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Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas

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1 Introduction

We consider the initial-boundary value problem on $\mathbf{R}_+ = (0,\infty)$ for a system of one-dimensional barotropic viscous flow in the Eulerian coordinate:

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, \quad (\tilde{x}, t) \in \mathbf{R}_+ \times \mathbf{R}_+ \\ (\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_{\tilde{x}} = \mu \tilde{u}_{\tilde{x}\tilde{x}} \\ (\tilde{\rho}, \tilde{u})|_{t=0} = (\tilde{\rho}_0, \tilde{u}_0)(\tilde{x}) \to (\rho_+, u_+) \quad \text{as} \quad \tilde{x} \to +\infty \\ (\tilde{\rho}, \tilde{u})|_{\tilde{x}=0} = (\rho_-, u_-), \end{cases} \tag{1.1}$$

where the conditions

$$\rho_\pm > 0, \ \tilde{\rho}_0(\tilde{x}) > 0 \ \text{ and } \ u_- > 0$$

and the compatibility conditions are assumed. Here $\tilde{\rho}$ is the density, $\tilde{u}$ is the velocity, and the pressure $\tilde{p}$ is given by $\tilde{p} = \tilde{\rho}^\gamma (\gamma \geq 1)$. Since $u_- > 0$, the flow with (ρ_-, u_-) goes into the region under consideration through the boundary $\tilde{x} = 0$, and hence the problem (1) is called the *inflow* problem. In the case of $u_- = 0$ the condition $\tilde{\rho}|_{\tilde{x}=0} = \rho_-$ is not imposed. The asymptotic behaviors in the case $u_- = 0$ are investigated in [3,7]. The present problem is treated in Matsumura and Nishihara [8].

We first rewrite (1.1) in the Lagrangian coordinate. The mass of the gas inflowed for $(0,t)$ is $\rho_- u_- t = \frac{u_-}{v_-} t, \ v_- := 1/\rho_-$. Hence, the problem (1.1) is transformed into that with the moving boundary $x = s_- t, \ s_- = -u_-/v_- (< 0)$ in the Lagrangean coordinate:

$$\begin{cases} v_t - u_x = 0, \quad (x,t) \in \{(x,t); x > s_- t, \ t > 0\} \\ u_t + p(v)_x = \mu (\frac{u_x}{v})_x, \quad p = p(v) = v^{-\gamma} \\ (v, u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) := (1/\rho_+, u_+) \quad \text{as} \quad x \to +\infty \\ (v, u)|_{x=0} = (v_-, u_-) = (1/\rho_-, u_-), \end{cases} \tag{1.2}$$

where $(v,u)(x,t) = (1/\tilde{\rho}, \tilde{u})(\tilde{x}, t)$.

Our aim is to investigate the asymptotic behaviors of the solution (v,u) to (1.2), equivalent to (1.1).

The characteristic speeds for the corresponding hyperbolic system are $\lambda_i(v) = (-1)^i \sqrt{-p'(v)}, \ i = 1, 2$. Comparing the speed s_- of moving boundary with the characteristic speed $\lambda_1(v)$, we devide the quarter space into three resions

$$\begin{array}{rcl} \Omega_{sub} & = & \{(v,u);\ 0 < u < c(v),\ v > 0,\ u > 0\} \\ \Gamma_{trans} & = & \{(v,u);\ u = c(v),\ v > 0,\ u > 0\} \\ \Omega_{super} & = & \{(v,u);\ u > c(v),\ v > 0,\ u > 0\}, \end{array} \tag{1.3}$$

where $c(v) = v\sqrt{-p'(v)} = \sqrt{\gamma}v^{-\frac{\gamma+1}{2}}(= \sqrt{\tilde{p}'(\tilde{\rho})})$ is the sound speed. So, we call them the subsonic, transonic and supersonic regions, respectively. See Figure 1.1.

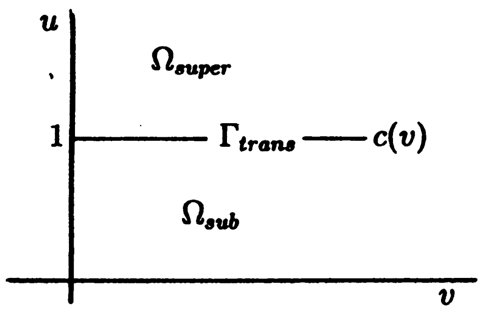

Figure 1.1($\gamma = 1$)

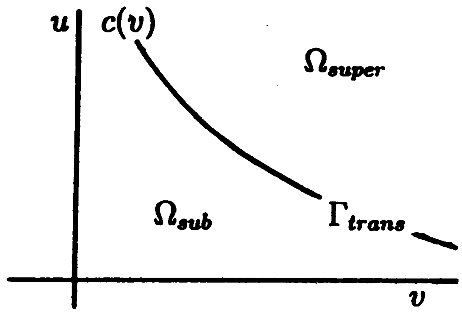

Figure 1.1($\gamma > 1$)

If $(v_-, u_-) \in \Omega_{sub}$, then $\lambda_1(v_-) < s_-(< 0)$, and hence the existence of a traveling wave solution $(V,U)(x - s_-t)$ with $(V,U)(0) = (v_-, u_-)$, $(V,U)(+\infty) = (v_+, u_+)$ is expected. Substitute this into $(1.2)_{1,2}$ (this means the first and second equations in (1.2)) to have

$$\begin{cases} -s_-V' - U' = 0, \quad ' = d/d\xi, \quad \xi = x - s_-t > 0 \\ -s_-U' + p(V)' = \mu(\dfrac{U'}{V})' \\ (V,U)(0) = (v_-, u_-), \quad (V,U)(+\infty) = (v_+, u_+). \end{cases} \tag{1.4}$$

When the solution (V,U) to (1.4) exists, it is called the boundary layer solution, or BL-solution simply. Seek for the condition for the existence. When (V,U) exists, the integration of (1.4) over $(0,\infty)$ and (ξ,∞) yields

$$\begin{cases} -s_-(v_+ - v_-) - (u_+ - u_-) = 0 \\ -s_-(u_+ - u_-) + p(v_+) - p(v_-) = -\mu\dfrac{U'(0)}{v_-} \end{cases} \tag{1.5}$$

and

$$\begin{cases} -s_-(V - v_+) - (U - u_+) = 0 \\ -s_-(U - u_+) + p(V) - p(v_+) = \mu\dfrac{U'}{V}. \end{cases} \tag{1.6}$$

