On large time behaviour of small solutions to the Vlasov-Poisson-Fokker-Planck equation

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1. Main Result

We consider the Cauchy problem for the Vlasov-Poisson-Fokker-Planck equation (without friction term)

\[ \partial_t f + u \cdot \nabla_x f - E(f) \cdot \nabla_u f - \Delta_u f = 0, \quad (x, u) \in \mathbb{R}^N \times \mathbb{R}^N, \quad t > 0 \]
\[ f|_{t=0} = f_0. \]

Here \( N \geq 2 \), \( f = f(x, u, t) \) is the unknown function, which describes the number density of particles at position \( x \in \mathbb{R}^N \) and time \( t \) with velocity \( u \in \mathbb{R}^N \) in a physical system under consideration; \( \nabla_x = (\partial_{x_1}, \cdots, \partial_{x_N}) \), \( \nabla_u = (\partial_{u_1}, \cdots, \partial_{u_N}) \); \( \Delta_u = \partial_{u_1}^2 + \cdots + \partial_{u_N}^2 \) is the Laplacian with respect to the variable \( u \); and

\[ E(f) = \frac{\omega}{|S^{N-1}|} \frac{x}{|x|^N} *_{x} \int_{\mathbb{R}^N} f(x, u, t) \, du, \quad \omega: a \text{ constant}, \]

\( |S^{N-1}| \) is the \( (N-1) \) dimensional volume of the \( N \)-dimensional unit sphere, and \( *_{x} \) denotes the convolution with respect to \( x \).

In this article we present the results on the large time behaviour of small solutions of (1), which were obtained in [1], and give some remarks. (The detailed proofs of theorems are thus found in [1].)

**Theorem 1** ([1]). Let \( n \) be an integer satisfying \( 0 \leq n \leq 3N - 5 \) and let \( r \) be an integer satisfying \( r \geq n + 3N + \frac{3}{2} \). Assume that for initial value \( f_0 \), the quantity

\[ I(f_0) = \|(1 + |x|^2 + |u|^2)^{r/2} f_0\|_{H^m(\mathbb{R}^N \times \mathbb{R}^N)} \]

is finite for some \( m > \left[ \frac{N}{2} - 1 \right] + N + 1 \). Here \( H^m(\mathbb{R}^N \times \mathbb{R}^N) \) denotes the \( L^2 \)-Sobolev space of order \( m \) and \( [q] \) denotes the largest integer less than or equal to \( q \). Then for any \( \varepsilon > 0 \), if \( I(f_0) \) is sufficiently small, there exists a unique global solution \( f(t) \) of (1) in \( C([0, \infty); H^m) \) and \( f(t) \) satisfies

\[ \lim_{t \to \infty} t^{\frac{n+1}{2} - \varepsilon} \| t^{2N} f(t^{3/2}x, t^{1/2}u, t) - \sum_{k=0}^{n} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha, \beta} g_{\alpha, \beta}(x, u) \|_{L_{x,u}^\infty} = 0. \]
Here \( g_{\alpha,\beta}(x, u) = \partial_x^\alpha (\partial_x + \partial_u)^\beta g(x, u) \) and \( g(x, u) = e^{-3|x-x_0|^2 - \frac{1}{4}|u|^2}; \) \( B_{\alpha,\beta} \) are constants determined by \( f_0 \) and the nonlinearity. In particular, \( B_{0,0} = \int f_0(x, u) \, dx \, du. \)

**Remark 2.** In Theorem 1 the range of \( n \) is restricted as \( 0 \leq n \leq 3N - 5. \) One can, however, obtain the asymptotics of \( f \) with error estimate of order \( O(t^{-\frac{n+1}{2}}}) \) for any nonnegative \( n \in \mathbb{Z} \), if the weight is taken large enough in such a way that \( r \geq n + 3N + \frac{3}{2}. \)

In fact, for \( n \) in the range in Theorem 1, the asymptotics is similar to that for the solution of the linear problem (i.e., the problem without the term \( E(f) \cdot \nabla u f \)). The only difference from the linear problem appears in the constants \( B_{\alpha,\beta}'s \); in the linear case \( B_{\alpha,\beta}'s \) are given by some moments of \( f_0 \) only, while in the nonlinear case \( B_{\alpha,\beta}'s \) also involve some additional terms depending on \( f_0 \) and the nonlinearity.

