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On large time behaviour of small solutions to the Vlasov-Poisson-Fokker-Planck equation

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1. Main Result

We consider the Cauchy problem for the Vlasov-Poisson-Fokker-Planck equation (without friction term)

$$(1)\quad \begin{aligned} &\partial_t f + u\cdot\nabla_x f - E(f)\cdot\nabla_u f - \Delta_u f = 0, \quad (x,u)\in \boldsymbol{R}^N\times\boldsymbol{R}^N,\ t>0 \\ &f|_{t=0} = f_0. \end{aligned}$$

Here $N\geq 2$, $f=f(x,u,t)$ is the unknown function, which describes the number density of particles at position $x\in\boldsymbol{R}^N$ and time t with velocity $u\in\boldsymbol{R}^N$ in a physical system under consideration ; $\nabla_x=(\partial_{x_1},\cdots,\partial_{x_N})$, $\nabla_u=(\partial_{u_1},\cdots,\partial_{u_N})$; $\Delta_u=\partial_{u_1}^2+\cdots+\partial_{u_N}^2$ is the Laplacian with respect to the variable u ; and

$$E(f)=\frac{\omega}{|S^{N-1}|}\frac{x}{|x|^N}*_x\int_{\boldsymbol{R}^N} f(x,u,t)\,du, \quad \omega\ :\ a\ constant,$$

$|S^{N-1}|$ is the $(N-1)$ dimensional volume of the N-dimensional unit sphere, and $*_x$ denotes the convolution with respect to x.

In this article we present the results on the large time behaviour of small solutions of (1), which were obtained in [1], and give some remarks. (The detailed proofs of theorems are thus found in [1].)

Theorem 1 ([1]). *Let n be an integer satisfying $0\leq n\leq 3N-5$ and let r be an integer satisfying $r\geq n+3N+\frac{3}{2}$. Assume that for initial value f_0, the quantity*

$$I(f_0)=\|(1+|x|^2+|u|^2)^{r/2}f_0\|_{H^m(\boldsymbol{R}^N\times\boldsymbol{R}^N)}$$

is finite for some $m>[\frac{N}{2}-1]+N+1$. Here $H^m(\boldsymbol{R}^N\times\boldsymbol{R}^N)$ denotes the L^2-Sobolev space of order m and $[q]$ denotes the largest integer less than or equal to q. Then for any $\varepsilon>0$, if $I(f_0)$ is sufficiently small, there exists a unique global solution $f(t)$ of (1) in $C([0,\infty);H^m)$ and $f(t)$ satisfies

$$\lim_{t\to\infty} t^{\frac{n+1}{2}-\varepsilon}\|t^{2N}f(t^{3/2}x,t^{1/2}u,t)-\sum_{k=0}^{n}t^{-\frac{k}{2}}\sum_{3|\alpha|+|\beta|=k}B_{\alpha,\beta}g_{\alpha,\beta}(x,u)\|_{L^\infty_{x,u}}=0.$$

Here $g_{\alpha,\beta}(x,u) = \partial_x^\alpha(\partial_x + \partial_u)^\beta g(x,u)$ *and* $g(x,u) = e^{-3|x-\frac{u}{2}|^2-\frac{1}{4}|u|^2}$; $B_{\alpha,\beta}$ *are constants determined by* f_0 *and the nonlinearity. In particular,* $B_{0,0} = \int f_0(x,u)\,dxdu$.

Remark 2. In Theorem 1 the range of n is restricted as $0 \leq n \leq 3N-5$. One can, however, obtain the asymptotics of f with error estimate of order $O(t^{-\frac{n+1}{2}})$ for any nonnegative $n \in \mathbf{Z}$, if the weight is taken large enough in such a way that $r \geq n + 3N + \frac{3}{2}$.

In fact, for n in the range in Theorem 1, the asymptotics is similar to that for the solution of the linear problem (i.e., the problem without the term $E(f) \cdot \nabla_u f$). The only difference from the linear problem appears in the constants $B_{\alpha,\beta}$'s ; in the linear case $B_{\alpha,\beta}$'s are given by some moments of f_0 only, while in the nonlinear case $B_{\alpha,\beta}$'s also involve some additional terms depending on f_0 and the nonlinearity.