From $(1.5)_1$ and $(1.6)_1$

$$s_- = -\frac{U(\xi)}{V(\xi)} = -\frac{u_-}{v_-} = -\frac{u_+}{v_+}, \tag{1.7}$$

and hence we define BL-line through $(v_-, u_-) \in \Omega_{sub}$ by

$$BL(v_-, u_-) = \{(v, u) \in \Omega_{sub} \cup \Gamma_{trans};\ \frac{u}{v} = \frac{u_-}{v_-} = -s_-\}.$$

Especially, denote $\Gamma_{trans} \cap BL(v_-, u_-) = \{(v_*, u_*)\}$. By (1.6) we have the ordinary differential equation of V:

$$\begin{cases} \mu \dfrac{dV}{d\xi} = \dfrac{V}{s_-}\{-s_-^2(V - v_+) - (p(V) - p(v_+))\} =: \dfrac{V}{s_-}h(V) \\ V(0) = v_-, \quad V(+\infty) = v_+. \end{cases} \tag{1.8}$$

Conversely we can show the existence of the solution V to (1.8) and hence U for $(v_+, u_+) \in BL(v_-, u_-)$. At (v_*, u_*), $-s_-^2 = -(u_*/v_*)^2 = -(c(v_*)/v_*)^2 = -\gamma v_*^{-\gamma-1} = p'(v_*)$. That is, $-s_-^2$ is the slope of the tangential line of $p(V)$ at $(v_*, p(v_*))$. Hence we find that $h(v_+) = 0$, $h(v) > 0$ and $\frac{dV}{d\xi} < 0$ for $v_+ < v < v_-$ if $v_+ < v_-$, and $h(v) < 0$ and $\frac{dV}{d\xi} > 0$ for $v_- < v_+ (\leq v_*)$ if $v_- < v_+$. Thus, we have the following lemma.

Lemma 1. 1 (Boundary Layer Solution) *Let $(v_-, u_-) \in \Omega_{sub}$ and $(v_+, u_+) \in BL(v_-, u_-)$. Then, there exists a unique solution $(V, U)(\xi)$ to (1.8), which satisfies*

$$\begin{aligned} &|(V(\xi) - v_+, U(\xi) - u_+)| \leq C\delta \exp(-c|\xi|) && \text{if} \quad v_+ < v_* \\ &|(V(\xi) - v_+, U(\xi) - u_+)| \leq C\delta |\xi|^{-1} && \text{if} \quad v_+ = v_*, \end{aligned}$$

where $\delta = |(v_+ - v_-, u_+ - u_-)|$.

On the other hand, since $0 > \lambda_1(v) > s_-$ in Ω_{super}, the 1-characteristic field is away from the moving boundary. Since $\lambda_2(v) > 0 > s_-$, the waves along the 2-characteristic field, of course, go away from the boundary. Hence, in these cases the behaviors of solutions are expected to be same as those for the Cauchy problem (See Matsumura and Nishihara [4,5,6]).

Hence, the large time behaviors to be expected devide the (v, u)-space as the following figure, Figure 1.2.

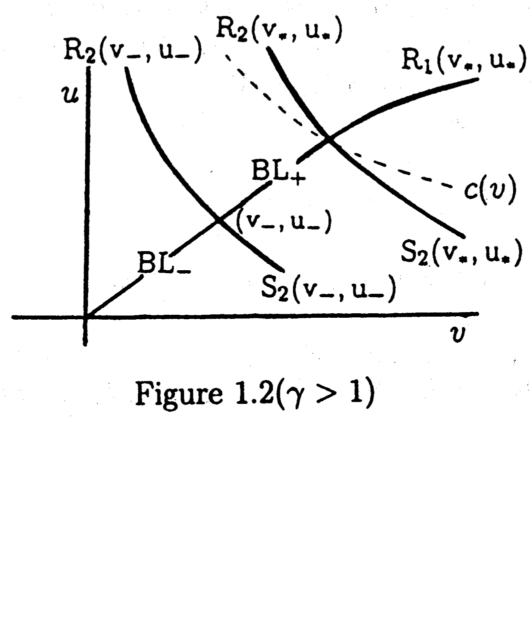

Figure 1.2($\gamma > 1$)

Here,

$$\begin{aligned}
&BL_+(v_-,u_-) = \{(v,u) \in BL(v_-,u_-); v_- < v \le v_*\}\\
&BL_-(v_-,u_-) = \{(v,u) \in BL(v_-,u_-); 0 < v < v_-\}\\
&R_1(v_*,u_*) = \{(v,u); u = u_* - \int_{v_*}^{v} \lambda_1(s)ds,\ v > v_*\}\\
&R_2(v_-,u_-) = \{(v,u); u = u_- - \int_{v_-}^{v} \lambda_2(s)ds,\ v < v_*\}\\
&R_2(v_*,u_*) = \{(v,u); u = u_* - \int_{v_*}^{v} \lambda_2(s)ds,\ v < v_*\}\\
&S_2(v_-,u_-) = \{(v,u); u = u_- - s_2(v-v_-),\ v > v_-\}\\
&S_2(v_*,u_*) = \{(v,u); u = u_* - s_*(v-v_*),\ v > v_*\},
\end{aligned}$$

together with $s_2 = \sqrt{-(p(v)-p(v_-))/(v-v_-)}$ and $s_* = \sqrt{-(p(v)-p(v_*))/(v-v_*)}$.

Our aim is to investigate the stability of the BL-solution or a superposition of the BL-solution and nonlinear waves. Our results are, roughly speaking, as follows.

(I) If $(v_+,u_+) \in BL_+(v_-,u_-)$, then the BL-solution is stable.

(II) If $(v_+,u_+) \in BL_-(v_-,u_-)$, then the BL-solution is stable provided that $|(v_+-v_-,u_+-u_-)|$ is small. That is, the BL-solution is necessary to be weak.

