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Kyoto University
A Study of the Relativistic Euler Equation

Tetu Makino (牧野 哲)
Faculty of Engineering,
Yamaguchi University

This is a joint work with Cheng-Hsiung Hsu (National Central University, Chungli, Taiwan) and Song-Sun Lin (National Chiao Tung University, Hsinchu, Taiwan).

1 Introduction

In this article we study the Cauchy problem to the one-dimensional relativistic Euler equation

\[
\begin{align*}
\frac{\partial}{\partial t}\frac{\rho + Pu^2/c^4}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{\rho + Pu/c^2}{1-u^2/c^2} &= 0, \\
\frac{\partial}{\partial t}\frac{(\rho + Pu/c^2)u}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{P + Pu^2}{1-u^2/c^2} &= 0, \\
\rho|_{t=0} &= \rho_0(x), \\
u|_{t=0} &= u_0(x).
\end{align*}
\]

Here \(c\) is a positive constant, the speed of light, and \(P\) is a given function of \(\rho\). The equation (1.1) governs the one dimensional motion of a perfect gas in the Minkowski space-time. When \(c \to \infty\), (1.1) tends to the usual Euler equation of gas dynamics

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (P + \rho u^2)_x &= 0.
\end{align*}
\]

Many mathematical investigations for this non-relativistic Euler equation were done. But the first mathematical investigation for the relativistic Euler equation (1.1) was done recently by Smoller and Temple [6]. They assume \(P = \sigma^2 \rho\), where \(\sigma\) is a positive constant < \(c\). Under this assumption, they showed that if the initial data \(\rho_0(x)\) and \(u_0(x)\) satisfy

\[
T.V. \log \rho_0 < \infty, \quad T.V. \log \frac{c + u_0}{c - u_0} < \infty,
\]

\(T.V.\) denotes the total variation.
then there exists a global weak solution to the Cauchy problem (1.1)(1.2). The result was obtained by Glimm's scheme and it is the relativistic version of Nishida's result [5] for the non-relativistic problem.

However we would like to consider a more realistic equation of states. We keep in mind the equation of state for a neutron stars, which is given by

\[ P = Kc^5 f(y), \quad \rho = Kc^3 g(y) \]

\[ f(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}} dq, \]

\[ g(y) = 3 \int_0^y q^2 \sqrt{1+q^2} dq. \]

For this equation of state, we have \( P \sim \frac{c^2}{3} \rho \) as \( \rho \to \infty \) but \( P \sim \frac{1}{5} K^{2/3} \rho^{5/3} \) as \( \rho \to 0 \). So we assume the following properties of the function \( P(\rho) \):

(A):

\[ P(\rho) > 0, \quad 0 < dP/d\rho < c^2, \quad 0 < d^2P/d\rho^2 \]

for \( \rho > 0 \), and

\[ P = A\rho^\gamma (1 + [\rho^{\gamma-1}/c^2]_1) \]

as \( \rho \to 0 \). Here \( A \) and \( \gamma \) are positive constants and

\[ \gamma = 1 + \frac{2}{2N+1}, \]

\( N \) being a positive integer, and \( [X]_1 \) denotes a convergent power series of the form \( \sum_{k \geq 1} a_k X^k \).

The result which we want to generalize to the relativistic problem is those by G.-Q. Chen et al [2]. So we assume that the initial data \( \rho_0(x), u_0(x) \) satisfy

\[ 0 \leq \rho_0(x) \leq M_0, \quad \left| \frac{c}{2} \log \frac{c+u_0(x)}{c-u_0(x)} \right| \leq M_0. \]

A weak solution of (1.1)(1.2) is defined as follows.

We write

\[ E = \frac{\rho + Pu^2/c^4}{1-u^2/c^2}, \]

\[ F = \frac{(\rho + P/c^2)u}{1-u^2/c^2}, \]

\[ G = \frac{P + \rho u^2}{1-u^2/c^2}, \]

\[ U = (E, F)^T, \quad f(U) = (F, G)^T. \]

Then (1.1) can be written as

\[ U_t + f(U)_x = 0. \]
Let us denote by $U_0(x)$ the initial data. Then a weak solution $U(t, x)$ is a bounded measurable function which satisfies
\[ \int \int (U\Phi_t + f(U)\Phi_x)dxdt + \int U_0(x)\Phi(0, x)dx = 0 \]
for any test function $\Phi \in C_0^\infty([0, +\infty) \times R)$.

2 Riemann problems

The Riemann problem is the problem to the special initial data of the form
\[ U_0(x) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0 \end{cases} \]
In order to solve this we introduce the Riemann invariants
\[ w = x + y, \quad z = x - y \]
where
\[ x = \frac{c}{2} \log \frac{c+u}{c-u}, \quad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho. \]
Then (1.1) is diagonalized as
\[ w_t + \lambda_2 w_x = 0, \quad z_t + \lambda_1 z_x = 0, \]
where
\[ \lambda_1 = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}, \quad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \sqrt{P'}u/c^2}. \]
The possible states $U = U_R$ connected to $U_L$ on the right by rarefaction waves are
\[ R_1 : \quad w = w_L, z > z_L \]
and
\[ R_2 : \quad w > w_L, z = z_L. \]
The Rankine Hugoniot jump condition
\[ \sigma[U] = [f(U)], \]
where $[U] = U_R - U_L$, $[f(U)] = f(U_R) - f(U_L)$, gives the shock curve
\[ \frac{(u_R - u_L)^2}{(1 - u_R^2/c^2)(1 - u_L^2/c^2)} = \frac{(\rho_R - \rho_L)(P_R - P_L)}{(\rho_L + P_L/c^2)(\rho_R + P_R/c^2)}. \]
Along this curve we have shocks
\[ S_1 : \quad \rho_L < \rho_R, u_R < u_L, \]
\[ S_2 : \quad \rho_R < \rho_L, u_R < u_L. \]
The Riemann problem can be solved uniquely by using these rarefaction waves and shock waves and vacuum state. The detailed discussion can be found in J. Chen [1].

If we look at a region of the form
\[ \Sigma_B = \{(w, z)| -B \leq z \leq w \leq B\}, \]
we have the following

**Proposition 1** If the initial data \( U_L, U_R \) belong to \( \Sigma_B \) for some large \( B \), then the solution of the Riemann problem is confined to \( \Sigma_B \).

Moreover if we consider the image of \( \Sigma_B \) in the \((E, F)\)-space, we have

**Proposition 2** The region \( \Sigma_B \) is convex in the \((E, F)\)-plane.

Proof. Let us consider the above hedge \( F = F(E) \) which corresponds to \( w = B, -B < z < B \). We have to show \( d^2F/dE^2 < 0 \). Along the hedge \( w = B \), we have
\[
u = c \tanh \frac{1}{c} (B - \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho),
\]
from which
\[
\frac{du}{d\rho} = -(1-u^2/c^2) \frac{\sqrt{P'}}{\rho + P/c^2}.
\]
By a direct calculation we have
\[
\frac{dF}{dE} = \frac{u - \sqrt{P'}}{1 - \sqrt{P'u/c^2}} = \lambda_1.
\]
Differentiating once more we have
\[
\frac{d^2F}{dE^2} = \frac{1 - u^2/c^2}{(1 - \sqrt{P'u/c^2})^4} \left( \frac{P''}{2\sqrt{P'}} + \left(1 - \frac{P'}{c^2}\right) \frac{\sqrt{P'}}{\rho + P/c^2}\right) < 0.
\]
This was to be seen. QED.

From Proposition 2, we have

**Proposition 3** If \( U(s), s \in [a, b], \) is confined to a region \( \Sigma_B \), then the average
\[
\frac{1}{b - a} \int_a^b U(s) ds
\]
belongs to \( \Sigma_B \).

Let us look at the shock wave which connects the left state \( U_L \) to the right state \( U_R \) with the shock speed \( \sigma \).

The right state \( U_R \) and \( \sigma \) are parametrized by \( \rho = \rho_R \). Then we have the following fact, which will be used in Section 4.
Proposition 4 Along $S_1(\rho_L < \rho)$, we have $d\sigma/d\rho < 0$, and along $S_2(\rho < \rho_L)$ we have $d\sigma/d\rho > 0$.

Proof. Without loss of generality we can assume $u_L = 0$. Then $u = u_R$ is given by

$$u = -\sqrt{\frac{[\rho][P]}{(\rho_L + P/c^2)(\rho + P_L/c^2)}},$$

where $[\rho] = \rho - \rho_L$, $[P] = P - P_L$. We have

$$\sigma = \frac{[F]}{[E]} = \frac{(\rho + P/c^2)u}{\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2)}.$$

By a direct but tedious computations, we have

$$\frac{d\sigma}{d\rho} = \frac{(\rho + P/c^2)(\rho_L + P_L/c^2)[\rho]X}{2(\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2))^2u(\rho_L + P/c^2)^2(\rho + P_L/c^2)^2},$$

$$X = (\rho + P_L/c^2)(\rho + P/c^2)P'[\rho] +$$

$$+ (\rho + P_L/c^2)(-(\rho + P_L/c^2) + [P]/c^2)[P] +$$


Since $P'' > 0$ we know $[P] \leq P'[\rho]$. Thus

$$X \geq (\rho + P_L/c^2)(\rho + P/c^2)[P] +$$

$$+ (\rho + P_L/c^2)(-(\rho_L + P_L/c^2) + [P]/c^2)[P] +$$

$$- (\rho_L + P/c^2)[P]^2/c^2$$

$$= [P][(\rho + P_L/c^2)([\rho] + [P]/c^2) + ([\rho] - [P]/c^2)[P]/c^2).$$

But

$$1 > \frac{[\rho] - [P]/c^2}{[\rho]} = 1 - P'(\rho_L + \theta(\rho - \rho_L))/c^2 > 0.$$ Using this, it is easy to see $X > 0$ both when $[\rho] > 0$ and when $[\rho] < 0$. Since $u < 0$, this completes the proof. QED.

3 Entropies

A pair of functions $\eta$ and $q$ is called an entropy-entropy flux if it satisfies the equation

$$D_U q = D_U \eta . D_U f.$$ (3.1)

Using the Riemann invariants, we can write (3.1) as

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \quad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$
By eliminating $q$ from the equation, we get the following second order equation:

$$
\frac{\partial^2 \eta}{\partial w \partial z} + Q \left( J \frac{\partial \eta}{\partial w} - \frac{1}{J} \frac{\partial \eta}{\partial z} \right) = 0,
$$

where

$$
Q = \frac{1}{4\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}),
$$

$$
J = \frac{1 - \sqrt{P'}u/c^2}{1 + \sqrt{P'}u/c^2}.
$$

Since this equation tends to the Euler-Poisson-Darboux equation

$$
\frac{\partial^2 \eta}{\partial w \partial z} + \frac{N}{w-z} \left( \frac{\partial \eta}{\partial w} - \frac{\partial \eta}{\partial z} \right) = 0
$$

as $c \to \infty$, we shall call (3.2) the relativistic Euler-Poisson-Darboux equation.

Among entropies of (3.3) when $c = \infty$ the kinetic energy

$$
\eta = \frac{1}{2} \rho u^2 + \frac{P}{\gamma - 1}
$$

plays an important role. Therefore we want to find an entropy of (3.2) which tends to (3.4) as $c \to \infty$. Let us look for an entropy-entropy flux of the form

$$
\eta = H(\rho, u^2), \quad q = Q(\rho, u^2)u.
$$

Inserting this to the equation it is easy to find an entropy-entropy flux

$$
\eta^* = -\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2 \frac{\rho + Pu^2/c^4}{1 - u^2/c^2},
$$

$$
q^* = \left( -\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2 \frac{\rho + P/c^2}{1 - u^2/c^2} \right) u,
$$

$$
\Psi = \exp(\int_1^K \frac{d\rho}{\rho + P/c^2} + K_0),
$$

where $K_0$ is determined so that $\eta^*$ tends to the kinetic energy (3.4) as $c = \infty$. We call the entropy $\eta^*$ defined by (3.5) the relativistic standard entropy. The important fact is

**Proposition 5** The Hessian $D_U^2 \eta^*$ is positive definite. For any fixed $B$ there is a positive constant $k$ such that

$$
(\xi|D_U^2 \eta^*(U)\xi) \geq k|\xi|^2,
$$

for any $U \in \Sigma_B$ and $\xi = (\xi_0, \xi_1)$ with $|\xi|^2 = \xi_0^2 + \xi_1^2$. 
Proof. The proof is due to direct but tedious calculations. We note

\[
\frac{\partial \rho}{\partial E} = \frac{1 + u^2/c^2}{1 - P'u^2/c^4}, \\
\frac{\partial u}{\partial E} = \frac{(1 + P'/c^2)(1 - u^2/c^2)u}{(\rho + P/c^2)(1 - P'u^2/c^4)}, \\
\frac{\partial \rho}{\partial F} = -\frac{2u/c^2}{1 - P'u^2/c^4}, \\
\frac{\partial u}{\partial F} = \frac{(1 - u^2/c^2)(1 + P'u^2/c^4)}{(\rho + P/c^2)(1 - P'u^2/c^4)}.
\]

Using these, we have

\[
\frac{\partial \eta^*}{\partial E} = \frac{\Psi}{(\rho + P/c^2)(1 - u^2/c^2)^{1/2}} + c^2, \\
\frac{\partial \eta^*}{\partial F} = \frac{\Psi u/c^2}{(\rho + P/c^2)(1 - u^2/c^2)^{1/2}}, \\
\frac{\partial^2 \eta^*}{\partial E^2} = \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2}(P' + 2P'u^2/c^2 + u^2), \\
\frac{\partial^2 \eta^*}{\partial E \partial F} = \frac{-\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2}(2P'/c^2 + 1 + P'u^2/c^4)u, \\
\frac{\partial^2 \eta^*}{\partial F^2} = \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2}(1 + 3P'u^2/c^4).
\]

Therefore we get

\[
(x|D^2_{\xi} \eta^* \xi) = \eta^*_{EE}\xi_0^2 + 2\eta^*_{EF}\xi_0\xi_1 + \eta^*_{FF}\xi_1^2 \\
= \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2}Z, \\
Z = (P' + 2P'u^2/c^2 + u^2)\xi_0^2 - 2(2P'/c^2 + 1 + P'u^2/c^4)u\xi_0\xi_1 + \xi_1^2 \\
\geq \frac{2P'(1 - u^2/c^2)^2(1 - P'u^2/c^4)}{A + C + \sqrt{(A - C)^2 + 4B^2}}(\xi_0^2 + \xi_1^2), \\
A = P' + 2P'u^2/c^2 + u^2, \\
B = (2P'/c^2 + 1 + P'u^2/c^4)u, \\
C = 1 + 3P'u^2/c^4.
\]

This completes the proof. QED.

4 Construction of approximate solutions

Let us construct approximate solutions using the Godunov scheme. The construction is similar if we use the Lax-Friedrichs scheme.
Suppose that the initial data $U_0(x)$ is confined to an invariant region $\Sigma_B$. Put $\Lambda_0 = \sup\{|\lambda_j(U)|; j = 1, 2, U \in \Sigma_B\}$. Fixing $\Lambda_1 > \Lambda_0$, we take mesh lengths $\Delta x, \Delta t$ such that $\Delta x = \Lambda_1 \Delta t$. We denote $\Delta = \Delta x$.