If n is beyond the range in Theorem 1, i.e., if $n \geq 3N-4$, then the effect of the nonlinearity becomes much stronger and the asymptotics is given by not only $t^{-\frac{k}{2}}$ and $g_{\alpha,\beta}$'s but also some terms with $\log t$ and other functions besides $g_{\alpha,\beta}$'s. For example, if $n = 3N-4$, then we have

$$\begin{aligned} t^{2N} f(t^{3/2}x, t^{1/2}u, t) \sim & \sum_{k=0}^{n-1} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha,\beta}\, g_{\alpha,\beta}(x,u) \\ & + t^{-\frac{n}{2}} \sum_{3|\alpha|+|\beta|=n} (B_{\alpha,\beta} + \widetilde{B}_{\alpha,\beta}\log t)\, g_{\alpha,\beta}(x,u) + h(g_{\alpha,\beta}, t) \\ & + O(t^{-\frac{n+1}{2}+\varepsilon}), \end{aligned}$$

where $B_{\alpha,\beta}$ and $\widetilde{B}_{\alpha,\beta}$ are some constants and $h(g_{\alpha,\beta}, t) = O(t^{-\frac{n}{2}})$. See the argument in Section 3 below as for the dependence of h on $g_{\alpha,\beta}$'s. In case $n \geq 3N-3$, the form of the asymptotics becomes more complicated.

Remark 3. We mention some related works on large time behaviour of solutions of (1). Carrillo, Soler and Vázquez [4] obtained the asymptotics to the first order for weak solutions of (1) belonging to certain classes. The proof is based on the re-scaling argument. Carrillo and Soler [3] then proved the existence of weak solutions belonging to the classes given in [4] for initial date small in some sense. Carpio [2] obtained the asymptotics to the second order for small solutions by a detailed analysis of the linear problem and using the re-scaling argument. We also mention the work by Ono and Strauss [8], in which sharp decay rates of small solutions were proved and also it was proved that small solutions approach to those of the corresponding linear problem with error estimate of $O(t^{-\frac{1}{2}})$.

2. Finite Dimensional Invariant Manifolds for the VPFP

We derive the long-time asymptotics given in Theorem 1 and Remark 2 by constructing finite dimensional invariant manifolds. To construct invariant manifolds, we change the variables into the "similarity" variables:

$$\tilde{t} = \log(t+1), \quad \tilde{x} = x/(t+1)^{3/2}, \quad \tilde{u} = u/(t+1)^{1/2},$$
$$f(x,u,t) = (t+1)^{-\gamma}\tilde{f}(x/(t+1)^{3/2}, u/(t+1)^{1/2}, \log(t+1)),$$

where $\gamma = \frac{N}{2} + 2$. Then the equation for $\tilde{f}$ is written, after omitting tildes, as

$$(2) \qquad \begin{aligned} &\partial_t f - (\tfrac{3}{2}x - u)\cdot\nabla_x f - \tfrac{1}{2}u\cdot\nabla_u f - \gamma f - E(f)\cdot\nabla_u f - \Delta_u f = 0, \\ &f|_{t=0} = f_0. \end{aligned}$$

We write the problem (2) in the form

$$\partial_t f = \mathcal{L}f + \mathcal{N}(f), \quad f(0) = f_0,$$

where $\mathcal{L}f = \Delta_u f + (\frac{3}{2}x - u)\cdot\nabla_x f + \frac{1}{2}u\cdot\nabla_u f + \gamma f$ and $\mathcal{N}(f) = E(f)\cdot\nabla_u f$.

We first consider the linear problem in the weighted space $X_r^{l,m}$ which is defined by

$$X_r^{l,m} = \{f(x,u) \in L^2(\boldsymbol{R}^N \times \boldsymbol{R}^N) : (1+|x|^2+|u|^2)^{r/2}\partial_x^\alpha\partial_u^\beta f \in L^2(\boldsymbol{R}^N \times \boldsymbol{R}^N),$$
$$0 \le |\alpha| \le l,\ 0 \le |\beta| \le m\},$$

where l, m and r are nonnegative integers.