(III) If $(v_+,u_+) \in BL_+R_2(v_-,u_-)$, then there exists $(\bar v,\bar u) \in BL_+(v_-,u_-)$ such that $(v_+,u_+) \in R_2(\bar v,\bar u)$, and the superposition of the BL-solution connecting (v_-,u_-) with $(\bar v,\bar u)$ and the 2-rarefaction wave connecting $(\bar v,\bar u)$ with (v_+,u_+) is stable provided that $|(v_+-\bar v,u_+-\bar u)|$ is small, where

$$BL_+R_2(v_-,u_-) = \{(v,u);\ u > -s_-v,\ \ u > u_- - \int_{v_-}^{v} \lambda_2(s)ds,\ \ u \le u_* - \int_{v_*}^{v} \lambda_2(s)ds\}.$$

That is, the rarefaction wave is weak, but the BL-solution is not necessarily weak.
(IV) If $(v_+,u_+) \in BL_-R_2(v_-,u_-)$, then the superposition of the BL-solution and the 2-rarefaction wave is stable provided that $|(v_+-v_-,u_+-u_-)|$ is small, where

$$BL_-R_2(v_-,u_-) = \{(v,u);\ u > -s_-v,\ \ u < u_- - \int_{v_-}^{v} \lambda_2(s)ds\}.$$

In this case, both the BL-solution and the rarefaction wave are weak.
(V) If $(v_*,u_*) \in BL_+(v_-,u_-)$, $(v_+,u_+) \in R_1R_2(v_*,u_*)$ and $|(v_+-v_*,u_+-u_*)|$ is small, then the superposition of the BL-solution, 1-rarefaction wave and 2-rarefaction wave is stable. Here,

$$R_1R_2(v_*,u_*) = \{(v,u);\ u > u_* - \int_{v_*}^{v} \lambda_i(s)ds,\ i = 1,2\}.$$

Similar to (III), the BL-solution is not necessarily weak.

In the proofs of the above assertions, the sign of $U_\xi = V_t$ is important, similar to those of the Cauchy problem. So, to show (I) and (II) are essential. The cases

(III), (V) are the applications to (I), and (IV) is to (II). In the next section we mainly state the stability thorems of the boundary layer solutions.

The other cases are still open. For example, when

$$(v_+, u_+) \in BL_-S_2(v_-, u_-) = \{(v, u);\ u < -s_-v,\ \ u < u_- - s_2(v - v_-)\},$$

the asmptotic state is conjectured to be $(V, U)(x-s_-t)+(V_2^S, U_2^S)(x-s_2t+\alpha)-(\bar{v}, \bar{u})$ together with a suitable shift α, where $(\bar{v}, \bar{u}) \in BL_-(v_-, u_-)$ such that $(v_+, u_+) \in S_2(\bar{v}, \bar{u})$, and (V, U) is the BL-solution connecting (v_-, u_-) with $(\bar{v}, \bar{u})$ and (V_2^S, U_2^S) is 2-viscous shock wave connecting $(\bar{v}, \bar{u})$ with (v_+, u_+). Even though the shift α is conjectured by the same way as in Matsumura and Mei [3], this case is not solved yet.

2 Stability of the boundary layer solution

2.1 The case $(v_+, u_+) \in BL_+(v_-, u_-)$

Assume that

$$(v_-, u_-) \in \Omega_{sub} \quad \text{and} \quad (v_+, u_+) \in BL_+(v_-, u_-), \tag{2.1}$$

then a boundary layer solution $(V, U)(\xi)$, $\xi = x-s_-t \geq 0$, $s_- = -u_-/v_-$ connecting (v_-, u_-) with (v_+, u_+) is uniquely determined in Lemma 1.1. Note that

$$U_\xi = -s_-V_\xi > 0, \tag{2.2}$$

which plays an important role in the a priori estimate. The perturbation $(\phi, \psi)(\xi, t)$ defined by

$$(v, u)(x, t) = (V, U)(\xi) + (\phi, \psi)(\xi, t), \tag{2.3}$$

satisfies

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, \quad \xi > 0, \quad t > 0 \\ \psi_t - s_-\psi_\xi + (p(V+\phi) - p(V))_\xi = \mu(\dfrac{U_\xi + \psi_\xi}{V+\phi} - \dfrac{U_\xi}{V})_\xi \\ (\phi, \psi)|_{\xi=0} = (0, 0) \\ (\phi, \psi)|_{t=0} = (\phi_0, \psi_0)(\xi) := (v_0 - V, u_0 - U)(\xi), \end{cases} \tag{2.4}$$

from (1.2) and (1.6).

To solve (2.4) we apply the L^2-energy method. The solution space is defined by

$$X_{m,M}(0, T) = \{(\phi, \psi) \in C([0, T]; H_0^1) \mid \phi_x \in L^2(0, T; L^2), \quad \psi_x \in L^2(0, T; H^1)$$
$$\text{with} \quad \sup_{[0,T]} \|(\phi, \psi)(t)\|_1 \leq M, \quad \inf_{\mathbf{R}_+\times[0,T]} (V+\phi)(\xi, t) \geq m\},$$

for positive constants m, M. Here, we denote $\|f\|_k = (\sum_{j=1}^k \|\partial_x^j f\|^2)^{1/2}$ and $\|f\| = (\int_0^\infty |f(x)|^2 dx)^{1/2}$. To obtain a time-global solution, we combine the time-local existence of the solution with the a priori estimates, which are given as follows.