Let us construct the approximate solution $U^\Delta(t, x)$. First we put

$$U_0^\Delta(x) = U_0(x) \chi[-1/\Delta, 1/\Delta].$$

We define

$$U^\Delta(+0, x) = \frac{1}{2\Delta x} \int_{2j\Delta x}^{(2j+2)\Delta x} U_0^\Delta(x) dx$$

for $2j\Delta x < x \leq (2j+2)\Delta x$. Solving the Riemann problem on each interval $[2(j-1)\Delta, 2(j+1)\Delta]$, we define $U^\Delta(t, x)$ for $0 \leq t < \Delta t$. Since the Courant-Friedrichs-Levi condition is satisfied, the wave from the center $2j\Delta$ does not intersect. If $U^\Delta(t, x)$ for $0 \leq t < n\Delta t$ has been defined, then we define

$$U^\Delta(n\Delta t, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U^\Delta(n\Delta t - 0, x) dx$$

for $2j\Delta < x \leq (2j+2)\Delta$. Solving the Riemann problem, we define $U^\Delta(t, x)$ for $n\Delta t \leq t < (n+1)\Delta t$.

By Proposition 1 and 3, it is inductively guaranteed that $U^\Delta$ remains in $\Sigma_B$, say,

**Proposition 6** The approximate solution $U^\Delta(t, x)$ satisfies $U^\Delta(t, x) \in \Sigma_B$, therefore,

$$0 \leq \rho^\Delta(t, x) \leq M, \quad \left| \frac{c}{2} \log \frac{c + u^\Delta(t,x)}{c - u^\Delta(t,x)} \right| \leq M.$$ 

Moreover we shall prove

**Proposition 7** For any test function $\Phi$ it holds that

$$\int \int (\Phi_i U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0, x) U_0^\Delta(x) dx = O(\Delta^{1/2}).$$

In order to prove Proposition 7, we prepare

**Proposition 8** For any shock wave from $U_L$ to $U_R$ with the shock speed $\sigma$ and for any convex entropy $\eta$, we have

$$\sigma[\eta] - [q] \geq 0,$$

where $[\eta] = \eta(U_R) - \eta(U_L), [q] = q(U_R) - q(U_L)$. 

Proof. The right state of shocks can be parametrized by $\rho = \rho_R$. Putting

$$Q(\rho) = \sigma[\eta] - [q],$$

we shall see $dQ/d\rho \geq 0$ along $S_1: [\rho] > 0$ and $dQ/d\rho \leq 0$ along $S_2: [\rho] < 0$. Using the equation (3.1) and the differentiation of the Rankine-Hugoniot condition, we have

$$\frac{dQ}{d\rho} = \frac{d\sigma}{d\rho}([\eta] - D_U \eta(U).[U])$$

$$= -\frac{d\sigma}{d\rho} \int_0^1 \theta(U - U_L)|D_U^2 \eta(U_L + \theta(U - U_L).(U - U_L))d\theta.$$

We supposed $D_U^2 \eta \geq 0$. By Proposition 4, we know $d\sigma/d\rho < 0$ on $S_1$ and $d\sigma/d\rho > 0$ on $S_2$. QED.

Proof of Proposition 7.

We fix $T$ to consider $U^\Delta$ on $0 \leq t \leq T$. First we shall show

$$\sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, (2j+1)\Delta)|^2 dx \leq C.$$  \hspace{1cm} (4.1)

Let us consider the standard entropy $\eta^*$. Then we have

$$0 = \int \eta^*(U(T, x))dx - \int \eta^*(U(0, x))dx + L + \Sigma,$$

$$L = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\eta^*(U(n\Delta t - 0, x) - \eta^*(U(n\Delta t + 0, (2j+1)\Delta)))dx,$$

$$\Sigma = \int_0^T \sum_{shocks} (\sigma[\eta^*] - [q^*])dt.$$

We write $U_0 = U(n\Delta t + 0, (2j+1)\Delta), U_1 = U(n\Delta t - 0, x)$. Since

$$U_0 = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U_1 dx,$$

we see

$$L = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \int_0^1 (1 - \theta)(U_1 - U_0)|D_U^2 \eta^*(U_0 + \theta(U_1 - U_0).(U_1 - U_0))d\theta dx$$

$$\geq 0.$$

On the other hand we have $\Sigma \geq 0$ from Proposition 8. Thus $L \leq C, \Sigma \leq C$. But from Proposition 5, we have $D_U^2 \eta^* \geq k$. Therefore

$$C \geq L \geq \frac{k}{2} \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U_1 - U_0|^2 dx.$$
Thus we get (4.1).

Now let us consider a test function $\Phi$. Put

$$J = \int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0, x) U_0^\Delta dx.$$ 

Since $U^\Delta$ is a weak solution on each time strip $n\Delta t < t < (n + 1)\Delta t$, we have

$$J = \sum \int \Phi(n\Delta t, x)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx$$ 

$$J = J_1 + J_2,$$

$$J_1 = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \Phi(n\Delta t, j\Delta)(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx,$$

$$J_2 = \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\Phi(t, x) - \Phi(n\Delta t, j\Delta))(U(n\Delta t - 0, x) - U(n\Delta t + 0, x))dx.$$ 

Since

$$U(n\Delta t + 0, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U(n\Delta t - 0, x)dx$$

for $2j\Delta < x < (2j+2)\Delta$, we see $J_1 = 0$. It follows from (4.1) that

$$|J_2| \leq C \Delta^{1/2} |\Phi|_{C^1} \left( \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, x)|^2 dx \right)^{1/2}$$

$$\leq C' \Delta^{1/2}.$$ 

Here we have used $T/\Delta t = O(1/\Delta)$. QED.

Summing up, we have the following theorem.

**Theorem 1** The approximate solution $U^\Delta(t, x)$ satisfies

$$0 \leq \rho^\Delta(t, x) \leq M,$$ 

$$|\frac{c}{2} \log \frac{c + u^\Delta(t, x)}{c - u^\Delta(t, x)}| \leq M$$

and

$$\int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0, x) U_0^\Delta(x) = O(\Delta^{1/2})$$

for any test function $\Phi$.

We expect that $U^\Delta$ tends to a weak solution everywhere. For the non-relativistic gas dynamics, this was done by DiPerna [3] and G.Q.Chen et al [2]. In their proof the Darboux formula

$$\eta = \int_z^w ((w - s)(s - z))^N \phi(s)ds$$

which gives solutions of the Euler-Poisson-Darboux equation (3.3), $\phi$ being arbitrary, plays an important role. Section 6 will be devoted to find such an integral formula for the relativistic Euler-Poisson-Darboux equation (3.2).
5 Remark

We note that

\[ \lambda_2 - \lambda_1 = \frac{\sqrt{P'}(1 - u^2/c^2)}{1 - u^2P/c^4} > 0, \]

\[ \frac{\partial \lambda_1}{\partial z} = \frac{1 - u^2/c^2}{2(1 - \sqrt{P'}u/c^2)}(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P}) > 0, \]

\[ \frac{\partial \lambda_2}{\partial w} = \frac{1 - u^2/c^2}{2(1 + \sqrt{P'}u/c^2)}(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P}) > 0 \]

for \( \rho > 0 \) and \( |u| < c \).

This says that the system is strictly hyperbolic and genuinely nonlinear on \( \rho > 0 \). Therefore the Glimm's theory can be applied if

\[ ||U_0(x) - U^*||_{L^\infty} + T.V.U_0 \]

is sufficiently small, where \( U^* \) is a constant state such that \( \rho^* > 0, |u^*| < c \). But the vacuum may not be covered by this application of the general theorem.

6 Generalized Darboux formula

In this section we seek an integration formula for solutions of the relativistic Euler-Poisson-Darboux equation. Let us introduce the variables

\[ x = \frac{c}{2} \log \frac{c+u}{c-u}, \quad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} \, d\rho. \]

Then the relativistic Euler-Poisson-Darboux equation is

\[ (EPD) \quad \eta_{xx} - \eta_{yy} + A(x, y)\eta_y + B(x, y)\eta_x = 0, \]

where

\[ A(x, y) = \frac{1}{\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P'')\frac{1 + P'u^2/c^4}{1 - P'u^2/c^4}, \]

\[ B(x, y) = -\frac{2u/c^2}{1 - P'u^2/c^4}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P''). \]

The coefficients \( A \) and \( B \) are of the form

\[ A = \frac{2N}{y} + a, \quad a = \frac{y}{c^2}(a_0 + [x^2/c^2, y^2/c^2]), \]

\[ B = -\frac{4N}{N + 1} \frac{x}{c^2}(1 + [x^2/c^2, y^2/c^2]), \]
where $[X, Y]_1$ denotes a convergent power series $\sum_{j+k \geq 1} c_{jk} X^j Y^k$. In order to remove the singularity in $A$, we use the trick of Weinstein [7]. We introduce the sequence of variables $\eta_j, j = 0, 1, ..., N$ by

$$\frac{\partial \eta_j}{\partial y} = y \eta_{j+1},$$

or

$$\eta_j(x, y) = I \eta_{j+1}(x, y) = \int_0^y Y \eta_{j+1}(x, Y) dY,$$

where $\eta_0 = \eta$. The sequence of formal integro-differential operators $L_j$ is defined by

$$L_j V = V_{xx} - V_{yy} + \left( \frac{2(N-j)}{y} + a \right) V_y + BV_x + j \tilde{a} V + \sum_{k=1}^{j} F_{jk} I^k V_x + \sum_{k=1}^{j} H_{jk} I^k V,$$

where

$$\tilde{a} = \frac{\partial a}{\partial y} + \frac{a}{y} = \frac{1}{c^2} \left[ \frac{x^2}{c^2}, \frac{y^2}{c^2} \right]_0.$$

The coefficients $F_{jk}$ and $H_{jk}$ are determined inductively by

$$F_{j+1,k} = \left\{ \begin{array}{ll}
F_{j1} + \frac{1}{y} \frac{\partial B}{\partial y} & \text{if } k = 1 \\
F_{jk} + \frac{1}{y} \frac{\partial}{\partial y} F_{j,k-1} & \text{if } k \geq 2
\end{array} \right.$$  

$$H_{j+1,k} = \left\{ \begin{array}{ll}
H_{j1} + j \frac{1}{y} \frac{\partial \tilde{a}}{\partial y} & \text{if } k = 1 \\
H_{jk} + \frac{1}{y} \frac{\partial}{\partial y} H_{j,k-1} & \text{if } k \geq 2
\end{array} \right.$$  

It is easy to see that $F_{jk}$ are of the form $\frac{x}{c^2} \left[ \frac{x^2}{c^2}, \frac{y^2}{c^2} \right]_0$ and $H_{jk}$ are of the form $\frac{1}{c^2} \left[ \frac{x^2}{c^2}, \frac{y^2}{c^2} \right]_0$. By the definition we have formally

$$\frac{1}{y} \frac{\partial}{\partial y}(L_j \eta_j) = L_{j+1} \eta_{j+1}.$$

Now we consider the equation $L_N V = 0$ for $V = \eta_N$ with the initial conditions

$$V = 0, \quad V_y = 2^{N+1} N! \phi(x), \quad \text{at } y = 0.$$

The problem is

$$(Q) \quad V_{yy} - V_{xx} = a V_y + B V_x + N \tilde{a} V +$$

$$+ \sum_{k=1}^{N} F_k I^k V_x + \sum_{k=1}^{N} H_k I^k V,$$

$$V = 0, \quad V_y = 2^{N+1} N! \phi(x) \quad \text{at } y = 0.$$
Proposition 9 If $\phi \in C^1(R)$, then the problem (Q) admits a unique solution $V$ in $C^2(R \times [0, \infty))$.

Proof. Let us denote by $H(x, y, V)$ the right hand side of the equation $L_N = 0$. Then (Q) is transformed to the integral equation

$$V(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2} \int_0^y \int_{x-y+Y}^{x+y-Y} H(X, Y, V)dXdY.$$ 

We can solve this integral equation by the iteration

$$V_0(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi,$$

$$V^{n+1}(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi)d\xi + \frac{1}{2} \int_0^y \int_{x-y+Y}^{x+y-Y} H(X, Y, V^n)dXdY.$$ 

Fixing $L$ arbitrarily, we consider $|x| \leq L$. Then it is easy to get the estimates

$$|V^{n+1}(x, y) - V^n(x, y)| \leq \frac{M^{n+1}y^{n+1}}{(n+1)!}.$$ 

Therefore $V^n$ tends to a limit $V$ uniformly on $|x| \leq L, 0 \leq y \leq L$. The limit is the unique solution of (Q). QED.

Now we put

$$\eta_N = V, \quad \eta_{N-k} = I \eta_{N-k+1}.$$ 

Since $\eta_{N-k}$ and its derivatives of order $\leq 2$ all vanish on $y = 0$ for $k \geq 1$, we see $\eta = \eta_0$ gives a solution of the relativistic Euler-Poisson-Darboux equation (EPD).

Next we give an integral formula for the solution $V$ of (Q).

Proposition 10 There is a $C^{N+2}$-function $G(x, y, \xi)$ of $|x| < \infty, y \geq 0, x - y \leq \xi \leq x + y$ such that the solution $V$ of (Q) satisfies

$$V(x, y) = \int_{x-y}^{x+y} G(x, y, \xi) \phi(\xi)d\xi.$$ 

(6.1) Moreover

$$G = 2^N N! + O(y/c^2),$$

$$\partial_x^p \partial_y^p \partial_\xi^q G = O(1/c^2) \quad \text{for } 1 \leq p_1 + p_2 + p_3 \leq N + 2$$

Proof. We consider the approximate solution $V^n(x, y)$ which appeared in the iteration of the proof of Proposition 9. By writing $H$ as

$$H = (aV)_y + (BV)_x + bV + \sum (F_k I^k V)_x + \sum \tilde{H}_k I^k V,$$
\[ b = N \tilde{a} - a_y - B_z = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \]
\[ \tilde{H}_k = H_k - (F_k)_x = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \]
it is easy to see inductively that there is a kernel \( G^n(x, y, \xi) \) such that
\[ V^n(x, y) = \int_{x-y}^{x+y} G^n(x, y, \xi) \phi(\xi) d\xi. \]
In fact \( G^0 = 2 \) and \( G^n \) are determined inductively by the formula
\[
G^{n+1} = 2 + \frac{1}{2} (G^n_I + G^n_{II} + G^n_{III} + \sum G^n_{IVk} + \sum G^n_{Vk}), \\
G_I = \int_{(x+y-\xi)/2}^{y} a(x-y+Y, Y) G(x-y+Y, Y, \xi) dY + \int_{(x-y+\xi)/2}^{y} a(x+y-Y, Y) G(x+y-Y, Y, \xi) dY, \\
G_{II} = \int_{(x+y-\xi)/2}^{y} B(x+y-Y, Y) G(x+y-Y, Y, \xi) dY - \int_{(x+y+\xi)/2}^{y} B(x-y+Y, Y) G(x-y+Y, Y, \xi) dY, \\
G_{III} = \int \int_{D(x, y, \xi)} b(X, Y) G(X, Y, \xi) dX dY,
\]
where
\[ D(x, y, \xi) = \{(X, Y)|X-Y \leq \xi \leq X+Y, x-y+Y \leq X \leq x+y-Y, 0 \leq Y \leq y\}, \]
\[ G_{IVk} = \int_{(x+y+\xi)/2}^{y} F_k(x+y-Y, Y) J^k G(x+y-Y, Y, \xi) dY + \int_{(x-y+\xi)/2}^{y} F_k(x-y+Y, Y) J^k G(x-y+Y, Y, \xi) dY, \]
where
\[ JG(x, y, \xi) = \int_{|x-\xi|}^{y} YG(x, Y, \xi) dY, \]
and
\[ G_{Vk} = \int \int_{D(x, y, \xi)} \tilde{H}_k(X, Y) J^k G(X, Y, \xi) dX dY. \]
It is easy to see inductively that
\[ |G^{n+1}(x, y, \xi) - G^n(x, y, \xi)| \leq \frac{M^{n+1} y^{n+1}}{(n+1)!}. \]
therefore $G^n$ converges to a limit $G$ uniformly and (6.1) holds. Moreover, we can differentiate $G^{n+1}$ supposing that $G^n$ is differentiable. In fact