We are given a nonnegative integer n and we fix this n hereafter. For this n we take the weight large enough in such a way that $r \ge n + 3N + \frac{3}{2}$. Then as for the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ in $X_r^{0,0}$, we have

$$\sigma(\mathcal{L}) \subset \{\sigma_k : k = 0, 1, \cdots, n\} \cup \{\mathrm{Re}\sigma \le \sigma_{n+1}\} \quad (\sigma_j = -(2N-\gamma) - \tfrac{j}{2}).$$

Here each of σ_k $(k = 0, 1, \cdots, n)$ is a semi-simple eigenvalue ; the associated eigenspace is spanned by functions $g_{\alpha,\beta}$'s with α and β satisfying $3|\alpha|+|\beta| = k$; and the eigenprojection P_k is given by

$$P_k f = \sum_{3|\alpha|+|\beta|=k} \langle f, g^*_{\alpha,\beta}\rangle g_{\alpha,\beta}.$$

Here $g^*_{\alpha,\beta}(x,u) = c_{\alpha,\beta}(\partial_x + 3\partial_u)^\alpha(\partial_x + 2\partial_u)^\beta g(x,u)$ denotes the adjoint eigenfunction ($c_{\alpha,\beta}$ is a constant) ; and the inner product $\langle\cdot,\cdot\rangle$ is defined by

$$\langle f,g\rangle = \int f(x,u)g(x,u)e^{\mu(x,u)}\,dxdu, \quad \mu(x,u) = 3|x - \tfrac{u}{2}|^2 + \tfrac{1}{4}|u|^2.$$

We denote by $\mathcal{P}_n = \sum_{k=0}^n P_k$ the projection onto the spectral subspace corressponding to discrete eigenvalues $\{\sigma_k\}_{k=0}^n$; and define $\mathcal{Q}_n$ by $\mathcal{Q}_n = I - \mathcal{P}_n$.

Then $X_r^{l,m}$ is decomposed into the direct sum :

$$X_r^{l,m} = Y_n \oplus Z, \quad Y_n \equiv \mathcal{P}_n X_r^{l,m}, \quad Z \equiv \mathcal{Q}_n X_r^{l,m},$$

and the solution $e^{t\mathcal{L}}f_0$ of the linear problem is decomposed as

$$e^{t\mathcal{L}}f_0 = y_n(t) + z(t), \quad y_n(t) \in Y_n, \quad z(t) \in Z,$$
$$y_n(t) = \sum_{k=0}^n e^{\sigma_k t}P_k f_0, \quad z(t) = \mathcal{Q}_n e^{t\mathcal{L}}f_0.$$

As for the part $z(t) = \mathcal{Q}_n e^{t\mathcal{L}}f_0$ on the subspace Z, the estimate

$$\|\mathcal{Q}_n e^{t\mathcal{L}}f_0\|_{X_r^{l,m}} \le C(1 + t^{-\frac{j}{2}})e^{\sigma_{n+1}t}\|f_0\|_{X_r^{l,m-j}}$$

holds for $l \ge 0$, $m \ge j$ and $j = 0,1$. Therefore, the large time behaviour of solutions of the linear problem is described, up to $O(e^{\sigma_{n+1}t})$, by the behaviour of solutions on the *finite dimensional invariant subspace* Y_n.

For the nonlinear problem we have the following theorem, from which the long-time asymptotics given in Theorem 1 and Remark 2 are obtained.

Theorem 4 ([1]). *Let $n \ge 0$ be an integer and let r be an integer satisfying $r \ge n + 3N + \frac{3}{2}$. Then for any fixed integers $m \ge 1$ and $l \ge [\frac{N}{2} - 1] + 1$, there exists a finite dimensional invariant manifold $\mathcal{M}$ for (2) in a neighborhood of the origin of $X_r^{m+l,m}$, i.e., there exist $\Phi \in C^1(Y_n; Z)$ and $R > 0$ such that $\Phi(0) = 0$, $D\Phi(0) = 0$ and*