Proposition 2. 1 (Local existence) *Let (ϕ_0, ψ_0) be in $H_0^1(\mathbf{R}_+)$. If $\|\phi_0, \psi_0\|_1 \le M$, and $\inf_{\mathbf{R}_+\times[0,T]}(V+\phi)(\xi,t) \ge m$, then there exists $t_0 = t_0(m, M) > 0$ such that (2.4) has a unique solution $(\phi, \psi) \in X_{\frac{1}{2}m,2M}(0, t_0)$.*

Proposition 2. 2 (A priori estimates) *Let (ϕ, ψ) be a solution to (2.4) in $X_{\frac{1}{2}m,\varepsilon}(0,T)$. Then, for a suitably small $\varepsilon > 0$, there exists a constant $C_0 > 0$ such that*

$$\|(\phi,\psi)(t)\|_1^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\sqrt{V_\xi}\phi(\tau)\|^2 + \|\phi_\xi(\tau)\|^2 + \|\psi_\xi(\tau)\|_1^2)d\tau \le C_0\|\phi_0, \psi_0\|_1^2.$$

Remark 2.1 If ε is suitably small, then $\inf_{\mathbf{R}_+\times[0,T]}(V+\phi)(\xi,t) \ge m/4$ is automatically satisfies by the Sobolev inequality. Hence we denote $X_{m,\varepsilon}(0,T)$ simply by $X_{\frac{1}{2}\varepsilon}(0,T)$.

The stability theorem is derived from these two Propositions in a standard way.

Theorem 2. 1 (Stability of BL-solution) *If $\|v_0 - V, u_0 - U\|_1$ is suitably small together with the compatibility condition $(v_0 - V, u_0 - U)(0) = (0,0)$, then there exists a unique solution (v,u) to (1.2), which satisfy $(v-V, u-U) \in C([0,\infty); H_0^1)$ and*

$$\sup_{\xi\ge 0}|(\phi,\psi)(\xi,t)| = \sup_{x \ge s_- t}|(v,u)(x,t) - (V,U)(x - s_- t)| \to 0 \quad \textit{as} \quad t \to \infty.$$

We first sketch the proof of the local existence theorem, Propositions 2.1.

By $(2.4)_1$, ϕ has the explicit form

$$\phi(\xi,t) = \begin{cases} \displaystyle\int_{t+\frac{\xi}{s_-}}^t \psi_\xi(\xi + s_-(t-\tau),\tau)d\tau, & 0 \le \xi \le -s_- t \\ \\ \displaystyle\phi_0(\xi + s_- t) + \int_0^t \psi_\xi(\xi + s_-(t-\tau),\tau)d\tau, & \xi \ge -s_- t. \end{cases} \tag{2.5}$$

Eq.$(2.4)_2$ is written as an initial-boundary value problem for the linear parabolic equatioin of ψ:

$$\begin{cases} \psi_t - \mu(\dfrac{\psi_\xi}{V+\phi})_\xi = g := g(\psi_\xi, \phi, \phi_\xi) \\ \psi(0,t) = 0 \\ \psi(\xi,0) = \psi_0(\xi), \end{cases} \tag{2.6}$$

where

$$g(\psi_\xi,\phi,\phi_\xi) = s_-\psi_\xi - (p(V+\phi) - p(V))_\xi + \mu(\frac{U_\xi}{V+\phi} - \frac{U_\xi}{V})_\xi.$$

To use the iteration method, we approximate $(\phi_0, \psi_0) \in H_0^1$ by $(\phi_{0k}, \psi_{0k}) \in H^2 \cap H_0^1$ such that

$$(\phi_{0k}, \psi_{0k}) \to (\phi_0, \psi_0) \ \text{ strongly in } \ H^1$$

as $k \to \infty$. We may assume

$$\|\phi_{0k}, \psi_{0k}\|_1 \le \frac{3}{2}M, \quad \inf_{\mathbf{R}_+}(V + \phi_{0k}) \ge \frac{2}{3}m$$

for any $k \ge 1$. Define the sequence $\{(\phi^{(n)}, \psi^{(n)})\} := \{(\phi_k^{(n)}, \psi_k^{(n)})\}$ for each k so that

$$(\phi^{(0)}, \psi^{(0)})(\xi, t) = (\phi_{0k}, \psi_{0k})(\xi),$$

and, for a given $(\phi^{(n-1)}, \psi^{(n-1)})(\xi, t)$, $\psi^{(n)}$ is a solution to

$$\begin{cases} \psi_t^{(n)} - \mu\left(\dfrac{\psi_\xi^{(n)}}{V + \phi^{(n-1)}}\right)_\xi = g^{(n-1)} := g(\psi(n-1)_\xi, \phi^{(n-1)}, \phi_\xi^{(n-1)}) \\ \psi^{(n)}(0,t) = 0 \\ \psi^{(n)}(\xi, 0) = \psi_{0k}(\xi), \end{cases} \tag{2.6$'$}$$

and

$$\phi^{(n)}(\xi, t) = \begin{cases} \displaystyle\int_{t+\frac{\xi}{s_-}}^t \psi_\xi^{(n)}(\xi + s_-(t-\tau), \tau)d\tau, & 0 \le \xi \le -s_- t \\ \\ \displaystyle\phi_{0k}(\xi + s_- t) + \int_0^t \psi_\xi^{(n)}(\xi + s_-(t-\tau), \tau)d\tau, & \xi \ge -s_- t. \end{cases} \tag{2.5$'$}$$

From the linear theory, if $g \in C^0([0,T]; L^2)$, $\psi_0 \in H^2 \cap H_0^1$, then there exists a unique solution ψ to (2.6) satisfying

$$\psi \in C([0,T]; H^2 \cap H_0^1) \cap C^1([0,T]; L^2) \cap L^2(0,T; H^3).$$

Using this, if $(\phi^{(n-1)}, \psi^{(n-1)}) \in X_{\frac{1}{2}m, 2M}$, then we have

$$\begin{aligned} \|(\phi^{(n)}, \psi^{(n)})(t)\|^2 &\le \left(\left(\frac{3}{2}M\right)^2 + C(m,M)t_0\right)\exp\left(C(m,M)t_0\right) \\ &\le (2M)^2 \quad \text{if} \quad 0 < t_0 := t_0(m,M) \ll 1 \end{aligned} \tag{2.7}$$

and also

$$\int_0^{t_0} \|\psi_\xi^{(n)}(\tau)\|_1^2 d\tau \le C(m,M)M^2.$$

Hence, direct estimates on (2.5)$'$ give

$$\left\|\int_{t+\frac{\xi}{s_-}}^t \psi_\xi^{(n)}(\xi + s_-(t-\tau), \tau)d\tau\right\|_1 \le C\sqrt{t_0}M$$

and

$$\left\|\int_0^t \psi_\xi^{(n)}(\xi + s_-(t-\tau), \tau)d\tau\right\|_1 \le C\sqrt{t_0}M.$$