$$G_{I,x} = \frac{1}{2} aG((x - y + \xi)/2, (-x + y + \xi)/2, \xi)$$
$$- \frac{1}{2} aG((x + y + \xi)/2, (x + y - \xi)/2, \xi) +$$
$$+ \int_{(-x+y+\xi)/2}^{y} (aG)_x(z - y + Y, Y, \xi)dY$$
$$+ \int_{(x+y-\xi)/2}^{y} (aG)_x(z - Y + Y, Y, \xi)dY,$$

$$G_{I,\xi} = -\frac{1}{2} aG((x - y + \xi)/2, (-x + y + \xi)/2, \xi) +$$
$$+ \frac{1}{2} aG((x + y + \xi)/2, (x + y - \xi)/2, \xi) +$$
$$+ \int_{(-x+y+\xi)/2}^{y} aG_\xi(z - y + Y, Y, \xi)dY +$$
$$+ \int_{(x+y-\xi)/2}^{y} aG_\xi(z + y - Y, Y, \xi)dY,$$

$$G_{I,y} = -\frac{1}{2} aG((x - y + \xi)/2, (-x + y + \xi)/2, \xi) +$$
$$- \frac{1}{2} aG((x + y + \xi)/2, (x + y - \xi)/2, \xi) +$$
$$+ 2aG(x, y, \xi) +$$
$$- \int_{(-x+y+\xi)/2}^{y} (aG)_x(z - y + Y, Y, \xi)dY +$$
$$+ \int_{(x+y-\xi)/2}^{y} (aG)_x(z + y - Y, Y, \xi)dY;$$

$$G_{II,x} = -\frac{1}{2} BG((x + y + \xi)/2, (x + y - \xi)/2, \xi) +$$
$$- \frac{1}{2} BG((x - y + \xi)/2, (-x + y + \xi)/2, \xi) +$$
$$+ \int_{(x+y-\xi)/2}^{y} (BG)_x(z + y - Y, Y, \xi)dY +$$
$$- \int_{(-x+y+\xi)/2}^{y} (BG)_x(z - y + Y, Y, \xi)dY,$$

$$G_{II,\xi} = \frac{1}{2} BG((x + y + \xi)/2, (x + y - \xi)/2, \xi) +$$
$$+ \frac{1}{2} BG((x - y + \xi)/2, (-x + y + \xi)/2, \xi) +$$
$$+ \int_{(x+y-\xi)/2}^{y} BG_\xi(z + y - Y, Y, \xi)dY +$$
$$- \int_{(-x+y+\xi)/2}^{y} BG_\xi(z - y + Y, Y, \xi)dY,$$
\[ G_{II,y} = \frac{1}{2}BG((x + y + \xi)/2, (x + y - \xi)/2, \xi) + \]
\[ + \frac{1}{2}BG((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \]
\[ + \int_{(x+y-\xi)/2}^{y} (BG)_{x}(x-y-Y, Y, \xi)dY + \]
\[ + \int_{(-x+y+\xi)/2}^{y} (BG)_{x}(x-y+Y, Y, \xi)dY; \]

\[ G_{III,x} = \int_{(x+y-\xi)/2}^{y} bG(x+y-Y, Y, \xi)dY - \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y, Y, \xi)dY, \]

\[ G_{III,\xi} = \int_{0}^{(x+y-\xi)/2} bG(\xi+Y, Y, \xi)dY + \int_{0}^{(-x+y+\xi)/2} bG(\xi-Y, Y, \xi)dY + \]
\[ + \int \int_{D(x,y,\xi)} bG(X,Y, \xi)dXdY, \]

\[ G_{III,y} = \int_{(x+y-\xi)/2}^{y} bG(x+y-Y, Y, \xi)dY + \int_{(-x+y+\xi)/2}^{y} bG(x-y+Y, Y, \xi)dY; \]

and the derivatives of \( G_{IVk} \) are similar to \( G_{II} \) and the derivatives of \( G_{IVk} \) are similar to \( G_{III} \). Then it is easy to see inductively that

\[ |G_{x}^{n+1} - G_{x}^{n}| + |G_{\xi}^{n+1} - G_{\xi}^{n}| + |G_{y}^{n+1} - G_{y}^{n}| \leq \frac{M^{n}y^{n}}{n!}. \]

Thus the limit \( G \) is differentiable. In a similar manner we see

\[ |G_{xx}^{n+1} - G_{xx}^{n}| + |G_{x\xi}^{n+1} - G_{x\xi}^{n}| + |G_{xy}^{n+1} - G_{xy}^{n}| + \]
\[ + |G_{\xi\xi}^{n+1} - G_{\xi\xi}^{n}| + |G_{\xi y}^{n+1} - G_{\xi y}^{n}| + |G_{yy}^{n+1} - G_{yy}^{n}| \leq \]
\[ \frac{M^{n-1}y^{n-1}}{(n-1)!}. \]

Thus \( G \) is twice continuously differentiable. In a similar manner we see that \( G \) is \( N + 2 \)-times continuously differentiable. The rough estimates stated in the propositions is obvious since the coefficients are all of \( O(1/c^{2}) \). QED.

The solution \( \eta_{N-k} \) enjoys an integral representation

\[ \eta_{N-k} = \int_{x-y}^{x+y} K_{N-k}(x, y, \xi)\phi(\xi)d\xi, \]

where

\[ K_{N-k}(x, y, \xi) = JK_{N-k+1}(x, y, \xi) = J^{k}G(x, y, \xi). \]

So the solution \( \eta \) of the relativistic Euler-Poisson-Darboux equation is given by

\[ \eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi)\phi(\xi)d\xi, \]
$K(x, y, \xi) = J^N G(x, y, \xi)$.

By induction we see

$$J^k G(x, y, \xi) = \frac{2^N N!}{2^k k!} (y^2 - (x - \xi)^2)^k (1 + O(y/c^2)).$$

Thus we have

**Proposition 11** There is a kernel $K(x, y, \xi)$ which is of $C^{N+2}$-class in $|x| < \infty, 0 \leq y, x - y \leq \xi \leq x + y$ such that

$$\eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi$$

gives a solution of the relativistic Euler-Poisson-Darboux equation for any smooth $\phi$. Moreover

$$K(x, y, \xi) = (y^2 - (x - \xi)^2)^N (1 + O(y/c^2)).$$

But in order to apply this integration formula, the generalized Darboux formula, to the study of the relativistic Euler equation, more detailed estimates of the remainder are necessary.

**Proposition 12** We have

$$G_y = O(y/c^2).$$

Proof. Since $a = O(y/c^2)$, it is clear that $G_{I, y} = O(y/c^2)$. Next we see

$$G_{II, y} = -B((x+y+\xi)/2, (x+y-\xi)/2) + B((x-y+\xi)/2, (-x+y+\xi)/2) + O(y/c^2).$$

On the other hand we can write

$$B = \frac{1}{c^2} B_0(x) + O(y^2/c^2)$$

and

$$\frac{x + y + \xi}{2} = x + \frac{y + Z}{2}, \quad \frac{x - y + \xi}{2} = x + \frac{-y + Z}{2}, \quad Z = \xi - x.$$ 

Therefore we see $G_{II, y} = O(y/c^2)$. It is clear that $G_{III, y} = O(y/c^2)$ and $G_{IV, y}, G_{V, y} = O(y^2/c^2)$. QED.

**Proposition 13** We have

$$G = 2^N N! + \frac{1}{c^2} C_0(x, c)(\xi - x) + O(y^2/c^2),$$

where $C_0(x, c)$ is a function of the form

$$[x^2/c^2]_0 + \frac{x}{c^2} [x^2/c^2]_0.$$
Proof. It is clear that $G_I = O(y^2/c^2)$ since $a = O(y/c^2)$. Next we see

$$G_{II} = 2^N N! \int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY + O(y^2/c^2),$$

since $G = 2^N N! + O(y/c^2)$. If we write

$$B = \frac{1}{c^2} B_0(x) + O(y^2/c^2), \quad Z = \xi - x$$

then we see

$$\int_{(x+y-\xi)/2}^{y} B(x+y-Y,Y)dY - \int_{(-x+y+\xi)/2}^{y} B(x-y+Y,Y)dY =$$

$$= \frac{1}{c^2} \left( \int_{x}^{x+\frac{y}{c^2}} B_0(s)ds - \int_{x+y+\xi}^{x+y+\xi+\frac{Z}{c^2}} B_0(s)ds \right) + O(y^2/c^2)$$

$$= \frac{1}{c^2} B_0(x) Z + O(y^2/c^2).$$

Note $|Z| \leq y$. It is clear that $G_{III}, G_{IVk}, G_{Vk} = O(y^2/c^2)$. QED.

**Proposition 14** We have

$$G_x + G_\xi = O(y/c^2).$$

Proof. First we see

$$G_{I,x} + G_{I,\xi} = \int_{(x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)(x+y-Y,Y,\xi)dY +$$

$$+ \int_{(x+y-\xi)/2}^{y} ((aG)_x + aG_\xi)(x-y+Y,Y,\xi)dY +$$

$$= O(y^2/c^2),$$

since $a, a_x = O(y/c^2)$. Next we see

$$G_{II,x} + G_{II,\xi} = \int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_\xi)(x+y-Y,Y,\xi)dY +$$

$$- \int_{(-x+y+\xi)/2}^{y} ((BG)_x + BG_\xi)(x-y+Y,Y,\xi)dY +$$

$$= O(y^2/c^2).$$

It is clear that $G_{III,x}, G_{III,\xi}, G_{Vk,x}, G_{Vk,\xi} = O(y/c^2)$. $G_{IVk,x} + G_{IVk,\xi}$ is estimated in a similar manner as $G_{II,x} + G_{II,\xi}$. QED.

**Proposition 15** We have

$$(G_x + G_\xi)_y = O(y/c^2).$$
Proof. First we see

\[(G_{I,x} + G_{I,\xi})_y = 2((aG)_x + aG_{\xi})(x, y, \xi) + \]
\[- \frac{1}{2}((aG)_x + aG_{\xi})((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \]
\[- \frac{1}{2}((aG)_x + aG_{\xi})((x + y + \xi)/2, (x + y - \xi)/2, \xi) + \]
\[- \int_{(-x+y+\xi)/2}^{y} ((aG)_x + aG_{\xi})_{x}(x-y+Y, Y, \xi) dY + \]
\[+ \int_{((x+y-\xi)/2}^{y} ((aG)_x + aG_{\xi})_{x}(x+y-Y, Y, \xi) dY = O(y/c^2), \]

since \(a, a_x = O(y/c^2)\). Next we see

\[(G_{II,x} + G_{II,\xi})_y = -\frac{1}{2}((BG)_x + BG_{\xi})((x + y + \xi)/2, (x + y - \xi)/2, \xi) + \]
\[+ \frac{1}{2}((BG)_x + BG_{\xi})((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \]
\[+ \int_{(x+y-\xi)/2}^{y} ((BG)_x + BG_{\xi})_{x}(x+y-Y, Y, \xi) dY + \]
\[+ \int_{((-x+y+\xi)/2}^{y} ((BG)_x + BG_{\xi})_{x}(x-y+Y, Y, \xi) dY = 2^{N-1}N!B_x((x-y+\xi)/2, (-x+y+\xi)/2) - 2^{N-1}N!B_x((x+y+\xi)/2, (x+y-\xi)/2) + \]
\[+ O(y/c^2), \]

since \(G = 2^N N! + O(y/c^2)\) and \(G_x + G_{\xi} = O(y/c^2)\). But

\[B_x = \frac{1}{c^2} B'_0(x) + O(y^2/c^2)\]

and

\[B_x((x - y + \xi)/2, (-x + y + \xi)/2) - B_x((x + y + \xi)/2, (x + y - \xi)/2) = \]
\[= \frac{1}{c^2} B''_0(x)(-y) + O(y^2/c^2) = O(y/c^2). \]

It is clear that

\[(G_{III,x} + G_{III,\xi})_y = \int_{(x+y-\xi)/2}^{y} ((bG)_x + bG_{\xi})(x + y - Y, Y, \xi) dY + \]
\[+ \int_{((-x+y+\xi)/2}^{y} ((bG)_x + bG_{\xi})(x - y + Y, Y, \xi) dY = O(y/c^2). \]
Similarly we can estimate \( (G_{IVk,x} + G_{IVk,\xi})_y, (G_{Vk,x} + G_{Vk,\xi})_y \) bearing in mind that \( (JG)_x + (JG)_\xi = J(G_x + G_\xi) \). QED.

**Proposition 16** We have

\[
G_x + G_\xi = \frac{1}{c^2} C_1(x, c)(\xi - x) + O(y^2/c^2),
\]

where \( C_1(x, c) \) is a function of the form

\[
[x^2/c^2]_0 + \frac{x}{c^2}[x^2/c^2]_0.
\]

Proof. We already observed that \( G_{Ix} + G_{I\xi} = O(y^2/c^2) \). Next we look at

\[
G_{II,x} + G_{II,\xi} = \int_{(x+y-\xi)/2}^{y} (BG)_x + BG_\xi)(x+y,Y,\xi)dY + \int_{(-x+y+\xi)/2}^{y} (BG)_x + BG_\xi)(x-y,Y,\xi)dY
\]

\[
= 2^N N! \int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B_x(x-y-Y,Y)dY + O(y^2/c^2),
\]

since \( G = 2 + O(y/c^2) \) and \( G_x + G_\xi = O(y/c^2) \). Bearing in mind that

\( B_y = O(y/c^2) \), we see

\[
\int_{(x+y-\xi)/2}^{y} B_x(x+y-Y,Y)dY = -2B_x(x, y) + B((x+y+\xi)/2, (x+y-\xi)/2) + B((x-y+\xi)/2, (-x+y+\xi)/2) + \frac{1}{c^2} B_0(x)Z + O(y^2/c^2).
\]

Next we look at

\[
G_{III,x} + G_{III,\xi} = \int_{(x+y-\xi)/2}^{y} bG(x+y-Y,Y,\xi)dY - \int_{(-x+y+\xi)/2}^{y} bG(x-y-Y,Y,\xi)dY + \int_{0}^{y} bG(\xi+Y,Y,\xi)dY - \int_{0}^{(-x+y+\xi)/2} bG(\xi-Y,Y,\xi)dY + \int_{D(x,y,\xi)} bG(X,Y,\xi)dXdY.
\]
we see

\[ b(x, y) = \frac{1}{c^2} b_0(x) + O(y^2/c^2), \]

\[ G_{III,x} + G_{III,\xi} = 2^N N! \left( \int_x^{x+y+z/2} b_0(s) ds - \int_x^{x+y-z/2} b_0(s) ds + \int_{x+y-z/2}^{x+y+z/2} b_0(s) ds - \int_{x+y+z/2}^{x+y+z} b_0(s) ds \right) + O(y^2/c^2) \]

\[ = \frac{2^N N!}{c^2} b_0(x) \left( \frac{y+Z}{2} - \frac{y-Z}{2} + \frac{y-Z}{2} - \frac{y+Z}{2} \right) + O(y^2/c^2) \]

\[ = O(y^2/c^2). \]

\( G_{IVk,x} + G_{IVk,\xi} \) can be estimated in a similar manner as \( G_{II,x} + G_{II,\xi} \).

Finally \( G_{Vk,x}, G_{Vk,\xi} = O(y^3/c^2) \) since \( J^k G = O(y^2/c^2) \) for \( k \geq 1 \). QED.

