
\[
\left(\frac{-x+y+a}{2}\right)^N = (y - \frac{x+y-a}{2})^N = y^N + O(1/n).
\]
Similar estimates hold for $\tilde{\eta}_n, \tilde{\sigma}_n$. Thus
\[
B_n = ny^{2N}A_1 + y^N A_2 + A_3,
\]
where
\[
A_1 = -G \frac{N}{2N+1}2^N N!(1 + L')\Phi(\beta) \int_{-1}^{\beta} \Phi + G \frac{N}{2N+1}2^N N!(1 + L')\Phi(\beta) \int_{-1}^{\beta} \Phi = \frac{N}{2N+1}2^N N!G(1 + L')(\int_{-1}^{\beta} \Phi)^2,
\]
\[
\beta = n(x + y - a).
\]
The estimates on $S_R$ can be obtained in a similar manner considering $\overline{\sigma}^3, \overline{\sigma}^4, \overline{\sigma}_n$. QED.

If we put

\[ \hat{B}_n^3 = \eta^3 n_n^6 - \eta_n^6 q^3, \]
\[ \hat{B}_n^4 = \eta^4 n_n^6 - \eta_n^6 q^4, \]

then the same estimates hold.

9 Compactness of $\eta_t + q_x$

Let us consider an entropy $\eta$ generated by $\phi$ through the generalized Darboux formula and its flux $q$. In this section we will prove

Lemma 1 Let $U^\Delta$ be the approximate solutions constructed in Section 4. Then $\eta(U^\Delta)_t + q(U^\Delta)_x$ lies in a compact subset of $\mathcal{H}_{loc}^{-1}(\Omega)$, $\Omega$ being a bounded open subset of $\{t \geq 0\}$.

Proof. Let $\Phi$ be a test function and we consider

\[ J = \int \int (\eta(U^\Delta)\Phi_t + q(U^\Delta)\Phi_x)dxdt \]
\[ = N + L + \Sigma, \]
\[ N = - \int \eta(U^\Delta(+0, x))\Phi(0, x)dx, \]
\[ L = \sum_n \int [\eta(U^\Delta(t, x)]_{t=n\Delta t-0}^{t=n\Delta t+0}\Phi(n\Delta t, x)dx, \]
\[ \Sigma = \int \sum_{\text{shock}} (\sigma[\eta] - [q])\Phi dt. \]

Since $U^\Delta$ is bounded, we see

\[ |N| \leq M||\Phi||_C. \]

Let us look at $L$. We see

\[ L = L_1 + L_2, \]
\[ L_1 = \sum_{j,n} \Phi(n\Delta t, (2j+1)\Delta x) \int_{2j\Delta x}^{(2j+2)\Delta x} \eta(U^\Delta))_{t=n\Delta t+0}^{t=n\Delta t-0}dx, \]
\[ L_2 = \sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} (\Phi(n\Delta t, x) - \Phi(n\Delta t, (2j+1)\Delta x) \times \times \eta(U^\Delta))_{t=n\Delta t+0}^{t=n\Delta t-0}dx. \]
We note

\[
\begin{align*}
[\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} &= D_U \eta(U^\Delta(n\Delta t+0), x)[U^\Delta] \\
&+ \int_0^1 (1-\theta)([U^\Delta]|D_U^2(U^\Delta(n\Delta t+0) + \theta[U^\Delta]).[U^\Delta])d\theta.
\end{align*}
\]

and

\[
\int_{2j\Delta x}^{(2j+2)\Delta x}[U^\Delta]dx = 0
\]

by the scheme. Therefore

\[
|L_1| \leq M||\Phi||_c \sum_{j,n} \int \int_0^1 (1-\theta)|F(\theta, \eta)|d\theta dx,
\]

where

\[
F(\theta, \eta) = ([U^\Delta]|D_U^2 \eta(U^\Delta(n\Delta t+0) + \theta[U^\Delta]).[U^\Delta]).
\]

By Proposition 22 we know \(|F(\theta, \eta)| \leq MF(\theta, \eta^*). But in the proof of Proposition 7 we know

\[
\sum_{j,n} \int \int_0^1 (1-\theta)F(\theta, \eta^*)d\theta dx \leq C.
\]

Thus we know

\[
|L_1| \leq M||\Phi||_c.
\]

In the proof of Proposition 7 we know

\[
\sum_{j,n} \int \int_{2j\Delta x}^{(2j+2)\Delta x}||U^\Delta||^2 dx \leq C.
\]