Hence, for a suitable small t_0 we have

$$\sup_{0\le t\le t_0}\|\phi^{(n)}(t)\|_1 \le 2M \quad \text{and} \quad \inf_{\mathbf{R}_+\times[0,t_0]}(V+\phi)(\xi,t) \ge \frac{1}{2}m. \tag{2.8}$$

By (2.7)-(2.8), $(\phi^{(n)},\psi^{(n)}) \in X_{\frac{1}{2}m,2M}(0,t_0)$. By a standard way, $(\phi^{(n)},\psi^{(n)})$ can be shown to be the Cauchy sequence in $C([0,t_0];H^1)$. Thus we have a solution $(\phi_k,\psi_k) \in X_{\frac{1}{2}m,2M}(0,t_0)$ to (2.5) and (2.6) by $\lim_{n\to\infty}(\phi^{(n)},\psi^{(n)}) = \lim_{n\to\infty}(\phi_k^{(n)},\psi_k^{(n)})$. Here, we note that

$$\begin{aligned}&\phi_k \in C([0,t_0];H^2\cap H_0^1) \quad \text{and}\\ &\psi_k \in C([0,t_0];H^2\cap H_0^1)\cap C^1([0,t_0];L^2)\cap L^2(0,t_0;H^3),\end{aligned}$$

since $g((\psi_k)_\xi,\phi_k,(\phi_k)_\xi) \in C([0,t_0];L^2)$ and $(\phi_{0k},\psi_{0k}) \in H^2\cap H_0^1$. Again, showing that (ϕ_k,ψ_k) is a Cauchy sequence in $C([0,t_0];H^1)$ (taking t_0 smaller than the previous one if necessary), we obtain the desired unique-local solution $(\phi,\psi) \in X_{\frac{1}{2}m,2M}(0,t_0)$. .

Next, we show the a priori estimate.

Let (ϕ,ψ) be a solution in $X_{\frac{1}{2}m,\varepsilon}(0,T) = X_\varepsilon(0,T)$. First, Multiply $(2.4)_1$ and $(2.4)_2$ by ψ and $-(p(V+\phi)-p(V))$, respectively, and add these two equations to have a divergence form

$$\begin{aligned}&\{\frac{1}{2}\psi^2+\Phi(v,V)\}_t\\ &+\{s_-\Phi(v,V)-\frac{s_-}{2}\psi^2+(p(v)-p(V))\phi-\mu(\frac{\psi_\xi^2}{v}-\frac{U_\xi}{V})\psi\}_\xi\\ &+\{\mu\frac{\psi_\xi^2}{v}-\mu s_-\frac{V_\xi\phi\psi_\xi}{vV}-s_-V_\xi(p(V+\phi)-p(V)-p'(V)\phi)\}=0,\end{aligned} \tag{2.9}$$

where

$$\Phi(v,V) = p(V)\phi - \int_V^{V+\phi} p(\eta)d\eta. \tag{2.10}$$

Here and after we will often use the notation $(v,u) = (V+\phi,U+\phi)$, though the unknown functions are ϕ and ψ. Since $p''(V)>0$, put

$$p(V+\phi)-p(V)-p'(V)\phi = f(v,V)\phi^2, \tag{2.11}$$

then $f(v,V)\ge 0$. Hence (2.2) is effective, and the last three terms in (2.9) are regarded as the quadratic equation:

$$\begin{aligned}Q &:= \mu\frac{\psi_\xi}{v}-\mu s_-\frac{V_\xi\phi\psi_\xi}{vV}-s_-V_\xi f(v,V)\phi^2\\ &= (\sqrt{\mu}\frac{\psi_\xi}{\sqrt{v}})^2-\frac{\sqrt{-\mu s_-V_\xi}}{V\sqrt{vf(v,V)}}\cdot\sqrt{\mu}\frac{\psi_\xi}{\sqrt{v}}\cdot\sqrt{-s_-V_\xi f(v,V)}\phi+(\sqrt{-s_-V_\xi f(v,V)}\phi)^2.\end{aligned}$$

The discriminant of Q is

$$D = \frac{-\mu s_- V_\xi}{V^2 v f(v, V)} - 4 = \frac{-h(V)}{V v f(v, V)} - 4. \tag{2.12}$$

Since $v_+ > v_-$,

$$-h(V) = s_-^2 (V - v_+) + p(V) - p(v_+) < p(V) = V^{-\gamma}. \tag{2.13}$$

Moreover, by putting $X = V/v$,

$$\begin{aligned} V v f(v, V) &= \frac{V v (v^{-\gamma} - V^{-\gamma} + \gamma V^{-\gamma-1}(v - V))}{(v - V)^2} \\ &= V^{-\gamma} \cdot \frac{X^{\gamma+1} - (\gamma+1)X + \gamma}{(X-1)^2} \geq \gamma V^{-\gamma}, \end{aligned} \tag{2.14}$$

because $X^{\gamma+1} - (\gamma+1)X + \gamma \geq \gamma(X-1)^2$ for $X \geq 0$. By (2.12)-(2.14),

$$D \leq \frac{V^{-\gamma}}{\gamma V^{-\gamma}} - 4 = \frac{1}{\gamma} - 4 \leq 3. \tag{2.15}$$

Thus, integrating (2.9) over $(0, \infty) \times (0, t)$, we have the following basic estimate.