**Proposition 17** We have

\[ (G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2). \]

Proof. First we see

\[ (G_{I,x} + G_{I,\xi})_x + (G_{I,x} + G_{I,\xi})_\xi = \]

\[ = \int_y^{x+y+\xi/2} ((aG)_{xx} + 2(aG_\xi)_x + aG_\xi\xi)(x-y+Y,Y,\xi) dY + \int_{x+y-\xi/2}^y ((aG)_{xx} + 2(aG_\xi)_x + aG_\xi\xi)(x+y-Y,Y,\xi) dY \]

\[ = O(y^2/c^2). \]

since \( a, a_x, a_{xx} = O(y/c^2) \). Next

\[ (G_{II,x} + G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi = \]

\[ = \int_y^{x+y-\xi/2} (((BG)_{x} + BG_\xi)_x + ((BG)_{x} + BG_\xi)_\xi)(x+y-Y,Y,\xi) dY + \int_{x+y+\xi/2}^y (((BG)_{x} + BG_\xi)_x + ((BG)_{x} + BG_\xi)_\xi)(x+y-Y,Y,\xi) dY \]

\[ = O(y/c^2). \]

It is easy to see

\[ (G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi = O(y/c^2). \]

The estimates of \( G_{IVk} \) and \( G_{Vk} \) can be seen similarly. QED.
Proposition 18 We have

\[(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = \frac{1}{c^2} C_2(x, c)(\xi - x) + O(y^2/c^2),\]

where \(C_2(x, c)\) is a function of the form

\[\left[\frac{x^2}{c^2}\right]_0 + \frac{x}{c^2} \left[\frac{x^2}{c^2}\right]_0.\]

Proof. We already observed that

\[(G_{I,x} + G_{I,\xi})_x + (G_{I,x} + G_{I,\xi})_\xi = O(y^2/c^2).\]

Next, bearing in mind that \(G_x + G_\xi = O(y/c^2)\) and \((G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2)\), we see

\[(G_{II,x} + G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi =\]

\[= \int_0^y (B_{xx}G + 2B_x(G_x + G_\xi) + B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi))(x + y - Y, Y, \xi)dY +\]

\[\int_{-x+y+\xi/2}^{y} (B_{xx}G + 2B_x(G_x + G_\xi) + B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi))(x - y + Y, Y, \xi)dY\]

\[= 2^N N! \int_{(x+y-\xi)/2}^{y} B_{xx}(x+y-Y)\,dY - 2^N N! \int_{(-x+y+\xi)/2}^{y} B_{xx}(x-y+Y)\,dY +\]

\[O(y^2/c^2).\]

The same discussion to that of the proof of Proposition 16 can be applied by replacing \(B\) by \(B_x\). Let us look at \((G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi\).

Note that

\[\begin{align*}
(bG)_x + bG_\xi &= b_xG + b(G_x + G_\xi) \\
&= 2^N N! b_x + O(y/c^2),
\end{align*}\]

\[bG = 2^N N! b + O(y/c^2).\]

Applying the discussion of the proof of Proposition 16 by replacing \(b\) by \(b_x\), we see

\[\begin{align*}
(G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi &=
2^N N! \int_{x+\frac{y}{2}}^{x+\frac{Z}{2}} b_0(s)\,ds - \int_{x+\frac{-y}{2}}^{x+\frac{Z}{2}} b_0(s)\,ds +
\end{align*}\]

\[\begin{align*}
&= -2^N N! b_0(x)Z + O(y^2/c^2).
\end{align*}\]

The estimates of \(G_{IV,k}, G_{V,k}\) are paralell. QED.
Proposition 19 We have

\[ G_{\xi} = \frac{1}{c^2} C_3(x, c) + O(y/c^2). \]

Proof. It is sufficient to note that

\[ G_{II,\xi} = 2^{N-1} N! (B((x + y + \xi)/2, (x + y - \xi)/2) + B((x - y + \xi)/2, (-x + y + \xi)/2)) + \]
\[ + \frac{2^{N-1} N!}{c^2} (B_0(x + \frac{y + Z}{2}) + B_0(x + \frac{-y + Z}{2})) + O(y/c^2) \]
\[ = \frac{2^N N!}{c^2} B_0(x) + O(y/c^2). \]

QED.

Proposition 20 We have

\[ (G_x + G_{\xi})_{\xi} = \frac{1}{c^2} C_4(x, c) + O(y/c^2). \]

Proof. We see

\[ (G_{I,x} + G_{I,\xi})_{\xi} = O(y/c^2) \]

by \( a, a_x = O(y/c^2) \). Next we see

\[ (G_{II,x} + G_{II,\xi})_{\xi} = \]
\[ = 2^N N! b((x - y + \xi)/2, (-x + y + \xi)/2) + O(y/c^2) \]
\[ = \frac{2^N N!}{c^2} b_0(x) + O(y/c^2). \]

And we see

\[ (G_{III,x} + G_{III,\xi})_{\xi} = \]
\[ = 2^N N! b((x - y + \xi)/2, (-x + y + \xi)/2) + O(y/c^2) \]
\[ = \frac{2^N N!}{c^2} b_0(x) + O(y/c^2). \]

Other terms can be estimated similarly. QED.

7 Estimates of the derivatives of entropies

Let us consider the entropy \( \eta \) generated by \( \phi \) of \( C^3 \)-class, that is,

\[ \eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi. \]
In this section we will find estimates of the derivatives of $\eta$ with respect to $E,F$. As auxiliary variables we introduce

$$R = y^{2N+1}, \quad M = xy^{2N+1}. \quad (7.1)$$

We are going to prove the following

**Proposition 21** We have

$$\frac{\partial \eta}{\partial M} = 2^{2N+1} \int_0^1 (s-s^2)^N D\phi(x+(2s-1)y)ds + O(y^2/c^2), \quad (7.2)$$

$$\frac{\partial \eta}{\partial R} = 2^{2N+1} \int_0^1 (s-s^2)^N \phi ds + 2^{2N+1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^2/c^2), \quad (7.3)$$

$$\frac{\partial^2 \eta}{\partial M^2} = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N D^2\phi ds + O(y^{-2N+1}/c^2), \quad (7.4)$$

$$\frac{\partial^2 \eta}{\partial R \partial M} = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D^2\phi ds + O(y^{-2N+1}/c^2), \quad (7.5)$$

$$\frac{\partial^2 \eta}{\partial R^2} = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N ((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) D^2\phi(x+(2s-1)y)ds + O(y^{-1}/c^2). \quad (7.6)$$

**Proof.** We write

$$\eta = 2R^{\frac{1}{2N+1}} \int_0^1 K\left(\frac{M}{R}, R^{\frac{1}{2N+1}}, \frac{M}{R}+(2s-1)R^{\frac{1}{2N+1}}\right) \phi\left(\frac{M}{R}+(2s-1)R\right) ds.$$

Differentiating $\eta$ with respect to $M$, we have

$$\frac{\partial \eta}{\partial M} = (1) + (2),$$

$$(1) = 2R^{\frac{2N}{2N+1}} \int_0^1 (K_x + K_\xi)(x,y,(2s-1)y) \phi(x+(2s-1)y) ds,$$

$$(2) = 2R^{\frac{-2N}{2N+1}} \int_0^1 K(x,y,(2s-1)y) D\phi(x+(2s-1)y) ds.$$

Since $K(x,y,\xi) = J^N G(x,y,\xi)$, i.e.

$$K(x,y,\xi) = \int_{|x-\xi|}^{Y_N} Y_1 G(x,Y_1,\xi) dY_1 \cdots \int_{|x-\xi|}^{Y_2} Y_1 G(x,Y_1,\xi) dY_2 \cdots Y_N.$$
by Proposition 16 we see

\[
(K_x + K_\xi)(x, y, x + (2s - 1)y) = \int_{|2s-1|y}^{y} Y_N \int_{|2s-1|y}^{Y_N} Y_{N-1} \cdots \int_{|2s-1|y}^{Y_2} Y_1 (G_x + G_\xi)(x, y_1, x + (2s - 1)y) dy_1 \cdots
\]

\[
= C_1(x, c) 2^{N+1} (2s-1)(1-(2s-1)^2)^N + O(y^{2N+2}/c^2)
\]

\[
= \frac{2^N C_1(x, c)}{(N+1)!c^2} y^{2N+1} \frac{d}{ds} (s-s^2)^{N+1} + O(y^{2N+2}/c^2).
\]

Therefore by integration by part we get

\[
(1) = R^{\frac{-2N}{2N+1}} y^{2N+2} \frac{2^N+1 C_1(x, c)}{(N+1)!c^2} \int_0^1 (s-s^2)^{N+1} D\phi ds + O(y^2/c^2)
\]

\[
= O(y^2/c^2).
\]

By Proposition 13 we see

\[
K(x, y, \xi) = \int_{|x-\xi|}^{y} Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1} \cdots \int_{|x-\xi|}^{Y_2} Y_1 G(x, y_1, \xi) dy_1 \cdots Y_N,
\]

\[
= 2^{2N} (s-s^2)^N y^{2N} + \frac{2^N C_0(x, c)}{N!c^2} (2s-1)(s-s^2)^N y^{2N+1} + O(y^{2N+2}/c^2).
\]

Therefore by integration by parts we get

\[
(2) = 2^{2N+1} R^{\frac{-2N}{2N+1}} y^{2N} \int_0^1 (s(1-s))^N D\phi(x+(2s-1)y) ds
\]

\[
+ R^{\frac{-2N}{2N+1}} O(y^{2N+2}/c^2).
\]

Thus we have (7.2). Next we show (7.3). We have

\[
\frac{\partial \eta}{\partial R} = (3) + (4) + (5),
\]

(3) = \[
\frac{2}{2N+1} R^{\frac{-2N}{2N+1}} \int_0^1 K(x, y, x + (2s - 1)y) \phi(x + (2s - 1)y) ds,
\]

(4) = \[
2R^{\frac{-2N}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{1}{2N+1} y(K_y + (2s-1)K_\xi)) \phi(x + (2s - 1)y) ds,
\]

(5) = \[
2R^{\frac{-2N}{2N+1}} \int_0^1 K(x, y, x + (2s - 1)y)(-x + \frac{y}{2N+1}(2s - 1)) D\phi(\ldots) ds.
\]

By Proposition 13 we get

(3) = \[
\frac{2^{2N+1}}{2N+1} \int_0^1 (s-s^2)^N \phi(\ldots) ds + O(y^2/c^2).
\]

As for (4) we use Proposition 16 and

\[
K_y + (2s - 1)K_\xi = y J^{N-1} G - (2s - 1)(\xi - x) G(x, |\xi - x|, \xi) J^{N-1} + (2s - 1)J^N G_\xi
\]
\[
2^{2N+1}N(s-s^2)^Ny^{2N-1} + \frac{2^{N-1}C_0(x,c)}{(N-1)!c^2} (2s-1)(s-s^2)^Ny^{2N} + \frac{2^NC_3(x,c)}{N!c^2} (2s-1)(s-s^2)^Ny^{2N} + O(y^{2N+1}/c^2)
\]

(See Proposition 19). Then by integration by parts we have
\[
(4) = \frac{2^{2N+2}N}{2N+1} \int_{0}^{1} (s-s^2)^N \phi(...)ds + O(y^2/c^2).
\]

As (2) we get
\[
(5) = 2^{2N+1} \int_{0}^{1} (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1))D\phi(...)ds + O(y^2/c^2).
\]

Thus we get (7.3).

Next we show (7.4). We have
\[
\frac{\partial^2 \eta}{\partial M^2} = (6) + (7) + (8),
\]
\[
(6) = 2R^{\frac{-4N-1}{2N+1}} \int_{0}^{1} ((K_x + K_\xi)_x + (K_x + K_\xi)_\xi)(x,y,...) \times \phi(...)ds,
\]
\[
(7) = 4R^{\frac{-4N-1}{2N+1}} \int_{0}^{1} (K_x + K_\xi)(x,y,...)D\phi(...)ds,
\]
\[
(8) = 2R^{\frac{-4N-1}{2N+1}} \int_{0}^{1} K(x,y,...)D^2\phi(...)ds.
\]

By Proposition 18 we have
\[
((K_x + K_\xi)_x + (K_x + K_\xi)_\xi)(x,y,x+(2s-1)y) = \frac{2^NC_2(x,c)}{N!c^2} (s-s^2)^N (2s-1)y^{2N+1} + O(y^{2N+2}/c^2).
\]

Thus by integration by parts we get
\[
(6) = O(y^{-2N+1}/c^2).
\]

By the same discussion as (1) we see (7) = O(y^{-2N+1}/c^2). By the same discussion as (2) we see
\[
(8) = 2^{2N+1}y^{-2N-1} \int_{0}^{1} (s-s^2)^N D^2\phi(...)ds + O(y^{-2N+1}/c^2).
\]

Thus we get (7.4).

Next we show (7.5). We see
\[
\frac{\partial^2 \eta}{\partial M \partial R} = (9) + (10) + (11) + (12) + (13) + (14),
\]
\[(9) = -\frac{4N}{2N+1}R_{\frac{2N}{2N+1}}^{-\frac{4N-1}{2N+1}} \int_{0}^{1} (K_{x} + K_{\xi})(x, y, \ldots)\phi(...)|_{0}^{1} ds,\]

\[(10) = 2R_{\frac{2N}{2N+1}}^{-\frac{4N-1}{2N+1}} \int_{0}^{1} (-x(K_{x} + K_{\xi})_{x} + (K_{x} + K_{\xi})_{\xi}) + \frac{y}{2N+1}((K_{x} + K_{\xi})_{y} + (2s - 1)(K_{x} + K_{\xi})_{\xi})\phi(...)|_{0}^{1} ds,\]

\[(11) = 2R_{\frac{2N}{2N+1}}^{-\frac{4N-1}{2N+1}} \int_{0}^{1} (K_{x} + K_{\xi})(-x + \frac{y}{2N+1}(2s - 1))D\phi|_{0}^{1} ds,\]

\[(12) = -\frac{4N}{2N+1}R_{\frac{2N}{2N+1}}^{-\frac{4N-1}{2N+1}} \int_{0}^{1} K D\phi|_{0}^{1} ds,\]

\[(13) = 2R_{\frac{2N}{2N+1}}^{-\frac{4N-1}{2N+1}} \int_{0}^{1} (-x(K_{x} + K_{\xi}) + \frac{y}{2N+1}(K_{y} + (2s - 1)K_{\xi}))D\phi|_{0}^{1} ds,\]

\[(14) = 2R_{\frac{2N}{2N+1}}^{-\frac{4N-1}{2N+1}} \int_{0}^{1} K(-x + \frac{y}{2N+1}(2s - 1))D^{2}\phi|_{0}^{1} ds.\]

We already know that \[(9) = O(y^{-2N+1}/c^{2}).\] (Recall (1).) Next we look at (10). The first term is \(O(y^{-2N+1}/c^{2}).\) (Recall (6)). By Proposition 16 and 20 we see

\[(K_{x} + K_{\xi})_{y} + (2s - 1)(K_{x} + K_{\xi})_{\xi} = \]

\[= \frac{2^{N-1}C_{1}(x,c)}{(N-1)!c^{2}}y^{2N}(2s - 1)(s - s^{2})^{N} + \]

\[+ \frac{2^{N}C_{4}(x,c)}{N!c^{2}}y^{2N}(2s - 1)(s - s^{2})^{N} + \]

\[- \frac{2^{N-1}C_{1}(x,c)}{(N-1)!c^{2}}y^{2N}(2s - 1)^{3}(s - s^{2})^{N-1} + O(y^{2N+1}/c^{2}).\]

By integration by parts we see \((10) = O(y^{-2N+1}/c^{2}).\) We already know \((11) = O(y^{-2N+1}/c^{2}).\) Clearly

\[(12) = -\frac{2^{2N+2}N}{2N+1}y^{-2N-1} \int_{0}^{1} (s - s^{2})^{N}D\phi|_{0}^{1} ds + O(y^{-2N+1}/c^{2}).\]

We see

\[(13) = O(y^{-2N+1}/c^{2}) + \frac{2}{2N+1}R_{\frac{2N}{2N+1}}^{-\frac{4N}{2N+1}} \int_{0}^{1} (K_{y} + (2s - 1)K_{\xi})D\phi|_{0}^{1} ds.\]

As (4) we have

\[(13) = \frac{2^{2N+2}N}{2N+1}y^{-2N-1} \int_{0}^{1} (s - s^{2})^{N}D\phi|_{0}^{1} ds + O(y^{-2N+1}/c^{2}).\]

Finally we see

\[(14) = 2^{2N+1}y^{-2N-1} \int_{0}^{1} (s - s^{2})^{N}(-x + (2s - 1)\frac{y}{2N+1})D^{2}\phi|_{0}^{1} ds + O(y^{-2N+1}/c^{2})\]
(Recall (5)). Summing up we get (7.5).