Therefore

\[
|L_2| \leq 2^\alpha||\Phi||_c \sum_n \int (\Delta x)^\alpha||[\eta(U^\Delta)]||dx
\]

\[
\leq 2^{\alpha-1}||\Phi||_c \sum_n \int ((\Delta x)^{\alpha+\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}}||[\eta(U^\Delta)]||^2)dx
\]

\[
\leq M||\Phi||_c ((\Delta x)^{\alpha-\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} \sum \int ||U^\Delta||^2 dx
\]

\[
\leq M'((\Delta x)^{\alpha-\frac{1}{2}})||\Phi||_c,
\]

where we use the boundedness of \(D_U \eta\) and \(n = O(1/(\Delta x))\). Next we look at \(\Sigma\). Along the shock we have

\[
\sigma[\eta(U)] = [q(U)]
\]

\[
= \int_{\rho L}^{\rho R} \left(-\frac{d\sigma}{d\rho}\right) \int_0^1 \theta(U - U_L|D_U^2 \eta(U_L + \theta(U - U_L))(U - U_L))d\theta).
\]
This implies

$$|\sigma[\eta] - [q]| \leq M(\sigma[\eta^*] - [q^*]).$$

But we know

$$\int \sum_{\text{shock}} (\sigma[\eta^*] - [q^*]) dt \leq C$$

in the proof of Proposition 7. Therefore

$$|\Sigma| \leq M||\Phi||_{c}.$$ 

Summing up, we know the compactness. QED.

10 Convergence of approximate solutions

We consider the approximate solutions $U^\Delta$ constructed in Section 4. Since $U^\Delta$ is bounded, there is a sequence $U^{\Delta_n}$ and a family of Young measures $\nu_{t,x}$ such that $\text{supp} \nu_{t,x} \subset \Sigma = \Sigma_B$ and for any continuous function $f$

$$f(U^{\Delta_n}(t,x)) \to \bar{f} = \text{<}\nu_{t,x}, f>$$

in $L^\infty$ weak star topology. By Lemma 1, we can apply the compensated compactness theory, and we can assume

$$(\eta q' - \eta'q)(U^{\Delta_n}) \to <\nu, q > <\nu, q'> - <\nu, \eta' > <\nu, q >$$

in $L^\infty$ weak star. Here $\eta, q; \eta', q'$ are arbitrary Darboux entropy pairs. Thus we have

Lemma 2 For any pairs $(\eta, q), (\eta', q')$ of Darboux entropies-entropy flux, the identity

$$<\nu, \eta q' - \eta'q > = <\nu, \eta > <\nu, q'> - <\nu, \eta' > <\nu, q >$$

holds a.e. $(t, x)$, where $\nu = \nu_{t,x}$.

Since entropies we will use are countably many, we can assume that the above identity holds outside a null set which is common to all $\eta$. We fix $(t, x)$ at which the identity holds, and we write $\nu = \nu_{t,x}$. Of course $\text{supp} \nu \subset \Sigma$. Suppose that $\text{supp} \nu \cap \{\rho > 0\} \neq \phi$. Let $\Sigma_0$ be the smallest triangle $\{z_0 \leq z \leq w \leq w_0\}$ such that $\text{supp} \nu \cap \{\rho > 0\} \subset \Sigma_0$. Let us denote by $P_0$ the state $(w_0, z_0)$. It will be verified that $\nu = \delta_{P_0}$. (the Dirac measure).

First we show

Proposition 27

$$P_0 \in \text{supp} \nu.$$
Proof. Suppose $P_0 \notin \text{supp. } \nu$. Since $\Sigma_0$ is the smallest triangle containing $\text{supp. } \nu \cap \{\rho > 0\}$, $w = w_0$ and $z = z_0$ intersect with $\text{supp. } \nu \cap \{\rho > 0\}$.

In neighborhoods of these intersection points, we have

\[
\eta^1 \geq \frac{1}{M} e^{k(w_0 - \epsilon)},
\eta^2 \geq \frac{1}{M} e^{-k(z_0 + \epsilon)}.
\]

(See Proposition 24). Since $\nu, \eta^1, \eta^2$ are nonnegative, we see

\[
<\nu, \eta^1> \geq \frac{1}{M} e^{k(w_0 - \epsilon)},
<\nu, \eta^2> \geq \frac{1}{M} e^{-k(z_0 + \epsilon)}.
\]

Since $P_0 \notin \text{supp. } \nu$, we have

\[
<\nu, \eta^1 q^1 - \eta^1 q^2> \leq Me^{k(w_0 - z_0 - \delta)}.
\]