Lemma 2. 1 (Basic estimate) *For the solution $(\phi, \psi) \in X_\varepsilon(0, T)$, it holds that*

$$\begin{aligned} &\frac{1}{2}\|\psi(t)\|^2 + \int_0^\infty \Phi(v, V)(\xi, t) d\xi \\ &+C^{-1} \int_0^t \int_0^\infty \{\frac{\psi_\xi^2}{v} + |\frac{V_\xi \phi \psi_\xi}{vV}| + (p(V+\phi) - p(V) - p'(V)\phi)V_\xi\} d\xi d\tau \\ &\leq \frac{1}{2}\|\psi_0\|^2 + \int_0^\infty \Phi(v_0, V)(\xi) d\xi \leq C\|\phi_0, \psi_0\|^2. \end{aligned}$$

Next, following [7], change ϕ to $\tilde{v} := v/V$. Since

$$p(V+\phi) - p(V) - p'(V)\phi = V^{-\gamma}(\tilde{v}^{-\gamma} - 1 + \gamma(\tilde{v} - 1))$$

and

$$\Phi(v, V) = V^{-\gamma+1}\tilde{\Phi}(\tilde{v}),$$

where

$$\tilde{\Phi}(\tilde{v}) = \begin{cases} \tilde{v} - 1 - \ln \tilde{v} \quad (\gamma = 1) \\ \tilde{v} - 1 + \dfrac{1}{\gamma - 1}(\tilde{v}^{-\gamma+1} - 1) \quad (\gamma > 1), \end{cases}$$

Lemma 2.1 is rewritten as follows.

Lemma 2. 2 *It follows that*

$$\frac{1}{2}\|\psi(t)\|^2 + \int_0^\infty V^{-\gamma+1}\tilde{\Phi}(\tilde{v}(\xi,t))d\xi$$
$$+C^{-1}\int_0^t\int_0^\infty \{\frac{\psi_\xi^2}{v} + |\frac{V_\xi\phi\psi_\xi}{vV}| + \frac{V_\xi}{V^\gamma}(\tilde{v}^{-\gamma} - 1 + \gamma(\tilde{v}-1))\}d\xi d\tau$$
$$\leq C\|\phi_0,\psi_0\|^2.$$

Eq. $(2.4)_2$ is also written as

$$(\mu\frac{\tilde{v}_\xi}{\tilde{v}} - \psi)_t - s_-(\mu\frac{\tilde{v}_\xi}{\tilde{v}} - \psi)_\xi + \frac{\gamma\tilde{v}_\xi}{V^\gamma\tilde{v}^{\gamma+1}} + \frac{\gamma V_\xi}{V^{\gamma+1}}(\tilde{v}^{-\gamma} - 1) = 0. \tag{2.16}$$

Multiplying (2.16) by $\tilde{v}_\xi/\tilde{v}$, we have a divergence form

$$\begin{aligned}&\{\frac{\mu}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2 - \psi(\frac{\tilde{v}_\xi}{\tilde{v}})\}_t \\ &+\{\psi\frac{v_t}{v} - \frac{\gamma h(V)}{s_-\mu V^\gamma}(\frac{\tilde{v}^{-\gamma}-1}{\gamma} + \ln\tilde{v}) - \frac{\mu s_-}{2}(\frac{\tilde{v}_\xi}{\tilde{v}})^2\}_\xi + \frac{\gamma\tilde{v}_\xi^2}{V^\gamma\tilde{v}^{\gamma+2}} \\ &= \frac{\psi_\xi^2}{v} + \frac{s_-\phi\psi_\xi V_\xi}{vV} - \frac{\gamma V_\xi}{s_-\mu}\frac{h'(V)V^\gamma - h(V)\gamma V^{\gamma-1}}{V^{2\gamma}}\{\frac{\tilde{v}^{-\gamma}-1}{\gamma} + \ln\tilde{v}\}.\end{aligned} \tag{2.17}$$

By (2.2)

$$|\text{the final term of (2.14)}| \leq C\frac{V_\xi}{V^\gamma}(\tilde{v}^{-\gamma} - 1 + \gamma(\tilde{v}-1)).$$

Hence, the right hand side of (2.14) is controllable by Lemma 2.2. Thus, integrating (2.14) over $(0,\infty)\times(0,t)$ yields the following lemma.

Lemma 2. 3 *It holds that*

$$\begin{aligned}&\|\frac{\tilde{v}_\xi}{\tilde{v}}(t)\|^2 + \int_0^t\int_0^\infty \frac{\tilde{v}_\xi^2}{\tilde{v}^{\gamma+2}}d\xi d\tau \\ &\leq C(\|\phi_0\|_1^2 + \|\psi_0\|^2) + C\int_0^t(\frac{\tilde{v}_\xi}{\tilde{v}})^2(0,\tau)d\tau.\end{aligned} \tag{2.18}$$

We note that the estimates until now have been obtained without smallness condition. Hence we wish to control the final term of (2.18), $C\int_0^t(\frac{\tilde{v}_\xi}{\tilde{v}})^2(0,\tau)d\tau$, without smallness condition, in a similar fashion to [6,7]. But, we could not do it. However, we can control it provided that the initial data is small. Since

$$(\frac{\tilde{v}_\xi}{\tilde{v}})^2(0,\tau) = \frac{1}{v_-^2}\phi_\xi^2(0,\tau) = \frac{1}{u_-^2}\psi_\xi^2(0,\tau) \leq C\|\psi_\xi(\tau)\|\|\psi_{\xi\xi}(\tau)\|, \tag{2.19}$$

it is necessary to estimate $\int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau$, which is controllable for small the initial data.

We now assume that

$$N(T) := \sup_{0\le t\le T} \|(\phi,\psi)(t)\|_1 \le \varepsilon \ll 1.$$

Multiplying $(2.4)_2$ by $-\psi_{\xi\xi}$, we have

$$(\frac{1}{2}\psi_\xi^2)_t + (-\psi_t\psi_\xi + \frac{s_-}{2}\psi_\xi^2)_\xi + \mu\frac{\psi_{\xi\xi}^2}{v}$$

$$= \{-\mu\frac{\psi_\xi(V_\xi+\phi_\xi)}{(V+\phi)^2} + \mu(\frac{U_\xi}{V+\phi} - \frac{U_\xi}{V})_\xi - (p(V+\phi)-p(V))_\xi\}(-\psi_{\xi\xi})$$

and, after integrating the resultant equation over $(0,\infty)\times(0,t)$,

$$\begin{aligned} &\|\psi_\xi(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\psi_{\xi\xi}(\tau)\|^2)d\tau \\ &\le C\|\psi_{0\xi}\|^2 + C\int_0^t\int_0^\infty (\phi_\xi^2 + V_\xi\phi^2 + \psi_\xi^2)d\xi d\tau. \end{aligned} \tag{2.20}$$