Next we show (7.6).

\[
\frac{\partial^2 \eta}{\partial R^2} = \frac{\partial}{\partial R}(3) + \frac{\partial}{\partial R}(4) + \frac{\partial}{\partial R}(5),
\]

\[
\frac{\partial}{\partial R}(3) = (15) + (16) + (17),
\]

\[
(15) = -\frac{4N}{(2N+1)^2} R^{-\frac{4N-1}{2N+1}} \int_0^1 K \phi ds,
\]

\[
(16) = \frac{2}{2N+1} R^{-\frac{4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1} (K_y + (2s-1)K_\xi)) \phi ds,
\]

\[
(17) = \frac{2}{2N+1} R^{-\frac{4N-1}{2N+1}} \int_0^1 K (-x + \frac{y}{2N+1} (2s-1)) D\phi ds,
\]

\[
\frac{\partial}{\partial R}(4) = (18) + (19) + (20),
\]

\[
(18) = -\frac{4N}{2N+1} R^{-\frac{4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1} (K_y + (2s-1)K_\xi)) \phi ds,
\]

\[
(19) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 K'' \phi ds,
\]

where

\[
K'' = x(K_x + K_\xi) + x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) +
\]

\[
+ \frac{y}{2N+1} ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) +
\]

\[
- \frac{xy}{2N+1} ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) +
\]

\[
+ \frac{y^2}{2N+1} ((K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi) +
\]

\[
+ \frac{y}{(2N+1)^2} (K_y + (2s-1)K_\xi),
\]

\[
(20) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1} (K_y + (2s-1)K_\xi))
\]

\[
\times (-x + \frac{y}{2N+1} (2s-1)) D\phi ds,
\]

\[
\frac{\partial}{\partial R}(5) = (21) + (22) + (23) + (24),
\]

\[
(21) = -\frac{4N}{2N+1} R^{-\frac{4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1} (2s-1)) D\phi ds,
\]

\[
(22) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1} (K_y + (2s-1)K_\xi))
\]

\[
\times (-x + \frac{y}{2N+1} (2s-1)) D\phi ds,
\]

\[
(23) = 2R^{-\frac{4N-1}{2N+1}} \int_0^1 K(x + \frac{y}{(2N+1)^2} (2s-1)) D\phi ds,
\]
First we see

\[ \frac{-2^{2N+2}N}{(2N+1)^2}y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2), \]

\[ \frac{2^{2N+2}N}{(2N+1)^2}y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2), \]

\[ \frac{2^{2N+1}}{2N+1}y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2). \]

Thus we have

\[ \frac{\partial}{\partial R}(3) = \frac{2^{2N+1}}{2N+1}y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2). \]

Since (18) is similar to (16), we have

\[ (18) = -\frac{2^{2N+3}N^2}{(2N+1)^2}y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2). \]

Next let us look at (19). We already know

\[ 2R^{-\frac{4N+1}{2N+1}} \int_0^1 x(K_x + K_\xi) \phi ds = O(y^{-2N+1}/c^2), \]

\[ 2R^{-\frac{4N+1}{2N+1}} \int_0^1 x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2). \]

Recalling (10), we see

\[ \frac{2}{2N+1}R^{-\frac{4N+1}{2N+1}} y \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2), \]

\[ \frac{2}{2N+1}R^{-\frac{4N+1}{2N+1}} xy \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2). \]

When \( N = 1 \), we have

\[ (K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi \]
\[ = 8(s-s^2) + \frac{C_3}{c^2}(2s-1)y - \frac{C_0}{c^2}(2s-1)^3y - \]
\[ - \frac{2C_3}{c^2}(2s-1)^3y + O(y^2/c^2). \]

When \( N \geq 2 \), there are bounded functions \( F_j(x, c) \) such that

\[ (K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi = \]
\[ = 2^{2N+1}N(2N-1)(s-s^2)^N y^{2N-2} + \frac{F_1(x, c)}{c^2} (2s-1)(s-s^2)^{N-1}y^{2N-1} + \]
\[ + \frac{F_2(x, c)}{c^2}(2s-1)(s-s^2)^{N-2}y^{2N-1} + \frac{F_3(x, c)}{c^2}(2s-1)^3(s-s^2)^{N-2}y^{2N-1} + \]
\[ + \frac{F_4(x, c)}{c^2}(2s-1)^3(s-s^2)^{N-1}y^{2N-1} + \frac{F_5(x, c)}{c^2}(2s-1)^5(s-s^2)^{N-2}y^{2N-1} + \]
\[ + O(y^{2N}/c^2). \]
Thus we see
\[
2R^{-\frac{4N+1}{2N+1}} \frac{y^2}{(2N+1)^2} \int_0^1 ((K_y + (2s - 1)K_\xi)_y + (2s - 1)(K_y + (2s - 1)K_\xi)_y) \phi ds
\]
\[= 2^{2N+3} \frac{N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds - \frac{2^{2N+2} N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds
\]
\[+ O(y^{-2N+1}/c^2).
\]
We have
\[
\frac{2}{(2N+1)^2} R^{-\frac{4N+1}{2N+1}} y \int_0^1 (K_y + (2s - 1)K_\xi) \phi ds
\]
\[= \frac{2^{2N+2} N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).
\]
Therefore
\[
(19) = \frac{2^{2N+3} N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).
\]
We see
\[
(20) = \frac{2^{2N+2} N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).
\]
Therefore
\[
\frac{\partial}{\partial R} (4) = \frac{2^{2N+2} N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).
\]
Next we see
\[
(21) = -\frac{2^{2N+2} N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),
\]
\[
(22) = \frac{2^{2N+2} N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),
\]
\[
(23) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{(2N+1)^2}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),
\]
\[
(24) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1))^2 D^2 \phi ds + O(y^{-2N+1}/c^2).
\]
Therefore we get
\[
\frac{\partial}{\partial R} (5) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{(2N+1)^2}(2s-1)) D\phi ds +
\]
\[+ 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1))^2 D^2 \phi ds + O(y^{-2N+1}/c^2)
\]
Summing up, we have
\[
\frac{\partial^2 \eta}{\partial R^2} = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds
\]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N \left(x + \frac{y}{2N+1} (2s-1)\right) D\phi ds \]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N \left(-x + \frac{y}{2N+1} (2s-1)\right)^2 D^2\phi ds \]
\[ = \frac{2^{2N+2} (N+1)}{(2N+1)^2} y^{-2N-1} \int_{0}^{1} (s-s^2)^N (2s-1) y D\phi ds + \]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N \left(-x + \frac{y}{2N+1} (2s-1)\right)^2 D^2\phi ds \]
\[ = \frac{2^{2N+3}}{(2N+1)^2} y^{-2N-1} \int_{0}^{1} (s-s^2)^{N+1} y^2 D^2\phi ds + \]
\[ + 2^{2N+1} y^{-2N-1} \int_{0}^{1} (s-s^2)^N \left(-x + \frac{y}{2N+1} (2s-1)\right)^2 D^2\phi ds. \]

Thus we get (7.6). QED.

Let us recall the standard entropy $\eta^\ast$. This is generated by

\[ \phi^\ast(x) = A' c^2 \left( \frac{1}{1-u^2/c^2} - \frac{1}{\sqrt{1-u^2/c^2}} \right), \]

where

\[ A' = (2N+1)^{-2N} ((2N+1)/(2N+3)A)^{\frac{2N+1}{2}} (2N-1)!/2^{N+1} N!. \]

We note that

\[ D^2\phi^\ast(x) = A' (1 + \frac{u^2/c^2}{1-u^2/c^2}) (2 - \sqrt{1-u^2/c^2}) \geq A'. \]

We are going to show that the Hessian $D_U^2 \eta^\ast$ dominates any $D_U^2 \eta$.

**Proposition 22** For each $\phi$ fixed in $C^3$ we have on each compact subset of $\{\rho \geq 0\}$

\[ |(\xi | D_U^2 \eta^\ast . \xi)| \leq C (\xi | D_U^2 \eta . \xi), \]

provided that $c$ is sufficiently large.

By the assumption we have

\[ R = y^{2N+1} = K \rho (1 + [\rho \frac{3N+1}{2} / c^2]_1), \]
\[ \frac{dR}{d\rho} = K + [\rho \frac{3N+1}{2} / c^2]_1, \]
\[ \frac{d^2R}{d\rho^2} = \frac{\rho^{1-2N}}{c^2} [\rho \frac{3N+1}{2} / c^2]_0, \]

where $K = ((2N+3)(2N+1)A)^{\frac{2N+1}{2}}$. Using these, we have

\[ \frac{\partial R}{\partial E} = \frac{dR}{d\rho} \frac{1+u^2/c^2}{1-Pu^2/c^4} \]
\[
\frac{\partial R}{\partial F} = -\frac{dR}{d\rho} \frac{2u/c^2}{1-Pu^2/c^4},
\]
\[
\frac{\partial M}{\partial E} = \frac{R}{\rho+P/c^2} \frac{1+P'/c^4}{1-Pu^2/c^4} - \frac{dR}{d\rho} \frac{2xu/c^2}{1-Pu^2/c^4},
\]
\[
\frac{\partial R}{\partial F} = K(1-2xu/c^2) + O(y^2/c^2).
\]

Differentiating once more, we see
\[
\frac{\partial^2 R}{\partial E^2} = -\frac{K^2}{y^{2N+1}} 2u^2(1-u^2/c^2)/c^2 + O(y^{-2N+1}/c^2),
\]
\[
\frac{\partial^2 M}{\partial E^2} = \frac{K^2}{y^{2N+1}} u(-2u^2/c^2 - 2ux(1-u^2/c^2)/c^2) + O(y^{-2N+1}/c^2),
\]
\[
\frac{\partial^2 R}{\partial E \partial F} = \frac{K^2}{y^{2N+1}} \frac{2u}{c^2} (1-u^2/c^2) + O(y^{-2N+1}/c^2),
\]
\[
\frac{\partial^2 M}{\partial E \partial F} = \frac{K^2}{y^{2N+1}} \frac{2u^2}{c^2} + 2xu(1-u^2/c^2)/c^2) + O(y^{-2N+1}/c^2),
\]
\[
\frac{\partial^2 R}{\partial F^2} = -\frac{2}{c^2} \frac{K^2}{y^{2N+1}} (1-u^2/c^2) + O(y^{-2N+1}/c^2),
\]
\[
\frac{\partial^2 M}{\partial F^2} = -\frac{K^2}{y^{2N+1}} 2(u + x(1-u^2/c^2))/c^2 + O(y^{-2N+1}/c^2).
\]

The chain rule gives
\[
\frac{\partial^2 \eta}{\partial E^2} = \left(\frac{\partial R}{\partial E}\right)^2 \frac{\partial^2 \eta}{\partial R^2} + 2 \frac{\partial R}{\partial E} \frac{\partial M}{\partial E} \frac{\partial^2 \eta}{\partial R \partial M} + \left(\frac{\partial M}{\partial E}\right)^2 \frac{\partial^2 \eta}{\partial M^2} + \frac{\partial^2 R}{\partial E \partial M} \frac{\partial \eta}{\partial R} + \frac{\partial^2 M}{\partial E \partial M} \frac{\partial \eta}{\partial M},
\]
and so on. Inserting (7.7) and (7.8) into (7.9), and using Proposition 21, we have
\[
(\xi|D^2_{\eta} \eta.\xi) = \frac{2^{2N+1} K^2}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi] D^2 \phi ds +
\]
\[
-\frac{2K^2}{y^{2N+1}} \frac{1}{c^2} (1-u^2/c^2)(u\xi_0 - \xi_1)^2 \frac{\partial \eta}{\partial R} +
\]
\[
-\frac{2K^2}{y^{2N+1}} \frac{1}{c^2} (u + x(1-u^2/c^2))(u\xi_0 - \xi_1)^2 \frac{\partial \eta}{\partial M} +
\]
\[
O(y^{-2N+1}/c^2),
\]
where
\[
Z[\xi] = Z_{00}\xi_0^2 + 2Z_{01}\xi_0\xi_1 + Z_{11}\xi_1^2.
\]
\[
Z_{00} = (1 + u^2/c^2)^2((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) + \\
+ 2(1 + u^2/c^2)(-u + x(1 + u^2/c^2))(-x + \frac{y}{2N+1}(2s-1)) \\
+ (-u + x(1 + u^2/c^2))^2,
\]
\[
Z_{01} = -2(1 + u^2/c^2)u/c^2((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) + \\
+ (1 + 3u^2/c^2 - 4x(1 + u^2/c^2)u/c^2)(-x + \frac{y}{2N+1}(2s-1)) + \\
+ (-u + x(1 + u^2/c^2))(1 - 2xu/c^2),
\]
\[
Z_{11} = \frac{4u^2}{c^4}((-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2}s(1-s)y^2) + \\
- \frac{4u}{c^2}(1 - 2xu/c^2)(-x + \frac{y}{2N+1}(2s-1)) + \\
+ (1 - 2xu/c^2)^2.
\]

It can be shown that

\[
Z[\xi] \geq \kappa s(1-s)y^2,
\]

where \(\kappa\) is a positive constant depending on the compact subset of \(\{\rho \geq 0\}\).