$$\mathcal{M} = \{y_n + \Phi(y_n); y_n \in Y_n, \|y_n\| \le R\},$$

where $Y_n = \mathcal{P}_n X_r^{m+l,m}$ and $Z = \mathcal{Q}_n X_r^{m+l,m}$; and $\mathcal{M}$ is invariant under semiflows defined by (2). Furthermore, solutions near the origin approach to $\mathcal{M}$ at a rate $O(e^{(\sigma_{n+1}+\varepsilon)t})$ as $t \to \infty$. More precisely, if $\|f_0\|_{X_r^{m+l,m}}$ is *sufficiently small, then there uniquely exists a solution $\bar{f}(t)$ of (2) on $\mathcal{M}$ such that*

$$\|f(t) - \bar{f}(t)\|_{X_r^{m+l,m}} \le Ce^{(\sigma_{n+1}+\varepsilon)t}. \tag{3}$$

Remark 5. Wayne [9] constructed finite dimensional invariant manifolds in Sobolev spaces with polynomial weights for certain semilinear heat equations on whole spaces by using the similarity variables transformation. The method in [9] is then extended to various contexts as in [5, 6, 7, 10].

3. Outline of Proof

We here outline how to obtain the long-time asymptotics given in Theorem 1 and Remark 2. (See [1] for the proof of Theorem 4.)

Our starting point is the estimate (3) in Theorem 4. We can rewrite the estimate (3) in the form

$$\|y_n(t) - \bar{y}_n(t)\|_{X_r^{m+l,m}} \le Ce^{(\sigma_{n+1}+\varepsilon)t} \tag{4}$$

and

$$\|z(t) - \Phi(\bar{y}_n(t))\|_{X_r^{m+l,m}} \le Ce^{(\sigma_{n+1}+\varepsilon)t}, \tag{5}$$

where

$$f(t) = y_n(t) + z(t), \quad \bar{f}(t) = \bar{y}_n(t) + \Phi(\bar{y}_n(t)), \quad y_n(t), \bar{y}_n(t) \in Y_n, \quad z(t) \in Z.$$

Thus, to obtain the asymptotics of $f(t)$ up to $O(e^{(\sigma_{n+1}+\varepsilon)t})$, it suffices to investigate the behaviour of $\bar{y}_n(t)$, which is governed by a system of finite number of ordinary differential equations. Since $\bar{y}_n(t)$ can be written as

$$\bar{y}_n(t) = \sum_{3|\alpha|+|\beta|\le n} y_{\alpha,\beta}(t) g_{\alpha,\beta}, \quad y_{\alpha,\beta} \in \mathbf{R},$$

the problem is reduced to the analysis of the behaviour of $y_{\alpha,\beta}$'s.

We now derive a system of ordinary differential equations for $y_{\alpha,\beta}$'s. Since $\bar{f}(t) = \bar{y}_n(t) + \Phi(\bar{y}_n(t))$ is a solution of (2) on $\mathcal{M}$, it satisfies

$$\partial_t \bar{f} = \mathcal{L}\bar{f} + \mathcal{N}(\bar{f}).$$

Taking the inner product of this equation with $g^*_{\alpha,\beta}$, we have

$$\dot{y}_{\alpha,\beta} = \sigma_k y_{\alpha,\beta} + H_{\alpha,\beta}(\bar{y}_n), \quad 3|\alpha| + |\beta| = k, \ 0 \le k \le n,$$

where $\dot{y} = \frac{dy}{dt}$ and $H_{\alpha,\beta}(\bar{y}_n) = \langle \mathcal{N}(\bar{y}_n + \Phi(\bar{y}_n)), g^*_{\alpha,\beta}\rangle$.

For $\alpha = \beta = 0$, one can easily verify that $H_{0,0}(\bar{y}_n) = 0$. Hence,

$$\dot{y}_{0,0} = \sigma_0 y_{0,0}, \quad i.e., \quad y_{0,0}(t) = e^{\sigma_0 t} y_{0,0}(0).$$