If \( n \) is beyond the range in Theorem 1, i.e., if \( n \geq 3N - 4 \), then the effect of the nonlinearity becomes much stronger and the asymptotics is given by not only \( t^{-\frac{n}{2}} \) and \( g_{\alpha,\beta}'s \) but also some terms with \( \log t \) and other functions besides \( g_{\alpha,\beta}'s \). For example, if \( n = 3N - 4 \), then we have

\[
t^{2N} f(t^{3/2} x, t^{1/2} u, t) \sim \sum_{k=0}^{n-1} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha,\beta} g_{\alpha,\beta}(x, u) + t^{-\frac{n}{2}} \sum_{3|\alpha|+|\beta|=n} (B_{\alpha,\beta} + \tilde{B}_{\alpha,\beta} \log t) g_{\alpha,\beta}(x, u) + h(g_{\alpha,\beta}, t) + O(t^{-\frac{n+1}{2}+\epsilon}),
\]

where \( B_{\alpha,\beta} \) and \( \tilde{B}_{\alpha,\beta} \) are some constants and \( h(g_{\alpha,\beta}, t) = O(t^{-\frac{n}{2}}) \). See the argument in Section 3 below as for the dependence of \( h \) on \( g_{\alpha,\beta}'s \). In case \( n \geq 3N - 3 \), the form of the asymptotics becomes more complicated.

**Remark 3.** We mention some related works on large time behaviour of solutions of (1). Carrillo, Soler and Vázquez [4] obtained the asymptotics to the first order for weak solutions of (1) belonging to certain classes. The proof is based on the re-scaling argument. Carrillo and Soler [3] then proved the existence of weak solutions belonging to the classes given in [4] for initial data small in some sense. Carpio [2] obtained the asymptotics to the second order for small solutions by a detailed analysis of the linear problem and using the re-scaling argument. We also mention the work by Ono and Strauss [8], in which sharp decay rates of small solutions were proved and also it was proved
that small solutions approach to those of the corresponding linear problem with error estimate of $O(t^{-\frac{1}{2}})$.

2. Finite Dimensional Invariant Manifolds for the VPFP

We derive the long-time asymptotics given in Theorem 1 and Remark 2 by constructing finite dimensional invariant manifolds. To construct invariant manifolds, we change the variables into the "similarity" variables:

$$\tilde{t} = \log(t + 1), \quad \tilde{x} = x/(t + 1)^{3/2}, \quad \tilde{u} = u/(t + 1)^{1/2},$$

$$f(x, u, t) = (t + 1)^{-\gamma} \tilde{f}(x/(t + 1)^{3/2}, u/(t + 1)^{1/2}, \log(t + 1)),$$

where $\gamma = \frac{N}{2} + 2$. Then the equation for $\tilde{f}$ is written, after omitting tildes, as

$$\partial_{t}f - (\frac{3}{2}x - u) \cdot \nabla_{x}f - \frac{1}{2}u \cdot \nabla_{u}f - \gamma f - E(f) \cdot \nabla_{u}f - \Delta_{u}f = 0,$$

$$f|_{t=0} = f_{0}.$$

We write the problem (2) in the form

$$\partial_{t}f = \mathcal{L}f + N(f), \quad f(0) = f_{0},$$

where $\mathcal{L}f = \Delta_{u}f + (\frac{3}{2}x - u) \cdot \nabla_{x}f + \frac{1}{2}u \cdot \nabla_{u}f + \gamma f$ and $N(f) = E(f) \cdot \nabla_{u}f$.

We first consider the linear problem in the weighted space $X^l_m$ which is defined by

$$X^l_m = \{f(x, u) \in L^2(R^N \times R^N) : (1 + |x|^2 + |u|^2)^{r/2} \partial_\alpha \partial_\beta f \in L^2(R^N \times R^N), 0 \leq |\alpha| \leq l, 0 \leq |\beta| \leq m\},$$

where $l$, $m$ and $r$ are nonnegative integers.