In fact we see

\[
Z_{00}Z_{11} - Z_{01}^2 = (1-u^2/c^2)\frac{4}{(2N+1)^2}s(1-s)y^2.
\]

On the other hand, we can estimate

\[
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1 - u^2/c^2)\frac{\partial \eta}{\partial R}| \leq \frac{\epsilon}{y^{2N+1}},
\]

\[
|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u + x(1 - u^2/c^2))\frac{\partial \eta}{\partial M}| \leq \frac{\epsilon}{y^{2N+1}},
\]

where \(\epsilon = K'/c^2\). Let us introduce the parameters

\[
\zeta_0 = \xi_0, \quad \zeta_1 = \xi_1 - u\xi_0.
\]

Then we have

\[
Z[\xi] = Q_{00}\zeta_0^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{11}\zeta_1^2,
\]

and

\[
Q_{00} = Q_{00}^{(1)}(x)(2s-1)y + Q_{00}^{(2)}(x, s)y^2,
\]

\[
Q_{01} = Q_{01}^{(1)}(x)(2s-1)y + Q_{01}^{(2)}(x, s)y^2,
\]

\[
Q_{11} = Z_{11} = 1 + O(1/c^2) > 0.
\]

Therefore if \(|D^2\phi| \leq C\), we see

\[
|(\xi|D^2_\eta\xi)| \leq \frac{2^{N+1}K^2C}{y^{2N+1}} \int_0^1 (s - s^2)^N Z[\xi] ds
\]
\[
\frac{12\epsilon}{y^{2N+1}} \int_0^1 (s-s^2)^N \zeta^2 ds + O(y^{-2N+1}/c^2)
\]
\[
\leq \frac{2^{2N+1}K^2C}{y^{2N+1}} \int_0^1 (s-s^2)^N (Q_{11}(1+\epsilon')\zeta_1^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{00} + O(y^{-2N+1}/c^2)).
\]

But since \(Q_{00}^{(0)} = Q_{01}^{(0)} = 0\), \(\int_0^1 (s-s^2)^N (2s-1) ds = 0\), we see
\[
\int_0^1 (s-s^2)^N (-2\epsilon'Q_{01}\zeta_0\zeta_1 - \epsilon'Q_{00}\zeta_0^2) ds = O(y^{-2N+1}/c^2).
\]

Therefore we get
\[
|\langle \xi | D_U^2 \eta \xi \rangle| \leq \frac{2^{2N+1}K^2C(1+\epsilon')}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi] ds + O(y^{-2N+1}/c^2).
\]

Similarly, if \(D^2 \phi^* \geq \mu\), we have
\[
|\langle \xi | D_U^2 \eta^* \xi \rangle| \geq \frac{2^{2N+1}K^2\mu(1-\epsilon'')}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi] ds + O(y^{-2N+1}/c^2).
\]

Thus we get
\[
|\langle \xi | D_U^2 \eta \xi \rangle| \leq \frac{C(1+\epsilon')}{\mu(1-\epsilon'')}(\xi | D_U^2 \eta^* \xi \rangle + O(y^{-2N+1}/c^2).
\]

But we know
\[
|\langle \xi | D_U^2 \eta \xi \rangle| \geq \kappa|\xi|^2 y^{-2N+1}.
\]

Hence if \(c\) is sufficiently large we get the required estimate. QED.

As for the first derivatives, the following conclusion is now clear.

**Proposition 23** On each compact subset of \(\{\rho \geq 0\}\), we have
\[
\left| \frac{\partial \eta}{\partial E} \right| + \left| \frac{\partial \eta}{\partial F} \right| \leq C.
\]

8 $Usefull$ $entropies$

Let us consider an entropy \(\eta\) generated by \(\phi\), that is,
\[
\eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi.
\]

The corresponding entropy flux \(q\) is given by integrating the different equations
\[
\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \quad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.
\]
We can solve these equations as

\[
q = \lambda_2 \eta - \int_{z}^{w} \frac{\partial \lambda_2}{\partial w} \eta dw
= \lambda_1 \eta + \int_{z}^{w} \frac{\partial \lambda_1}{\partial z} \eta dz.
\]

Thus we get the formula

\[
q(x, y) = \int_{x-y}^{x+y} L(x, y, \xi) \phi(\xi) d\xi,
\]

(8.2)

where

\[
L(x, y, \xi) = \lambda_1 K(x, y, \xi) + L_1(x, y, \xi)
= \lambda_2 K(x, y, \xi) + L_2(x, y, \xi),
\]

\[
L_1(x, y, \xi) = 2 \int_{(x+y-\xi)/2}^{y} \mu_1(x + y - Y, Y, \xi) K(x+y-Y, Y, \xi) dY,
\]

\[
L_2(x, y, \xi) = -2 \int_{(-x+y+\xi)/2}^{y} \mu_2(x - y + Y, Y) K(x-y+Y, Y, \xi) dY,
\]

\[
\mu_1(x, y) = \frac{\partial \lambda_1}{\partial z},
= \frac{1 - u^2/c^2}{2(1 - \sqrt{P} u/c^2)} \left( 1 - \frac{P'}{c^2} + \frac{\rho + P/c^2}{2P'} \right)^N + O(1/c^2),
\]

\[
\mu_2(x, y) = \frac{\partial \lambda_2}{\partial w},
= \frac{1 - u^2/c^2}{2(1 + \sqrt{P} u/c^2)} \left( 1 - \frac{P'}{c^2} + \frac{\rho + P/c^2}{2P'} \right)^N + O(1/c^2).
\]

In this section we will construct various kinds of usefull entropies.

1) Let us put

\[
\eta^1_k(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) k^{N+1} e^{k\xi} d\xi,
\]

\[
\eta^2_k(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) k^{N+1} e^{-k\xi} d\xi.
\]

**Proposition 24** If $1/c^2$ is sufficiently small, we have

\[
\eta_k^1 > 0, \quad \eta_k^2 > 0 \quad \text{for } y > 0,
\]

\[
\eta_k^1 = 2^N N! y^N (1 + O(y/c^2)) e^{k(x+y)} (1 + O(1/k)), \quad \eta_k^2 = 2^N N! y^N (1 + O(y/c^2)) e^{-k(x-y)} (1 + O(1/k))
\]

(8.3)
uniformly on each compact subset of \( \{ y > 0 \} \). Moreover

\[
q_k^1 = \eta_k^1(\lambda_2 + O(1/k)), \\
q_k^2 = \eta_k^2(\lambda_1 + O(1/k))
\]

(8.5)

uniformly on each compact subset of \( \{ y \geq 0 \} \) and

\[
\eta_k^2 q_k^1 - \eta_k^1 q_k^2 = (2^N N!)^2 y^{2(N-1)} \left( \frac{1}{2N+1} + O(1/c^2) \right) e^{2ky} (y+O(1/k))^3.
\]

(8.6)

Proof. Since \( K = (1 + O(y/c^2))(y^2 - (x - \xi)^2)^N \), we see

\[
\eta_k^1 = (1 + O(y/c^2)) \int_{x-y}^{x+y} (y^2 - (x - \xi)^2)^N k^{N+1} e^{k\xi} d\xi
\]

\[
= (1 + O(y/c^2)) 2^{2N+1} y^N e^{ky} f(ky)
\]

where

\[
f(r) = r^{N+1} e^{-r} \int_0^1 (s(1-s))^N e^{2rs} ds
\]

\[
= e^{r} \int_0^r (\sigma(1-\frac{\sigma}{r}))^N e^{-2\sigma} d\sigma.
\]

It is easy to see

\[
e^{-r} f(r) = 2^{-(N+1)} N! + O(1/r)
\]

This implies (8.4). We note

\[
\eta^1 = (1 + O(1/c^2)) 2^N N! y^{N-1} e^{k(x+y)} (y + O(1/k))
\]

\[
\eta^2 = (1 + O(1/c^2)) 2^N N! y^{N-1} e^{-k(x-y)} (y + O(1/k))
\]

uniformly on \( \{ y \geq 0 \} \). Let us consider the flux. We have

\[
L_2(x, y, \xi) = -2 \int_{(x+y+\xi)/2}^{x+y} \mu_2(x - y + Y, Y) K(x - y + Y, Y, \xi) dY
\]

\[
= -2(\frac{N}{2N+1} + O(1/c^2)) \int_{(x+y+\xi)/2}^{y} (Y^2 - (x - y + Y - \xi)^2)^N dY
\]

\[
= -\left( \frac{N}{(2N+1)(N+1)} + O(1/c^2) \right) (y - x + \xi)^N (y + x - \xi)^{N+1},
\]

\[
q^1 - \lambda_2 \eta^1 = -\left( \frac{N}{(2N+1)(N+1)} + O(1/c^2) \right) \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi.
\]

But

\[
0 \leq \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi
\]

\[
= (N+1) k^N \int_{x-y}^{x+y} (y^2 - (x - \xi)^2)^N e^{k\xi} d\xi
\]
\[- Nk^{N} \int_{x-y}^{x+y} (y - x + \xi)^{N-1}(y + x - \xi)^{N+1}e^{k\xi}d\xi \leq (N + 1) \frac{1}{k} \int_{x-y}^{x+y} (y^{2} - (x - \xi)^{2})^{N}k^{N+1}e^{k\xi}d\xi.\]

Thus
\[q^{1} - \lambda_{2}\eta^{1} = O(1/k)\eta^{1}.\]

Since
\[\lambda_{2} - \lambda_{1} = \frac{\sqrt{P'}(1 - u^{2}/c^{2})}{1 - P'u^{2}/c^{4}} = (\frac{1}{2N+1} + O(1/c^{2}))y,\]
we have
\[\eta^{2}q^{1} - \eta^{1}q^{2} = \eta^{1}\eta^{2}((\frac{1}{2N+1} + O(1/c^{2}))y + O(1/k)).\]

This implies (8.6). QED.

2) Let \(\psi\) be a function in \(C_{0}^{\infty}(-1,1)\) such that \(\psi \geq 0, \int \psi = 1\). We put
\[\phi^{3}_{n}(x) = \psi_{n}(x) = n\psi(n(x-a)),\]
\[\phi^{4}_{n}(x) = -D\psi_{n}(x),\]
\[\eta^{3}_{n}(x, y) = \int_{x-y}^{x+y} K(x, y, \xi)\phi^{3}_{n}(\xi)d\xi,\]
\[\eta^{4}_{n}(x, y) = \int_{x-y}^{x+y} K(x, y, \xi)\phi^{4}_{n}(\xi)d\xi,\]
\[\eta^{3}(x, y) = K(x, y, a)X,\]
\[\eta^{4}(x, y) = K_{\xi}(x, y, a)X,\]
\[q^{3}(x, y) = L(x, y, a)X,\]
\[q^{4}(x, y) = L_{\xi}(x, y, a)X,\]
\[X = 1 \quad (x - y < a < x + y)\]
\[= \frac{1}{2} \quad (|x - a| = y)\]
\[= 0 \quad (|x - a| > y).\]

**Proposition 25** As \(n \to \infty\), we have
\[\eta^{3}_{n} \to \eta^{3}, \quad q^{3}_{n} \to q^{3}, \quad \eta^{4}_{n} \to \eta^{4}, \quad q^{4}_{n} \to q^{4}.\]

Moreover
\[|\eta^{3}_{n}| \leq My^{2N}, \quad |q^{3}_{n}| \leq My^{2N}(|x| + y), \quad (8.7)\]
\[|\eta^{4}_{n}| \leq My^{2N-1}, \quad |q^{4}_{n}| \leq My^{2N-1}(|x| + y), \quad (8.8)\]
\[\eta^{3}q^{4} - \eta^{4}q^{3} = \frac{N}{(2N+1)(N+1)}(1 + O(1/c^{2}))(y^{2} - (x - a)^{2})^{2N}(8.9)\]
Proof. We note

\begin{align*}
K\xi &= -(\xi - x)G(x, |\xi - x|, \xi) \frac{1}{2^{N-1}(N-1)!}(y^2 - (x - \xi)^2)^{N-1} + J^N G\xi \\
&= (2N(x - \xi) + O(1/c^2)(\xi - x)^2)(y^2 - (x - \xi)^2)^{N-1} + O(1/c^2)(y^2 - (x - \xi)^2)^N,
\end{align*}

\begin{align*}
L_{1,\xi} &= 2\int_{(x+y-\xi)/2}^{y} \mu_1(x + y - Y, Y)K_{\xi}(x + y - Y, \xi)dY.
\end{align*}

The estimates (8.7), (8.8) can be seen easily. Let us consider

$$
\eta^3 q^4 - \eta^4 q^3 = (KL_{\xi} - L K\xi)(x, y, a).
$$

Suppose \(x - a \geq 0\). Then

\begin{align*}
\frac{1}{2}(KL_{\xi} - L K\xi) &= K \int_{(x+y-a)/2}^{y} \mu_1 K\xi(x + y - Y, Y, a)dY - \\
&- K\xi \int_{(x+y-a)/2}^{y} \mu_1 K(x + y - Y, Y, a)dY.
\end{align*}

We note

\begin{align*}
0 \leq \frac{x + y - a}{2} \leq x - y + Y - a \leq x - a \leq y.
\end{align*}

Hence we have

\begin{align*}
\int_{(x+y-a)/2}^{y} \mu_1 K\xi(x + y - Y, Y, a)dY \\
&= \left(\frac{N}{2(2N+1)} + O(1/c^2))2N \int_{(x+y-a)/2}^{y} (x + y - Y - a)(Y^2 - (x + y - Y - a)^2)^{N-1}dY + \\
&+ O(1/c^2)) \int_{(x+y-a)/2}^{y} (Y^2 - (x + y - Y - a)^2)^N dY
\right.
\end{align*}

\begin{align*}
&= \left(\frac{N^2}{2(2N+1)} + O(1/c^2)))(x + y - a)^{N-1}(-x + y + a)^N \frac{1}{N(N+1)}(y + (2N+1)(x - a)) + \\
&+ O(1/c^2)(y^2 - (x - a)^2)^N.
\end{align*}

Thus

\begin{align*}
K \int_{(x+y-a)/2}^{y} \mu_1 K\xi dY \\
&= \left(\frac{N}{2(2N+1)(N+1)} + O(1/c^2)))(y^2 - (x - a)^2)^{2N-1}(-x + y + a)(y + (2N+1)(x - a))
\right.
\end{align*}

\begin{align*}
&+ O(1/c^2)(y^2 - (x - a)^2)^{2N}.
\end{align*}

Also we have

\begin{align*}
K\xi \int_{(x+y-a)/2}^{y} \mu_1 KdY \\
&= \left(\frac{N^2}{(2N + 1)(N+1)} + O(1/c^2)))(x - a)(-x + y + a)(y^2 - (x - a)^2)^{2N-1}
\right.
\end{align*}

\begin{align*}
&+ O(1/c^2)(-x + y + a)(y^2 - (x - a)^2)^{2N}.
\end{align*}
\[ \frac{1}{2}(KL_{\zeta} - LK_{\xi}) = \left( \frac{N}{2(2N+1)(N+1)} + O(1/c^2) \right) (y^2 - (x-a)^2)^{2N}. \]

Here we have used

\[
\begin{align*}
0 & \leq (x-a)(y-(x-a)) \leq y^2 - (x-a)^2, \\
0 & \leq (y-x+a)(y+(2N+1)(x-a)) \\
& \leq (2N+1)(y^2 - (x-a)^2)
\end{align*}
\]

provided that \(0 \leq x-a \leq y\). When \(x-a \leq 0\), we can discuss in a similar manner by using \(L_2\). QED.