Taking $2\epsilon < \delta$, we have

\[
\left| \frac{<\nu, q^1>}{<\nu, \eta^1>} - \frac{<\nu, q^2>}{<\nu, \eta^2>} \right| = \frac{<\nu, \eta^2 q^1 - \eta^1 q^2>}{<\nu, \eta^1> <\nu, \eta^2>} \leq Me^{-k(\delta - 2\epsilon)} \rightarrow 0
\]

as $k \rightarrow \infty$. Let $\beta$ be a sufficiently small positive number, and we put

\[
\Sigma_2 = \{z_0 \leq z \leq w < w_0 - \beta\}
\Sigma_3 = \{z_0 \leq z \leq w \leq w_0, w_0 - \beta \leq w\}.
\]

Then

\[
\eta^1 e^{-kw} = (1 + O(1/c^2))2^N N! y^{N-1}(y + O(1/k))
\]

is bounded on $\Sigma_0$ and we have

\[
<\nu_{|\Sigma_2}, \eta^1> \leq Me^{k(w_0 - \beta)}.
\]

Taking $\epsilon = \beta/2$, we know

\[
\frac{<\nu_{|\Sigma_2}, \eta^1>}{<\nu, \eta^1>} \leq Me^{-\beta k/2} \rightarrow 0.
\]

Since $\partial \lambda_2/\partial w > 0$, we know

\[
\lambda_2(w, z) \geq \lambda_2(w_0 - \beta, z_0).
\]
on $\Sigma_3$. Therefore we have

$$
\frac{<\nu,q^1>}{<\nu,\eta^1>} = \frac{<\nu|\Sigma_2,\eta^1\lambda_2>}{<\nu,\eta^1>} + \frac{<\nu|\Sigma_3,\eta^1\lambda_2>}{<\nu,\eta^1>} + O(1/k)
$$

$$
\geq o(1) + \lambda_2(w_0 - \beta, z_0)
$$

Similarly we see

$$
\frac{<\nu,q^2>}{<\nu,\eta^2>} \leq o(1) + \lambda_1(w_0, z_0 + \beta).
$$

Therefore we have

$$
\lambda_2(w_0 - \beta, z_0) - \lambda_1(w_0, z_0 + \beta) \leq 0 + o(1).
$$

Passing to the limit, we know

$$
\lambda_2(w_0, z_0) \leq \lambda_1(w_0, z_0).
$$

But this means $P_0 \in \{\rho = 0\}$, a contradiction. QED.

Let us fix $a$ such that $z_0 < a < w_0$. We have

$$
<\nu,B_n^3> = <\nu,\eta^3> <\nu,q_n^5> - <\nu,\eta_n^5><\nu,q^3>,
$$

$$
<\nu,B_n^4> = <\nu,\eta^4> <\nu,q_n^5> - <\nu,\eta_n^5><\nu,q^4>,
$$

$$
<\nu,\eta^3q^4 - \eta^4q^3> = <\nu,\eta^3> <\nu,q^4> - <\nu,\eta^4> <\nu,q^3>,
$$

$$
<\nu,B_n> = <\nu,\eta_n^5><\nu,q_n^6> - <\nu,\eta_n^6><\nu,q_n^5>.
$$

From (8.8) we know

$$
<\nu,\eta^3q^4 - \eta^4q^3> > 0
$$

and from (8.10) we know

$$
<\nu,B_n^3> \rightarrow 0
$$

Using these we can prove the following propositions. Proofs can be found in Chen et al [2].

**Proposition 28** As $n \rightarrow \infty$, $<\nu,\eta_n^5>, <\nu,q_n^5>, <\nu,q_n^6>,$ are bounded.

**Proposition 29** As $n \rightarrow \infty$, we have $<\nu,B_n> \rightarrow 0$.

Now, taking

$$
\Phi_0(x) = \begin{cases} 
  e^{-\frac{|x|}{1-2x}} & \text{if } |x| < 1 \\
  0 & \text{if } |x| \geq 1
\end{cases}
$$
we put
\[ \Phi(x) = \frac{1}{\beta}(\Phi_0(\frac{x+\beta}{\beta}) - \Phi_0(\frac{x-\beta}{\beta})) \]
for the generating function of \( \eta_n^5 \). Here \( \beta = (1 - \alpha)/2 \). We put
\[ S_+ = \{ z \leq w, |w - a| \leq \frac{1 - 3\alpha}{n} \}, \]
\[ S_- = \{ z \leq w, |z - a| \leq \frac{1 - 3\alpha}{n} \}. \]

**Proposition 30** As \( n \to \infty \), we have
\[ <\nu|_{S_+}, ny^{2N}> + <\nu|_{S_-}, ny^{2N}> \to 0. \]

**Proof.** Put \( S'_{L} = S_+ \cap S_L, S'_{R} = S_- \cap S_R \). It is sufficient to prove that
\[ <\nu|_{S'_{L}}, ny^{2N}> + <\nu|_{S'_{R}}, ny^{2N}> \to 0. \]