We are given a nonnegative integer $n$ and we fix this $n$ hereafter. For this $n$ we take the weight large enough in such a way that $r \geq n + 3N + \frac{3}{2}$. Then as for the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ in $X^0_0$, we have

$$\sigma(\mathcal{L}) \subset \{\sigma_k : k = 0, 1, \cdots, n\} \cup \{\text{Re}\sigma \leq \sigma_{n+1}\} \quad (\sigma_j = -(2N - \gamma) - \frac{j}{2}).$$

Here each of $\sigma_k$ ($k = 0, 1, \cdots, n$) is a semi-simple eigenvalue; the associated eigenspace is spanned by functions $g_{\alpha, \beta}$'s with $\alpha$ and $\beta$ satisfying $3|\alpha| + |\beta| = k$; and the eigenprojection $P_k$ is given by

$$P_kf = \sum_{3|\alpha| + |\beta| = k} \langle f, g_{\alpha, \beta}^* \rangle g_{\alpha, \beta}.$$
Here \( g_{\alpha,\beta}(x, u) = c_{\alpha,\beta}(\partial_x + 3\partial_u)^{\alpha}(\partial_x + 2\partial_u)^{\beta}g(x, u) \) denotes the adjoint eigenfunction \((c_{\alpha,\beta} \text{ is a constant})\); and the inner product \((\cdot, \cdot)\) is defined by

\[
\langle f, g \rangle = \int f(x, u)g(x, u)e^{\mu(x, u)}
\]

\[
\mu(x, u) = 3|x - \frac{u}{2}|^2 + \frac{1}{4}|u|^2.
\]

We denote by \( P_n = \sum_{k=0}^{n} P_k \) the projection onto the spectral subspace corresponding to discrete eigenvalues \( \{\sigma_k\}_{k=0}^{n} \); and define \( Q_n \) by \( Q_n = I - P_n \).

Then \( X_{r}^{l,m} \) is decomposed into the direct sum:

\[
X_{r}^{l,m} = Y_n \oplus Z, \quad Y_n \equiv P_n X_{r}^{l,m}, \quad Z \equiv Q_n X_{r}^{l,m},
\]

and the solution \( e^{tL}f_0 \) of the linear problem is decomposed as

\[
e^{tL}f_0 = y_n(t) + z(t), \quad y_n(t) \in Y_n, \quad z(t) \in Z,
\]

\[
y_n(t) = \sum_{k=0}^{n} e^{\sigma_k t} P_k f_0, \quad z(t) = Q_n e^{tL}f_0.
\]

As for the part \( z(t) = Q_n e^{tL}f_0 \) on the subspace \( Z \), the estimate

\[
\|Q_n e^{tL}f_0\|_{X_{r}^{l,m}} \leq C(1 + t^{-\frac{l}{2}}) e^{\sigma_{n+1} t} \|f_0\|_{X_{r}^{l,m-j}}
\]

holds for \( l \geq 0 \), \( m \geq j \) and \( j = 0, 1 \). Therefore, the large time behaviour of solutions of the linear problem is described, up to \( O(e^{\sigma_{n+1} t}) \), by the behaviour of solutions on the finite dimensional invariant subspace \( Y_n \).

For the nonlinear problem we have the following theorem, from which the long-time asymptotics given in Theorem 1 and Remark 2 are obtained.

**Theorem 4** ([1]). Let \( n \geq 0 \) be an integer and let \( r \) be an integer satisfying \( r \geq n + 3N + \frac{3}{2} \). Then for any fixed integers \( m \geq 1 \) and \( l \geq \left[\frac{N}{2} - 1\right] + 1 \), there exists a finite dimensional invariant manifold \( M \) for (2) in a neighborhood of the origin of \( X_{r}^{m+l,m} \), i.e., there exist \( \Phi \in C^1(Y_n; Z) \) and \( R > 0 \) such that \( \Phi(0) = 0 \), \( D\Phi(0) = 0 \) and

\[
M = \{y_n + \Phi(y_n); y_n \in Y_n, \|y_n\| \leq R\},
\]

where \( Y_n = P_n X_{r}^{m+l,m} \) and \( Z = Q_n X_{r}^{m+l,m} \); and \( M \) is invariant under semiflows defined by (2). Furthermore, solutions near the origin approach to \( M \) at a rate \( O(e^{(\sigma_{n+1} + \epsilon)t}) \) as \( t \to \infty \). More precisely, if \( \|f_0\|_{X_{r}^{m+l,m}} \) is
sufficiently small, then there uniquely exists a solution \( \tilde{f}(t) \) of (2) on \( \mathcal{M} \) such that

\[
\|f(t) - \tilde{f}(t)\|_{X_{r}^{m+l,m}} \leq C e^{(\sigma_{n+1}+\epsilon)t}.
\]