3) Let \(\Phi\) be a function in \(C_0^\infty(-1,1)\) such that \(\int \Phi = 0\) and the support \(\text{supp}\Phi\) is \([-1+\alpha,1+\alpha]\), where \(\alpha\) is a small positive number. We put

\[
\psi_n(x) = n\Phi(n(x-a)),
\]

\[
\eta_n^5(x,y) = \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi,
\]

\[
q_n^5(x,y) = \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\psi_n(\xi) d\xi;
\]

\[
\hat{\Phi}(x) = \frac{d}{dx}(x\int_{-1}^{x}\Phi),
\]

\[
\hat{\psi}_n(x) = n\hat{\Phi}(n(x-a)),
\]

\[
\eta_n^6(x,y) = \int_{x-y}^{x+y} K(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi) d\xi,
\]

\[
q_n^6(x,y) = \int_{x-y}^{x+y} L(x,y,\xi) D^{N+1}\hat{\psi}_n(\xi) d\xi;
\]

\[
B_n^3 = \eta_n^5 q_n^5 - \eta_n^5 q_n^3,
\]

\[
B_n^4 = \eta_n^6 q_n^5 - \eta_n^5 q_n^4,
\]

\[
B_n = \eta_n^5 q_n^6 - \eta_n^6 q_n^5.
\]

Let us divide the domain \(\Sigma = \{-B \leq x-y \leq x+y \leq B\}\) into the following 5 parts.

\[
S_0 = \left\{ -\frac{1}{n} < x+y-a \leq \frac{1}{n}, -\frac{1}{n} \leq x-y-a < \frac{1}{n} \right\} \cap \Sigma,
\]

\[
S_1 = \left\{ \frac{1}{n} < x+y-a, x-y-a < -\frac{1}{n} \right\} \cap \Sigma,
\]

\[
S_L = \left\{ -\frac{1}{n} < x+y-a \leq \frac{1}{n}, x-y-a < -\frac{1}{n} \right\} \cap \Sigma,
\]

\[
S_R = \left\{ \frac{1}{n} < x+y-a, -\frac{1}{n} \leq x-y-a < \frac{1}{n} \right\} \cap \Sigma,
\]

\[
S = \Sigma - (S_0 \cup S_1 \cup S_L \cup S_R).
\]
Proposition 26 We have

\[ |B_n^3| \leq M/n, \quad |B_n^4| \leq M \]  (8.10)
on \Sigma, and

\[ |B_n| \leq M/n \]  (8.11)
on S_0 \cup S_1 \cup S. Moreover, on S_L, we have

\[ B_n = ny^{2N}A_1 + y^N A_2 + A_3, \]  (8.12)

where

\[ A_1 = \left( \frac{N(2^N N!)^2}{2N+1} + O(1/c^2) \right) \left( \int_{-1}^{n(x+y-a)} \Phi \right)^2, \]

\[ |A_2| \leq M \left( | \int_{-1}^{n(x+y-a)} \Phi | + |\Phi(n(x+y-a))| \right), \]

\[ |A_3| \leq \frac{M}{n}. \]

On S_R, we have

\[ B_n = ny^{2N}C_1 + y^N C_2 + C_3, \]

\[ C_1 = \left( \frac{N(2^N N!)^2}{2N+1} + O(1/c^2) \right) \left( \int_{-1}^{n(x-y-a)} \Phi \right)^2, \]

\[ |C_2| \leq M \left( | \int_{-1}^{n(x-y-a)} \Phi | + |\Phi(n(x-y-a))| \right), \]

\[ |C_3| \leq \frac{M}{n}. \]

Proof. For the simplicity, we write \( \eta_n = \eta_n^5, q_n = q_n^5, \hat{\eta}_n = \eta_n^6, \hat{q}_n = q_n^6. \)

It is easy to see inductively that, for \( G_j = J^jG = K_{N-j} \), we have

\[ \partial_\xi^p G_j = J \partial_\xi^p G_{j-1} \]

for \( j \geq p + 1 \) and

\[ \partial_\xi^p G_p = (-1)^p (\xi - x)^p G(x, |\xi - x|, \xi) + J \partial_\xi^p G_{p-1}. \]

Therefore

\[ \partial_\xi^p K = \partial_\xi^p G_N(x, y, \xi) = 0 \]

for \( p \leq N - 1 \) and \( y = |x - \xi| \). Thus by integration by parts we have

\[ \eta_n = (-1)^N \partial_\xi^N K(x, y, x + y) \psi_n(x + y) + \]

\[ - (-1)^N \partial_\xi^N K(x, y, x - y) \psi_n(x - y) + \]

\[ + F_n^1(x, y), \]

\[ F_n^1(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} K(x, y, \xi) \psi_n(\xi) d\xi. \]
We see
\[ \partial_{\xi}^{p} L_{2}(x, y, \xi) = -2 \int_{(-x+y+\xi)/2}^{y} \mu_{2} \partial_{\xi}^{p} K(x-y+Y, Y, \xi) dY \]
for \( p \leq N-1 \). Therefore
\[ \partial_{\xi}^{p} L_{2}(x, y, x+y) = \partial_{\xi}^{p} L_{2}(x, y, x-y) = 0 \]
for \( p \leq N-1 \). Moreover we see
\[ \partial_{\xi}^{N} L_{2}(x, y, x+y) = 0. \]

Therefore by integration by parts we have
\[
\sigma_{n}(x, y) = q_{n}(x, y) - \lambda_{2} \eta_{n}(x, y) =\]
\[ = \frac{(-1)^{N}}{2} \partial_{\xi}^{N} L_{2}(x, y, x-y) \psi_{n}(x-y) +
\]
\[ + F_{n}^{2}(x, y), \]
\[ F_{n}^{2}(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_{\xi}^{N+1} L_{2}(x, y, \xi) \psi_{n}(\xi) d\xi. \]

Similarly
\[
\bar{\sigma}_{n}(x, y) = q_{n}(x, y) - \lambda_{1} \eta_{n}(x, y) =\]
\[ = \frac{(-1)^{N}}{2} \partial_{\xi}^{N} L_{1}(x, y, x+y) \psi_{n}(x+y) +
\]
\[ + \bar{F}_{n}^{2}(x, y), \]
\[ \bar{F}_{n}^{2}(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_{\xi}^{N+1} L_{1}(x, y, \xi) \psi_{n}(\xi) d\xi. \]

We note
\[ \partial_{\xi}^{N} K(x, y, \xi) = (-1)^{N} (\xi-x)^{N} G(x, |x-\xi|, \xi) + J \partial_{\xi}^{N} G_{n-1}. \]

It is easy to see inductively that
\[ \partial_{\xi}^{p+1} G_{p}(x, y, \xi) = \frac{(-1)^{p} p (p+1)}{2} (\xi-x)^{p-1} G(x, |x-\xi|, \xi) + \]
\[ + (\xi-x)^{p} H_{p}(x, \xi) + J \partial_{\xi}^{p} G_{p-1}, \]

where \( H_{p} = O(1/c^{2}) \). Therefore
\[ \partial_{\xi}^{N+1} K(x, y, \xi) = (-1)^{N} \frac{N(N+1)}{2} (\xi-x)^{N-1} G(x, |\xi-x|, \xi) +
\]
\[ + (\xi-x)^{N} H_{N}(x, \xi) + J \partial_{\xi}^{N} G_{N-1}. \]

1) Suppose \((x, y) \in S\). Then it is clear that \( \eta^{3}, \eta^{4}, q^{3}, q^{4}, \eta_{n}, q_{n}, B_{n}^{3}, B_{n}^{4}, B_{n} \) all vanish.
2) Suppose $(x, y) \in S_0$. Then we see

\[ \eta^3 = K(x, y, a) \]
\[ = O((y^2 - (x - a)^2)^N) \]
\[ = O(n^{-2N}), \]
\[ \eta^4 = K_\xi(x, y, a) \]
\[ = O(|x - a|(y^2 - (x - a)^2)^{N-1}) + O((y^2 - (x - a)^2)^N) \]
\[ = O(n^{-2N+1}), \]
\[ \sigma^3 = L_2(x, y, a) \]
\[ = -2 \int_{(-x+y+a)/2}^{y} \mu_2 K(x-y+Y, Y, a) dY \]
\[ = O(n^{-2N-1}), \]
\[ \sigma^4 = L_{2,\xi}(x, y, a) \]
\[ = -2 \int_{(-x+y+a)/2}^{y} \mu_2 K_\xi(x-y+Y, Y, a) dY \]
\[ = O(n^{-2N}). \]

Since $y = O(1/n)$ and $\psi_n = O(n)$, we see

\[ (-1)^N \partial_\xi^N K(x, y, x+y) \psi_n(x+y) + \]
\[ (-1)^N \partial_\xi^N K(x, y, x-y) \psi_n(x-y) = \]
\[ = O(n^{-N+1}). \]

Since $F_{n}^1 = O(1)$, we have $\eta_n = O(1)$. We see

\[ \partial_\xi^N L_2(x, y, x-y) = -2 \int_0^y \mu_2 \partial_\xi^N K(x-y+Y, Y, x-y) dY = O(n^{-N-1}). \]

Therefore

\[ (-1)^N \partial_\xi^N L_2(x, y, x-y) \psi_n(x-y) = O(n^{-N}). \]

Since

\[ \partial_\xi^{N+1} L_2(x, y, \xi) = \mu_2 \partial_\xi^N K((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \]
\[ - 2 \int_{(-x+y+\xi)/2}^{y} \partial_\xi^{N+1} K(x-y+Y, Y, \xi) dY \]
\[ = O((-x+y+\xi)^N) + O(x+y-\xi), \]

we see

\[ F_{n}^2(x, y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x, y, \xi) \psi_n(\xi) d\xi \]
\[ = O(n^{-1}). \]
Hence $\sigma_n = O(n^{-1})$. Therefore

\[
\begin{align*}
B_n^3 &= \eta^3 \sigma_n - \eta_n \sigma^3 = O(n^{-2N-1}), \\
B_n^4 &= \eta^4 \sigma_n - \eta_n \sigma^4 = O(n^{-2N}), \\
B_n &= \eta_n \hat{\sigma}_n - \hat{\eta}_n \sigma_n = O(n^{-1}).
\end{align*}
\]

3) Suppose $(x, y) \in S_1$, where $x + y > a + \frac{1}{n}$ and $x - y < a - \frac{1}{n}$. Then $\psi_n(x+y) = \psi_n(x-y) = \hat{\psi}_n(x+y) = \hat{\psi}_n(x-y) = 0$. So, $\eta_n = F_n^1, \sigma_n = F_n^2$, and so on. But

\[
F_n^1(x,y) = (-1)^{N+1} \int_{x-y}^{x+y} \partial^{N+1}_{\xi} K(x,y,\xi) \psi_n(\xi)d\xi
\]

\[
= (-1)^{N+1} \int_{-1}^{1} (\partial^{N+1}_{\xi} K(x,y,a+\frac{s}{n}) - \partial^{N+1}_{\xi} K(x,y,a)) \Phi(s)ds
\]

\[
= O(1/n)
\]

since $\int \Phi = 0$ and $\partial^{N+1}_{\xi} K$ is Lipschitz continuous. Same estimates hold for $F_n^2, \hat{F}_n^1, \hat{F}_n^2$. Thus

\[
\begin{align*}
B_n^3 &= \eta^3 F_n^2 - F_n^1 \sigma^3 = O(1/n), \\
B_n^4 &= \eta^4 F_n^2 - F_n^1 \sigma^4 = O(1/n), \\
B_n &= F_n^1 \hat{F}_n^2 - \hat{F}_n^1 F_n^2 = O(1/n^2).
\end{align*}
\]

4) Suppose $(x, y) \in S_L$, where $|x + y - a| \leq 1/n$. It is easy to see $\eta^3 = O(n^{-N}), \eta^4 = O(n^{-N+1}), \sigma^3 = O(n^{-N-1}), \sigma^4 = O(n^{-N})$. Since $n(x-y-a) < -1$, we have $\psi_n(x-y) = 0$. Thus $\eta_n = O(n), \sigma_n = F_n^2 = O(1)$. Therefore

\[
\begin{align*}
B_n^3 &= \eta^3 \sigma_n - \eta_n \sigma^3 = O(n^{-N}), \\
B_n^4 &= \eta^4 \sigma_n - \eta_n \sigma^4 = O(n^{-N}).
\end{align*}
\]

Let us estimate $B_n = \eta_n \hat{\sigma}_n - \hat{\eta}_n \sigma_n$. Since

\[
\partial^{N+1}_{\xi} K = (-1)^N \frac{N(N+1)}{2} (\xi - x)^{N-1} G(x, |x-\xi|, \xi) + \quad + (\xi - x)^N H_N(x, \xi) + J \partial^{N} G_{N-1},
\]

we have

\[
F_n^1 = (-1)^{N+1} \int_{x-y}^{x+y} \partial^{N+1}_{\xi} K(x,y,\xi) \psi_n(\xi)d\xi =
\]

\[
= (-1)^{N+1}((-1)^N \frac{N^2}{2} 2^N N!(a-x)^{N-1} + F'(x,a)) \int_{-1}^{n(x+y-a)} \Phi + O(1/n)
\]

\[
= \frac{N(N+1)}{2} 2^N N! y^{N-1} (1 + F''(x,y,a)) \int_{-1}^{n(x+y-a)} \Phi + O(1/n),
\]

where $F' = O(1/c^2)|x - a|^N, F'' = O(1/c^2).$ On the other hand
\[
\partial_\xi^K(x, y, x + y) = (-1)^N y^N G(x, y, x + y).
\]
Hence
\[
\eta_n = ny^N G(x, y, x + y) \Phi(n(x + y - a)) + \frac{N(N + 1)}{2} 2^N N! y^{N-1} (1 + F''(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi + O(1/n).
\]
Since
\[
\partial_\xi^{N+1} L_2(x, y, \xi) = \mu_2 \partial_\xi^K((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + 2 \int_{(-x+y+\xi)/2}^{y} \mu_2 \partial_\xi^{N+1} K(x-y+\xi, \xi) dY = \left(\frac{N}{2N+1} + O(1/c^2)\right)(-1)^N \left(-\frac{x+y+\xi}{2}\right)^N \times G((x+y+\xi)/2, (-x+y+\xi)/2, \xi) + O(x+y-\xi),
\]
we see
\[
\sigma_n = F_n^2 = (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x, y, \xi) \psi_n(\xi) d\xi = -\frac{N}{2N+1} 2^N N! y^N (1 + L'(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi + O(1/n),
\]
where $L' = O(1/c^2).$ Here we have used
\[
\left(-\frac{x+y+a}{2}\right)^N = (y - \frac{x+y-a}{2})^N = y^N + O(1/n).
\]
Similar estimates hold for $\hat{\eta}_n, \hat{\sigma}_n.$ Thus
\[
B_n = n y^{2N} A_1 + y^N A_2 + A_3,
\]
where
\[
A_1 = -G \frac{N}{2N+1} 2^N N! (1 + L') \Phi(\beta) \int_{-1}^{\beta} \Phi + G \frac{N}{2N+1} 2^N N! (1 + L') \Phi(\beta) \int_{-1}^{\beta} \Phi = \frac{N}{2N+1} 2^N N! (1 + L') (\int_{-1}^{\beta} \Phi)^2,
\]
\[
\beta = n(x + y - a).
\]
The estimates on $S_R$ can be obtained in a similar manner considering $\overline{\sigma}^3, \overline{\sigma}^4, \sigma_n$. QED.

If we put

$$\tilde{B}_n^3 = \eta^3 \eta_n^6 - \eta_n^6 q^3,$$
$$\tilde{B}_n^4 = \eta^4 q_n^6 - \eta_n^6 q^4,$$

then the same estimates hold.

9 Compactness of $\eta_t + q_x$

Let us consider an entropy $\eta$ generated by $\phi$ through the generalized Darboux formula and its flux $q$. In this section we will prove

Lemma 1 Let $U^\Delta$ be the approximate solutions constructed in Section 4. Then $\eta(U^\Delta)_t + q(U^\Delta)_x$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$, $\Omega$ being a bounded open subset of $\{t \geq 0\}$.