Here, we have estimated the amount $(\phi_\xi\psi_\xi)^2$ as

$$\begin{aligned} \int_0^t\int_0^\infty (\phi_\xi\psi_\xi)^2 d\xi d\tau &\le \int_0^t \|\psi_\xi\|\|\psi_{\xi\xi}\|\|\phi_\xi\|^2 d\tau \\ &\le \nu\int_0^t \|\psi_{\xi\xi}\|^2 d\tau + C_\nu N(T)^2\int_0^t \|\phi_\xi(\tau)\|^2 d\tau \end{aligned}$$

for a small constant $\nu > 0$. By Lemma 2.1, (2.20) is reduced to

$$\begin{aligned} &\|\psi_\xi(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\psi_{\xi\xi}(\tau)\|^2)d\tau \\ &\le C(\|\phi_0\|^2 + \|\psi_0\|_1^2) + C\int_0^t \|\phi_\xi(\tau)\|^2 d\tau. \end{aligned} \tag{2.21}$$

For a small constant $\lambda > 0$, $(2.18) + (2.21)\cdot\lambda$ together with (2.19) yields

$$\begin{aligned} &\|\frac{\tilde v_\xi}{\tilde v}(t)\|^2 + \lambda\|\psi_\xi(t)\|^2 + \int_0^t (\|\tilde v_\xi(\tau)\|^2 + \lambda\psi_\xi(0,\tau)^2 + \lambda\|\psi_{\xi\xi}(\tau)\|^2)d\tau \\ &\le C\|\phi_0,\psi_0\|_1^2 + C\int_0^t ((\frac{\tilde v_\xi}{\tilde v})^2(0,\tau) + \lambda\|\phi_\xi(\tau)\|^2)d\tau \\ &\le C\|\phi_0,\psi_0\|^2 + \int_0^t (\nu\|\psi_{\xi\xi}(\tau)\|^2 + C_\nu\|\psi_\xi(\tau)\|^2 + C\lambda\|\phi_\xi(\tau)\|^2)d\tau. \end{aligned}$$

since

$$\|\tilde{v}_\xi(t)\|^2 = \int_0^\infty (\frac{\phi_\xi}{V} - \frac{V_\xi \phi}{V^2})^2 d\xi$$
$$\geq \int_0^\infty (\frac{\phi_\xi^2}{2V^2} - \frac{(V_\xi\phi)^2}{V^4}) d\xi \geq c_0\|\phi_\xi(t)\|^2 - C\|\phi(t)\|^2$$

and

$$\int_0^t \|\tilde{v}_\xi(\tau)\|^2 d\tau \geq c_0 \int_0^t \|\phi_\xi(\tau)\|^2 d\tau - C\int_0^t \int_0^\infty V_\xi \phi^2 d\xi d\tau,$$

we fix λ such that $C\lambda \leq c_0/2$ and ν such that $\nu \leq \lambda/2$. Then, the following lemma holds.

Lemma 2. 4 *If $N(T) = \sup_{0\leq t\leq T} \|(\phi,\psi)(t)\|_1$ is suitably small, then*

$$\|(\phi_\xi,\psi_\xi)(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|(\phi_\xi,\psi_{\xi\xi})(\tau)\|^2) d\tau \leq C\|\phi_0,\psi_0\|_1^2.$$

Combining Lemmas 2.1-2.4 completes the proof of Proposition 2.2.

We briefly mention the case $(v_+,u_+) \in BL_+R_2(v_-,u_-)$, $R_1(v_*,u_*)$ or $R_1R_2(v_*,u_*)$. For example, for $(v_+,u_+) \in BL_+R_2(v_-,u_-)$, there is a unique $(\bar{v},\bar{u}) \in BL_+$ such that $(v_+,u_+) \in R_2(\bar{v},\bar{u})$, and there exist a BL-solution $(V_0,U_0)(x-s_-t)$ connecting (v_-,u_-) with $(\bar{v},\bar{u})$ and a 2-rarefaction wave $(v_2^R,u_2^R)(x/t)$ connecting $(\bar{v},\bar{u})$ with (v_+,u_+). The behavior of solution (v,u) to (1.2) is expected to be

$$(v,u)(x,t) \sim (V_0(x-s_-t) + v_2^R(x/t) - \bar{v}, U_0(x-s_-t) + u_2^R(x/t) - \bar{u}) \qquad (2.22)$$

as $t\to\infty$. To show (2.22) we first construct a smooth approximate rarefaction wave $(V_2,U_2)(x,t)$ satisfying

$$\begin{cases} V_{2t} - U_{2x} = 0 \\ U_{2t} + p(V_2)_x = 0 \end{cases}$$

with $U_{2x} = V_{2t} > 0$ and $\lim_{t\to\infty}\sup|(V_2,U_2)(x,t) - (v_2^R,u_2^R)(x/t)| = 0$(See [5,6,7]). Then, the perturbation $(\phi,\psi)(\xi,t) = (v-(V_0+V_1-\bar{v}), u-(U_0+U_1-\bar{u})) =: (v-V, u-U)$ satisfies

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0 \\ \psi_t - s_-\psi_\xi + (p(V+\phi)-p(V))_\xi = \mu(\dfrac{U_{0\xi}+\psi_\xi}{V_0+\phi} - \dfrac{U_{0\xi}}{V_0})_\xi \\ \qquad\qquad -(p(V)-p(V_0)-p(V_1)+p(\bar{v}))_\xi \\ (\phi,\psi)|_{\xi=0} = (V_1-\bar{v}, U_1-\bar{u})|_{\xi=0} =: (b_V,b_U)(t) \\ (\phi,\psi)|_{t=0} = (v_0 - V|_{t=0}, u_0 - U|_{t=0}) =: (\phi_0,\psi_0)(\xi). \end{cases} \qquad (2.23)$$

Since the last term of $(2.23)_2$ and the boundary value $(b_V,b_U)(t)$ are small as $t\to\infty$ if $|(v_+-\bar{v},u_+-\bar{u})| \ll 1$, we can treat (2.23) as essentially same as (2.4). In particular, since $U_\xi = U_{0\xi} + U_{1\xi} > 0$, the basic estimate similar to Lemma 2.1 is obtained, and hence the stability theorem for $(V,U) = (V_0+V_1-\bar{v}, U_0+U_1-\bar{u})$ holds provided that $|(v_+-\bar{v},u_+-\bar{u})| \ll 1$. We omit the details.