From (8.11) we have
\[ <\nu|_{S_+}, ny^{2N}A_1 + y^N A_2 > + <\nu|_{S_-}, ny^{2N}C_1 + y^N C_2 > \to 0. \]

Note
\[ A_1 = (\frac{N(2^N N!)^2}{2N + 1} + O(1/c^2))(\int_{-1}^{n(x+y-a)} \Phi)^2 \geq \frac{1}{M_0} > 0 \]
on \( S'_{L} \). Put
\[ E_n = \{ 0 \leq y \leq (\frac{1}{n})^\mu \}, \]
where \( \mu \) is a positive parameter. Then \( |y^N A_2| \leq M(1/n)^{\mu N} = o(1) \) on \( S_L \cap E_n \) and \( |y^N A_2| \leq Mny^{2N}(1/n)^{1-\mu N} \) on \( S_L - E_n \). Choose \( d_n \searrow 0 \) such that
\[ \int_{-1+\alpha}^{1-\alpha-d_n} \Phi = -\int_{1-\alpha-d_n}^{1-\alpha} \Phi \geq (1/n)^{\mu_0}. \]

Then
\[ (\int_{-1}^{H} \Phi)^2 \geq (1/n)^{2\mu_0} \]
for \( |H| \leq 1 - \alpha - d_n \), and
\[ |\Phi(H)| + |\int_{-1}^{H} \Phi| = o(1) \]
for \( 1 - \alpha - d_n \leq |H| \leq 1 \). Put
\[ S'^{n}_+ = S_L \cap \{|w - a| \leq \frac{1 - \alpha - d_n}{n} \}. \]

Then \( S'_{L} \subset S'^{n}_+ \subset S_L \) and
\[ |y^N A_2| = o(1) \]
on $S_L - S_+^n$ and
\[ ny^{2N} A_1 + y^N A_2 \geq ny^{2N}(\frac{1}{M}(1/n)^{2\mu_0} - M(1/n)^{1-\mu N}) \geq 0 \]
on $S_+^n - E_n$. Here we take $0 < 2\mu_0 < 1 - \mu N$. Then
\[ <\nu|_{S_L}, ny^{2N} A_1 + y^N A_2 > = <\nu|_{S_L\cap E_n}, ny^{2N} A_1 > + \]
\[ + <\nu|_{S_L - E_n}, ny^{2N} A_1 + y^N A_2 > + o(1) \]
\[ \geq \frac{1}{M_0} <\nu|_{(S_L\cap E_n), ny^{2N} > + \]
\[ + <\nu|_{S_L - S_+^n \cap E_n}, ny^{2N} A_1 > + \]
\[ + <\nu|_{S_L' - E_n}, ny^{2N} A_1 + y^N A_2 > + \]
\[ + <\nu|_{S_+^n - S_L' - E_n}, ny^{2N} A_1 > + \]
\[ + o(1) \]
\[ \geq \frac{1}{M_0} <\nu|_{S_L\cap E_n}, ny^{2N} > + \]
\[ + <\nu|_{S_L' - E_n}, ny^{2N} > + \]
\[ + o(1). \]

Similarly we know
\[ <\nu|_{S_R}, ny^{2N} C_1 + y^N C_2 > \geq \frac{1}{2M_0} <\nu|_{S_R'}, ny^{2N} > + o(1) \]
Thus we see
\[ <\nu|_{S_L'}, ny^{2N} > + <\nu|_{S_R'}, ny^{2N} > \to 0. \]

QED.

**Proposition 31** We have
\[ \nu|_{\{\rho > 0\}} = \delta_{P_0}. \]

Proof. Proposition 30 says that the projections $P_w \tilde{\nu}, P_z \tilde{\nu}$ of the measure $\tilde{\nu} = y^{2N} \nu$ admits the Lebesgue lower derivatives which vanish at any $a$. Therefore we can claim that
\[ \text{supp.} \nu \cap \{\rho > 0\} = \{P_0\}. \]
Since $\nu$ is a probability measure, we have

$$\nu|_{\{\rho>0\}} = C\delta_{P_0}.$$  

But

$$C(\eta^3 q^4 - \eta^4 q^3) = C^2(\eta^3 q^4 - \eta^4 q^3)$$

at $P_0$. Hence $C = 1$. QED.

Summing up we get the final

**Theorem 2** For any $M_0$ there is a positive number $\epsilon_0$ such that if the initial data satisfy

$$0 \leq \rho_0(x) \leq M_0, \quad \frac{c}{2} \log \frac{c + u_0(x)}{c - u_0(x)} \leq M_0.$$

and if $1/c^2 \leq \epsilon_0$, then a subsequence of the approximate solutions $U^\Delta$ converges a.e. to a limit $U$ which is a weak solution of the relativistic Euler equation.

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参考文献