Proof. Let $\Phi$ be a test function and we consider

$$J = \int \int (\eta(U^\Delta)_t + q(U^\Delta)_x) \Phi_t + q(U^\Delta)_x) dx \, dt = N + L + \Sigma,$$

$$N = - \int \eta(U^\Delta(+0,x)) \Phi(0,x) dx,$$

$$L = \sum_n \int \eta(t,x) \Phi(n\Delta t, x) dx,$$

$$\Sigma = \int \sum_{\text{shock}} (\sigma[\eta] - [q]) \Phi dt.$$

Since $U^\Delta$ is bounded, we see

$$|N| \leq M\|\Phi\|_C.$$

Let us look at $L$. We see

$$L = L_1 + L_2,$$

$$L_1 = \sum_{j,n} \Phi(n\Delta t, (2j+1)\Delta x) \int_{2j\Delta x}^{(2j+2)\Delta x} [\eta(U^\Delta)]_{t=n\Delta t-0}^{t=n\Delta t+0} dx,$$

$$L_2 = \sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} (\Phi(n\Delta t, x) - \Phi(n\Delta t, (2j+1)\Delta x) \times$$

$$\times [\eta(U^\Delta)]_{t=n\Delta t-0}^{t=n\Delta t+0} dx.$$
We note
\[
[\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} = D_U \eta(U^\Delta(n\Delta t + 0), x)[U^\Delta]
\]
\[
+ \int_0^1 (1 - \theta) \left[ (U^\Delta)D_U^2 \eta(U^\Delta(n\Delta t + 0) + \theta[U^\Delta]).[U^\Delta] \right] d\theta.
\]
and
\[
\int_{2j\Delta x}^{(2j+2)\Delta x} [U^\Delta] dx = 0
\]
by the scheme. Therefore
\[
|L_1| \leq M||\Phi||_C \sum_{j,n} \int \int_0^1 (1 - \theta)|F(\theta, \eta)|d\theta dx,
\]
where
\[
F(\theta, \eta) = \left[ (U^\Delta)D_U^2 \eta(U^\Delta(n\Delta t + 0) + \theta[U^\Delta]).[U^\Delta] \right].
\]
By Proposition 22 we know \(|F(\theta, \eta)| \leq MF(\theta, \eta^*)\). But in the proof of Proposition 7 we know
\[
\sum_{j,n} \int \int_0^1 (1 - \theta)F(\theta, \eta^*)d\theta dx \leq C.
\]
Thus we know
\[
|L_1| \leq M||\Phi||_C.
\]
In the proof of Proposition 7 we know
\[
\sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} [U^\Delta]^2 dx \leq C.
\]
Therefore
\[
|L_2| \leq 2^\alpha||\Phi||\|C^\alpha\| \sum_n \int (\Delta x)^\alpha ||\eta(U^\Delta)|| dx
\]
\[
\leq 2^{\alpha-1}||\Phi||\|C^\alpha\| \sum_n \int ((\Delta x)^{\alpha+\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} ||\eta(U^\Delta)||^2) dx
\]
\[
\leq M||\Phi||\|C^\alpha\|((\Delta x)^{\alpha+\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} \sum\int ||U^\Delta||^2 dx
\]
\[
\leq M^')(\Delta x)^{\alpha-\frac{1}{2}} ||\Phi||_C^\alpha,
\]
where we use the boundedness of \(D_U \eta\) and \(n = O(1/(\Delta x))\). Next we look at \(\Sigma\). Along the shock we have
\[
\sigma[\eta(U)] = [g(U)]
\]
\[
= \int_{\rho_L}^{\rho_L} \left(-\frac{d\sigma}{d\rho} \int_0^1 \theta(U - U_L)D_U^2 \eta(U_L + \theta(U - U_L))(U - U_L))d\theta\right).
\]
This implies

$$|\sigma[\eta] - [q]| \leq M(|\sigma[\eta^*] - [q^*]|).$$

But we know

$$\int \sum_{\text{shock}} (\sigma[\eta^*] - [q^*])dt \leq C$$

in the proof of Proposition 7. Therefore

$$|\Sigma| \leq M||\Phi||_C.$$ 

Summing up, we know the compactness. QED.

10 Convergence of approximate solutions

We consider the approximate solutions $U^\Delta$ constructed in Section 4. Since $U^\Delta$ is bounded, there is a sequence $U^{\Delta_n}$ and a family of Young measures $\nu_{t,x}$ such that $supp \nu_{t,x} \subset \Sigma = \Sigma_B$ and for any continuous function $f$

$$f(U^{\Delta_n}(t,x)) \rightarrow \bar{f} = \langle \nu_{t,x}, f \rangle$$

in $L^\infty$ weak star topology. By Lemma 1, we can apply the compensated compactness theory, and we can assume

$$(\eta q' - \eta' q)(U^{\Delta_n}) \rightarrow \langle \nu, q \rangle < \nu, q' > - \langle \nu, \eta' \rangle < \nu, q >$$

in $L^\infty$ weak star. Here $\eta, q; \eta', q'$ are arbitrary Darboux entropy pairs. Thus we have

Lemma 2 For any pairs $(\eta, q), (\eta', q')$ of Darboux entropies-entropy flux, the identity

$$\langle \nu, \eta q' - \eta' q \rangle = \langle \nu, \eta \rangle < \nu, q' > - \langle \nu, \eta' \rangle < \nu, q >$$

holds a.e.-$(t,x)$, where $\nu = \nu_{t,x}$.

Since entropies we will use are countably many, we can assume that the above identity holds outside a null set which is common to all $\eta$. We fix $(t, x)$ at which the identity holds, and we write $\nu = \nu_{t,x}$. Of course $supp.\nu \subset \Sigma$. Suppose that $supp.\nu \cap \{\rho > 0\} \neq \emptyset$. Let $\Sigma_0$ be the smallest triangle $\{z_0 \leq z \leq w \leq w_0\}$ such that $supp.\nu \cap \{\rho > 0\} \subset \Sigma_0$. Let us denote by $P_0$ the state $(w_0, z_0)$. It will be verified that $\nu = \delta_{P_0}$. (the Dirac measure).

First we show

Proposition 27

$$P_0 \in supp.\nu.$$
Proof. Suppose $P_0 \notin \text{supp.}\nu$. Since $\Sigma_0$ is the smallest triangle containing $\text{supp.}\nu \cap \{\rho > 0\}$, $w = w_0$ and $z = z_0$ intersect with $\text{supp.}\nu \cap \{\rho > 0\}$ neighborhoods of these intersection points we have

\[
\eta^1 \geq \frac{1}{M} e^{k(w_0 - \epsilon)},
\]
\[
\eta^2 \geq \frac{1}{M} e^{-k(z_0 + \epsilon)}.
\]

(See Proposition 24). Since $\nu, \eta^1, \eta^2$ are nonnegative, we see

\[
<\nu, \eta^1> \geq \frac{1}{M} e^{k(w_0 - \epsilon)},
\]
\[
<\nu, \eta^2> \geq \frac{1}{M} e^{-k(z_0 + \epsilon)}.
\]

Since $P_0 \notin \text{supp.}\nu$, we have

\[
<\nu, \eta^2 q^1 - \eta^1 q^2> \leq M e^{k(w_0 - z_0 - \delta)}.
\]

Taking $2\epsilon < \delta$, we have

\[
|<\nu, q^1> - <\nu, q^2>| \leq M e^{-k(\delta - 2\epsilon)} \rightarrow 0
\]

as $k \rightarrow \infty$. Let $\beta$ be a sufficiently small positive number, and we put

\[
\Sigma_2 = \{z_0 \leq z \leq w < w_0 - \beta\},
\]
\[
\Sigma_3 = \{z_0 \leq z \leq w \leq w_0, w_0 - \beta \leq w\}.
\]

Then

\[
\eta^1 e^{-kw} = (1 + O(1/c^2)) 2^N N! y^{N-1}(y + O(1/k))
\]

is bounded on $\Sigma_0$ and we have

\[
<\nu|_{\Sigma_2}, \eta^1> \leq Me^{k(w_0 - \beta)}.
\]

Taking $\epsilon = \beta/2$, we know

\[
\frac{<\nu|_{\Sigma_2}, \eta^1>}{<\nu, \eta^1>} \leq Me^{-\beta k/2} \rightarrow 0.
\]

Since $\partial \lambda_2 / \partial w > 0$, we know

\[
\lambda_2(w, z) \geq \lambda_2(w_0 - \beta, z_0)
\]
on $\Sigma_3$. Therefore we have
\[
\frac{<\nu, q^1>}{<\nu, \eta^1>} = \frac{<\nu|\Sigma_2, \eta^1\lambda_2>}{<\nu, \eta^1>} + \frac{<\nu|\Sigma_3, \eta^1\lambda_2>}{<\nu, \eta^1>} + O(1/k)
\]
\[
\geq o(1) + \lambda_2(w_0 - \beta, z_0)
\]
Similarly we see
\[
\frac{<\nu, q^2>}{<\nu, \eta^2>} \leq o(1) + \lambda_1(w_0, z_0 + \beta).
\]
Therefore we have
\[
\lambda_2(w_0 - \beta, z_0) - \lambda_1(w_0, z_0 + \beta) \leq 0 + o(1).
\]
Passing to the limit, we know
\[
\lambda_2(w_0, z_0) \leq \lambda_1(w_0, z_0).
\]
But this means $P_0 \in \{\rho = 0\}$, a contradiction. QED.

Let us fix $a$ such that $z_0 < a < w_0$. We have
\[
<\nu, B_n^3> = <\nu, \eta^3><\nu, q_n^5> - <\nu, \eta^5><\nu, q^3>^3,
\]
\[
<\nu, B_n^4> = <\nu, \eta^4><\nu, q_n^5> - <\nu, \eta^5><\nu, q^4>^3,
\]
\[
<\nu, \eta^3 q^4 - \eta^4 q^3> = <\nu, \eta^3><\nu, q^4> - <\nu, \eta^4><\nu, q^3>^3,
\]
\[
<\nu, B_n> = <\nu, \eta_n^5><\nu, q_n^6> - <\nu, \eta_n^6><\nu, q_n^5>.
\]
From (8.8) we know
\[
<\nu, \eta^3 q^4 - \eta^4 q^3>> 0
\]
and from (8.10) we know
\[
<\nu, B_n^3> \rightarrow 0
\]
Using these we can prove the following propositions. Proofs can be found in Chen et al [2].

**Proposition 28** As $n \rightarrow \infty$, $<\nu, \eta_n^5>, <\nu, q_n^5>, <\nu, q_n^6>, <\nu, q_n^6> >$ are bounded.

**Proposition 29** As $n \rightarrow \infty$, we have $<\nu, B_n> \rightarrow 0$.

Now, taking
\[
\Phi_0(x) = \begin{cases} 
  e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1 \\
  0 & \text{if } |x| \geq 1 
\end{cases}
\]
we put

$$\Phi(x) = \frac{1}{\beta}(\Phi_0\left(\frac{x+\beta}{\beta}\right) - \Phi_0\left(\frac{x-\beta}{\beta}\right))$$

for the generating function of $\eta_n^5$. Here $\beta = (1 - \alpha)/2$. We put

$$S_+ = \{z \leq w, |w - a| \leq \frac{1 - 3\alpha}{n}\},$$

$$S_- = \{z \leq w, |z - a| \leq \frac{1 - 3\alpha}{n}\}.$$

**Proposition 30** As $n \to \infty$, we have

$$\langle \nu |_{S_+}, ny^{2N} > + \langle \nu |_{S_-}, ny^{2N} \rangle \to 0.$$

**Proof.** Put $S_L' = S_+ \cap S_L, S_R' = S_- \cap S_R$. It is sufficient to prove that

$$\langle \nu |_{S_L'}, ny^{2N} > + \langle \nu |_{S_R'}, ny^{2N} \rangle \to 0.$$

From (8.11) we have

$$\langle \nu |_{S_L}, ny^{2N} A_1 + y^N A_2 > + \langle \nu |_{S_R}, ny^{2N} C_1 + y^N C_2 \rangle \to 0.$$

Note

$$A_1 = \left(\frac{N(2^{N}N!)^2}{2N+1} + O(1/c^2)\right)(\int_{-1}^{x+y-a})^2 \geq \frac{1}{M_0} > 0$$

on $S_L'$. Put

$$E_n = \{0 \leq y \leq (\frac{1}{n})^\mu\},$$

where $\mu$ is a positive parameter. Then $|y^N A_2| \leq M(1/n)^{\mu N} = o(1)$ on $S_L \cap E_n$ and $|y^N A_2| \leq Mny^{2N}(1/n)^{1-\mu N}$ on $S_L - E_n$. Choose $d_n \searrow 0$ such that

$$\int_{-1+\alpha}^{1-\alpha-d_n} \Phi = -\int_{1-\alpha}^{1-\alpha-d_n} \Phi \geq (1/n)^{\mu_0}. $$

Then

$$(\int_{-1}^{H} \Phi)^2 \geq (1/n)^{2\mu_0}$$

for $|H| \leq 1 - \alpha - d_n$, and

$$|\Phi(H)| + |\int_{-1}^{H} \Phi| = o(1)$$

for $1 - \alpha - d_n \leq |H| \leq 1$. Put

$$S_n^+ = S_L \cap \{|w - a| \leq \frac{1 - \alpha - d_n}{n}\}.$$

Then $S_L' \subset S_n^+ \subset S_L$ and

$$|y^N A_2| = o(1)$$
on $S_L - S^*_L$ and

$$ny^{2N} A_1 + y^N A_2 \geq ny^{2N}(\frac{1}{M}(1/n)^{2\mu_0} - M(1/n)^{1-\mu N})$$

$$\geq 0$$

on $S^*_L - E_n$. Here we take $0 < 2\mu_0 < 1 - \mu N$. Then

$$<\nu|_{S_L}, ny^{2N} A_1 + y^N A_2 > = <\nu|_{S_L \cap E_n}, ny^{2N} A_1 > +$$

$$+ <\nu|_{S_L - E_n}, ny^{2N} A_1 + y^N A_2 > +$$

$$+ o(1)$$

$$\geq \frac{1}{M_0} <\nu|_{S'_L \cap E_n}, ny^{2N} > +$$

$$+ <\nu|_{S_L - S^*_L \cap E_n}, ny^{2N} A_1 > +$$

$$+ <\nu|_{S'_L - E_n}, ny^{2N} A_1 + y^N A_2 > +$$

$$+ <\nu|_{S_L^* - S'_L - E_n}, ny^{2N} A_1 + y^N A_2 > +$$

$$+ o(1)$$

$$\geq \frac{1}{M_0} <\nu|_{S'_L \cap E_n}, ny^{2N} > +$$

$$+ <\nu|_{S_L^* - S'_L - E_n}, ny^{2N} > +$$

$$+ o(1).$$

Similarly we know

$$<\nu|_{S^*_R}, ny^{2N} C_1 + y^N C_2 > \geq \frac{1}{2M_0} <\nu|_{S'_R}, ny^{2N} > + o(1)$$

. Thus we see

$$<\nu|_{S'_L}, ny^{2N} > + <\nu|_{S'_R}, ny^{2N} > \rightarrow 0.$$

QED.

**Proposition 31** We have

$$\nu|_{\{\rho > 0\}} = \delta_{P_0}.$$

Proof. Proposition 30 says that the projections $P_w \tilde{\nu}, P_z \tilde{\nu}$ of the measure $\tilde{\nu} = y^{2N} \nu$ admits the Lebesgue lower derivatives which vanish at any $a$. Therefore we can claim that

$$\text{supp.} \nu \cap \{\rho > 0\} = \{P_0\}.$$
Since $\nu$ is a probability measure, we have 
\[ \nu|_{\{\rho > 0\}} = C \delta_{P_0}. \]

But 
\[ C(\eta^3 q^4 - \eta^4 q^3) = C^2(\eta^3 q^4 - \eta^4 q^3) \]
at $P_0$. Hence $C = 1$. QED.

Summing up we get the final

**Theorem 2** For any $M_0$ there is a positive number $\epsilon_0$ such that if the initial data satisfy 
\[ 0 \leq \rho_0(x) \leq M_0, \quad \left| \frac{c}{2} \log \frac{c + u_0(x)}{c - u_0(x)} \right| \leq M_0. \]
and if $1/c^2 \leq \epsilon_0$, then a subsequence of the approximate solutions $U^\Delta$ converges a.e. to a limit $U$ which is a weak solution of the relativistic Euler equation.

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### 参考文献