Remark 5. Wayne [9] constructed finite dimensional invariant manifolds in Sobolev spaces with polynomial weights for certain semilinear heat equations on whole spaces by using the similarity variables transformation. The method in [9] is then extended to various contexts as in [5, 6, 7, 10].

3. Outline of Proof

We here outline how to obtain the long-time asymptotics given in Theorem 1 and Remark 2. (See [1] for the proof of Theorem 4.)

Our starting point is the estimate (3) in Theorem 4. We can rewrite the estimate (3) in the form

\[
\|y_{n}(t) - \tilde{y}_{n}(t)\|_{X_{r}^{m+l,m}} \leq C e^{(\sigma_{n+1}+\epsilon)t}
\]

and

\[
\|z(t) - \Phi(\tilde{y}_{n}(t))\|_{X_{r}^{m+l,m}} \leq C e^{(\sigma_{n+1}+\epsilon)t},
\]

where

\[
f(t) = y_{n}(t) + z(t), \quad \tilde{f}(t) = \tilde{y}_{n}(t) + \Phi(\tilde{y}_{n}(t)), \quad y_{n}(t), \tilde{y}_{n}(t) \in Y_{n}, \quad z(t) \in Z.
\]

Thus, to obtain the asymptotics of \( f(t) \) up to \( O(e^{(\sigma_{n+1}+\epsilon)t}) \), it suffices to investigate the behaviour of \( \tilde{y}_{n}(t) \), which is governed by a system of finite number of ordinary differential equations. Since \( \tilde{y}_{n}(t) \) can be written as

\[
\tilde{y}_{n}(t) = \sum_{3|\alpha|+|\beta| \leq n} y_{\alpha,\beta}(t) g_{\alpha,\beta}, \quad y_{\alpha,\beta} \in \mathbb{R},
\]

the problem is reduced to the analysis of the behaviour of \( y_{\alpha,\beta}'s \).

We now derive a system of ordinary differential equations for \( y_{\alpha,\beta}'s \). Since \( \tilde{f}(t) = \tilde{y}_{n}(t) + \Phi(\tilde{y}_{n}(t)) \) is a solution of (2) on \( \mathcal{M} \), it satisfies

\[
\partial_{t}\tilde{f} = \mathcal{L}\tilde{f} + N(\tilde{f}).
\]

Taking the inner product of this equation with \( g_{\alpha,\beta}^{\ast} \), we have

\[
\dot{y}_{\alpha,\beta} = \sigma_{k} y_{\alpha,\beta} + H_{\alpha,\beta}(\tilde{y}_{n}), \quad 3|\alpha| + |\beta| = k, \quad 0 \leq k \leq n,
\]

Where
where $\dot{y} = \frac{dy}{dt}$ and $H_{\alpha,\beta}(\bar{y}_n) = \langle \mathcal{N}(\bar{y}_n + \Phi(\bar{y}_n)), g_{\alpha,\beta}^* \rangle$.

For $\alpha = \beta = 0$, one can easily verify that $H_{0,0}(\bar{y}_n) = 0$. Hence,

$$\dot{y}_{0,0} = \sigma_0 y_{0,0}, \ i.e., \ y_{0,0}(t) = e^{\sigma_0 t} y_{0,0}(0).$$