2.2 The case $(v_+, u_+) \in BL_-(v_-, u_-)$

In this subsection we assume that

$$(v_-, u_-) \in \Omega_{sub} \quad \text{and} \quad (v_+, u_+) \in BL_-(v_-, u_-).$$

The situations are all same as the case $(v_+, u_+) \in BL_+(v_-, u_-)$ except for $\mu V_\xi = \frac{V}{s_-}h(V) < 0$. Hence, the perturbation (ϕ, ψ) satisfies (2.4), but the proof of Lemma 2.1 is not available. In this case, we have, from (2.9),

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty (\frac{1}{2}\psi^2 + \Phi(v, V))d\xi + \int_0^\infty \mu\frac{\psi_\xi^2}{v}d\xi \\
&\leq C\int_0^\infty |V_\xi|\phi^2 d\xi + \nu\int_0^\infty \psi_\xi^2 d\xi + C_\nu\int_0^\infty |V_\xi|^2\phi^2 d\xi
\end{aligned}$$

for a small constant $\nu > 0$. If $\delta = |(v_+ - v_-, u_+ - u_-)| \ll 1$ and $\|(\phi, \psi)(t)\|_1 \leq \varepsilon \ll 1$, then

$$\|(\phi, \psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau \leq C\|\phi_0, \psi_0\|^2 + C\int_0^t\int_0^\infty |V_\xi|\phi(\xi, \tau)^2 d\xi d\tau. \tag{2.24}$$

The estimate of the last term is a key point. Applying the idea by Kawashima and Nikkuni [1], we have

$$\phi(\xi, t) = \phi(0, t) + \int_0^\xi \phi_\xi(\eta, t)d\eta \leq \xi^{1/2}\|\phi_\xi(t)\|,$$

and

$$C\int_0^t\int_0^\infty |V_\xi|\phi(\xi, \tau)^2 d\xi d\tau \leq C\int_0^t \|\phi_\xi(\tau)\|^2 \int_0^\infty \xi(-V_\xi(\xi))d\xi d\tau \leq C\delta\int_0^t \|\phi_\xi(\tau)\|^2 d\tau.$$

Thus we have

$$\|(\phi, \psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau \leq C(\|\phi_0, \psi_0\|^2 + \delta\int_0^t \|\phi_\xi(\tau)\|^2 d\tau). \tag{2.25}$$

Moreover, we seek for the estimates of higher order derivatives.
Similar to the proof of Lemma 2.3, we can show

$$\|\phi_\xi(t)\|^2 + \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \leq C(\|\phi_0, \psi_0\|_1^2 + \int_0^t \phi_\xi(0, t)^2 d\tau + \int_0^t\int_0^\infty |V_\xi|\phi^2 d\xi d\tau). \tag{2.26}$$

Since

$$C\int_0^t \phi_\xi(0, \tau)^2 d\tau = \frac{C}{s_-^2}\int_0^t \psi_\xi(0, \tau)^2 \leq \nu\int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau + C_\nu\int_0^t \|\psi_\xi(\tau)\|^2 d\tau,$$

(2.26) yields

$$\begin{aligned}
&\|\phi_\xi(t)\|^2 + \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \\
&\leq (\|\phi_0, \psi_0\|_1^2) + \int_0^t \{\nu\|\psi_{\xi\xi}(\tau)\|^2 + C\delta\|\phi_\xi(\tau)\|^2 + C_\nu\|\psi_\xi(\tau)\|^2\}d\tau.
\end{aligned} \tag{2.27}$$

Further we have, by multiplying $(2.4)_1$ by $-\psi_{\xi\xi}$,

$$\begin{aligned} &\|\psi_\xi(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\psi_{\xi\xi}(\tau)\|^2) d\tau \\ &\leq C\|\phi_0, \psi_0\|_1^2 + C\int_0^t (\|\phi_\xi(\tau)\|^2 + \|\psi_\xi(\tau)\|^2) d\tau. \end{aligned} \tag{2.28}$$

Add (2.27) to $\lambda\cdot$(2.28) for a fixed number $\lambda > 0$ such as $1 - C\lambda \geq 1/2$ and $\nu = \lambda/2$, then

$$\begin{aligned} &\|(\phi_\xi, \psi_\xi)(t)\|^2 + \int_0^t (\psi_\xi(0,\tau)^2 + \|\phi_\xi(\tau)\|^2 + \|\psi_{\xi\xi}(\tau)\|^2) d\tau \\ &\leq C(\|\phi_0, \psi_0\|_1^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau). \end{aligned} \tag{2.29}$$

Again, add $(2.29)\cdot\lambda(\lambda > 0)$ to (2.25), then

$$\begin{aligned} &\|(\phi, \psi)(t)\|^2 + \lambda\|(\phi_\xi, \psi_\xi)(t)\|^2 \\ &+ \int_0^t \{(1 - C\lambda)\|\psi_\xi(\tau)\|^2 + \lambda(\psi_\xi(0,\tau)^2 + \|\phi_\xi(\tau)\|^2 + \|\psi_{\xi\xi}(\tau)\|^2\} d\tau \\ &\leq C\|\phi_0, \psi_0\|_1^2 + C\delta \int_0^t \|\phi_\xi(\tau)\|^2 d\tau. \end{aligned}$$

Taking λ as $1 - C\lambda \geq 1/2$ and restrict δ as $\lambda - C\delta \geq \lambda/2$, we obtain the desired a priori estimate.

Thus we reach the following theorem.

Theorem 2. 2 *If $|v_+ - v_-, u_+ - u_-| + \|v_0 - V, u_0 - U\|_1$ is suitably small with the compatibility condition $(v_0 - V, u_0 - U)(0) = (0,0)$, then there exists a unique solution (v, u) to (1.2), which satisfies $(v - V, u - U) \in C([0,\infty); H_0^1)$ and*

$$\sup_{x \geq s_- t} |(v,u)(x,t) - (V,U)(x - s_- t)| \to 0 \quad as \quad t \to \infty.$$

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