Recall that $\sigma_0 = -(2N - \gamma) = -(\frac{3}{2}N - 2) < 0$. For $(\alpha, \beta) \neq (0,0)$, we have, by the variation of constants formula,

$$y_{\alpha,\beta}(t) = e^{\sigma_k t} y_{\alpha,\beta}(0) + e^{\sigma_k t} \int_0^t e^{-\sigma_k s} H_{\alpha,\beta}(\bar{y}_n(s))\, ds \tag{6}$$

with $k = 3|\alpha| + |\beta|$, $1 \leq k \leq n$. Since $\sigma_k = \sigma_0 - \frac{k}{2}$, one can expect that $y_{\alpha,\beta}(t)$ decays strictly faster than $y_{0,0}(t)$. Therefore, the slowest term in $H_{\alpha,\beta}(\bar{y}_n(s))$ behaves like $e^{2\sigma_0 s}$, since the lowest order terms of $H_{\alpha,\beta}(\bar{y}_n)$ are quadratic in $\{y_{\alpha,\beta}\}$. As a result, the integrand in (6) behaves like $e^{(2\sigma_0 - \sigma_k)s}$.

Now let $n \leq 3N - 5$. This is just equivalent to $|\sigma_n| < 2|\sigma_0|$ (and to $|\sigma_{n+1}| \leq 2|\sigma_0|$). It then follows that for $3|\alpha| + |\beta| = k$, $0 \leq k \leq n$,

$$y_{\alpha,\beta}(t) \sim const. e^{\sigma_k t} + O(e^{2\sigma_0 t}),$$

where $const.$ depends on $y_{\alpha,\beta}(0)$ and $H_{\alpha,\beta}$. We can also obtain

$$\|z(t)\|_{X_r^{m+l,m}} \leq C e^{(\sigma_{n+1}+\epsilon)t}.$$

Therefore,

$$\tilde{f}(\tilde{t}) \sim \sum_{k=0}^{n} e^{\sigma_k \tilde{t}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha,\beta} g_{\alpha,\beta} + O(e^{(\sigma_{n+1}+\epsilon)\tilde{t}}).$$

Here we write the solution of (2) and the time variable with tildes. Since the similarity variables $\tilde{f}$ and $\tilde{t}$ are connected with the original variables f and t by $\tilde{t} = \log t$ and $\tilde{f} = t^{\gamma} f$, we obtain the asymptotics given in Theorem 1 for $n \leq 3N - 5$.

We next consider higher order asymptotics. In higher order cases, the estimates (4), (5) and equations for $y_{\alpha,\beta}$'s, of course, take the same forms. Let $n \geq 3N - 4$. Then $|\sigma_n| \geq 2|\sigma_0|$ and $|\sigma_{n+1}| > 2|\sigma_0|$. Therefore, the integrand in (6) does not decay as $s \to \infty$ for some α and β, and the effect of the inhomogeneous term is no longer weak. Also, one must take the effect of $\Phi(\bar{y}_n(t))$ into account, and, thus, the form of the asymptotics becomes complicated.

For example, if $n = 3N - 4$, then we have $\sigma_n = 2\sigma_0$ and, therefore, the integrand in (6) with $3|\alpha| + |\beta| = n$ is of $O(1)$. It then follows that for $3|\alpha| + |\beta| = n$,

$$y_{\alpha,\beta}(t) \sim c_1 e^{\sigma_n t} + c_2 t e^{\sigma_n t} + O(e^{\sigma_{n+1} t}),$$

where c_1 and c_2 are some constants. One can also see that $\Phi_{\alpha,\beta}(\bar{y}_n(t)) = O(e^{\sigma_n t})$. Combining these with (4) and (5), we see that, in the original variables,

$$\begin{aligned} t^{2N} f \;\sim\; & \sum_{k=0}^{n-1} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha,\beta} g_{\alpha,\beta} \\ & + t^{-\frac{n}{2}} \sum_{3|\alpha|+|\beta|=n} (B_{\alpha,\beta} + \widetilde{B}_{\alpha,\beta} \log t)\, g_{\alpha,\beta} + h(\bar{y}_n(t)) + O(t^{-\frac{n+1}{2}+\epsilon}), \end{aligned}$$

where $B_{\alpha,\beta}$ and $\widetilde{B}_{\alpha,\beta}$ are some constants and $h(\bar{y}_n(t)) = O(t^{-\frac{n}{2}})$. This gives the asymptotics presented in Remark 2 for $n = 3N - 4$. For $n \geq 3N - 3$, it is possible to obtain the asymptotics in a similar manner as above, but the form of the asymptotics becomes more complicated.

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