Recall that $\sigma_0 = -(2N - \gamma) = -\left(\frac{3}{2}N - 2\right) < 0$. For $(\alpha, \beta) \neq (0, 0)$, we have, by the variation of constants formula,

$$y_{\alpha,\beta}(t) = e^{\sigma_k t} y_{\alpha,\beta}(0) + e^{\sigma_k t} \int_0^t e^{-\sigma_k s} H_{\alpha,\beta}(\bar{y}_n(s)) \, ds$$

with $k = 3|\alpha| + |\beta|, 1 \leq k \leq n$. Since $\sigma_k = \sigma_0 - \frac{k}{2}$, one can expect that $y_{\alpha,\beta}(t)$ decays strictly faster than $y_{0,0}(t)$. Therefore, the slowest term in $H_{\alpha,\beta}(\bar{y}_n(s))$ behaves like $e^{2\sigma_0 s}$, since the lowest order terms of $H_{\alpha,\beta}(\bar{y}_n)$ are quadratic in $\{y_{\alpha,\beta}\}$. As a result, the integrand in (6) behaves like $e^{(2\sigma_0 - \sigma_k)s}$.

Now let $n \leq 3N - 5$. This is just equivalent to $|\sigma_n| < 2|\sigma_0|$ (and to $|\sigma_{n+1}| \leq 2|\sigma_0|$). It then follows that for $3|\alpha| + |\beta| = k, 0 \leq k \leq n$,

$$y_{\alpha,\beta}(t) \sim \text{const.} e^{\sigma_k t} + O(e^{2\sigma_0 t}),$$

where $\text{const.}$ depends on $y_{\alpha,\beta}(0)$ and $H_{\alpha,\beta}$. We can also obtain

$$\|z(t)\|_{\mathcal{X}_{m+1}^{n+1}} \leq C e^{(\sigma_{n+1} + \epsilon)t}.$$ 

Therefore,

$$\tilde{f}(\tilde{t}) \sim \sum_{k=0}^n e^{\sigma_k \tilde{t}} \sum_{3|\alpha| + |\beta| = k} B_{\alpha,\beta} g_{\alpha,\beta} + O(e^{(\sigma_{n+1} + \epsilon)\tilde{t}}).$$

Here we write the solution of (2) and the time variable with tildes. Since the similarity variables $\tilde{f}$ and $\tilde{t}$ are connected with the original variables $f$ and $t$ by $\tilde{t} = \log t$ and $\tilde{f} = t^n f$, we obtain the asymptotics given in Theorem 1 for $n \leq 3N - 5$.

We next consider higher order asymptotics. In higher order cases, the estimates (4), (5) and equations for $y_{\alpha,\beta}$'s, of course, take the same forms. Let $n \geq 3N - 4$. Then $|\sigma_n| \geq 2|\sigma_0|$ and $|\sigma_{n+1}| > 2|\sigma_0|$. Therefore, the integrand in (6) does not decay as $s \to \infty$ for some $\alpha$ and $\beta$, and the effect of the inhomogeneous term is no longer weak. Also, one must take the effect of $\Phi(\bar{y}_n(t))$ into account, and, thus, the form of the asymptotics becomes complicated.
For example, if $n = 3N - 4$, then we have $\sigma_n = 2\sigma_0$ and, therefore, the integrand in (6) with $3|\alpha| + |\beta| = n$ is of $O(1)$. It then follows that for $3|\alpha| + |\beta| = n$,

$$y_{\alpha, \beta}(t) \sim c_1 e^{\sigma_n t} + c_2 t e^{\sigma_n t} + O(e^{\sigma_{n+1} t}),$$

where $c_1$ and $c_2$ are some constants. One can also see that $\Phi_{\alpha, \beta}(\bar{y}_n(t)) = O(e^{\sigma_n t})$. Combining these with (4) and (5), we see that, in the original variables,

$$t^{2N} f \sim \sum_{k=0}^{n-1} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha, \beta} g_{\alpha, \beta} + t^{-\frac{n}{2}} \sum_{3|\alpha|+|\beta|=n} (B_{\alpha, \beta} + \tilde{B}_{\alpha, \beta} \log t) g_{\alpha, \beta} + h(\bar{y}_n(t)) + O(t^{-\frac{n}{2} + 1 + \varepsilon}),$$

where $B_{\alpha, \beta}$ and $\tilde{B}_{\alpha, \beta}$ are some constants and $h(\bar{y}_n(t)) = O(t^{-\frac{n}{2}})$. This gives the asymptotics presented in Remark 2 for $n = 3N - 4$. For $n \geq 3N - 3$, it is possible to obtain the asymptotics in a similar manner as above, but the form of the asymptotics becomes more complicated.

References


