<table>
<thead>
<tr>
<th>Title</th>
<th>On large time behaviour of small solutions to the Vlasov-Poisson-Fokker-Planck equation (Mathematical Analysis in Fluid and Gas Dynamics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kagei, Yoshiyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1225: 114-121</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/41373">http://hdl.handle.net/2433/41373</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On large time behaviour of small solutions to the Vlasov-Poisson-Fokker-Planck equation

Yoshiyuki Kagei (隠居良行)
Graduate School of Mathematics, Kyushu University
(九大・数理)

1. Main Result

We consider the Cauchy problem for the Vlasov-Poisson-Fokker-Planck equation (without friction term)

\[ \partial_t f + u \cdot \nabla_x f - E(f) \cdot \nabla_u f - \Delta_u f = 0, \quad (x, u) \in \mathbb{R}^N \times \mathbb{R}^N, \quad t > 0 \]
\[ f|_{t=0} = f_0. \]

Here \( N \geq 2 \), \( f = f(x, u, t) \) is the unknown function, which describes the number density of particles at position \( x \in \mathbb{R}^N \) and time \( t \) with velocity \( u \in \mathbb{R}^N \) in a physical system under consideration; \( \nabla_x = (\partial_{x_1}, \ldots, \partial_{x_N}) \), \( \nabla_u = (\partial_{u_1}, \ldots, \partial_{u_N}) \); \( \Delta_u = \partial_{u_1}^2 + \cdots + \partial_{u_N}^2 \) is the Laplacian with respect to the variable \( u \); and

\[ E(f) = \frac{\omega}{|S^{N-1}|} \frac{x}{|x|^N} \ast_x \int_{\mathbb{R}^N} f(x, u, t) \, du, \quad \omega : \text{a constant}, \]

\( |S^{N-1}| \) is the \((N-1)\) dimensional volume of the \( N \)-dimensional unit sphere, and \( \ast_x \) denotes the convolution with respect to \( x \).

In this article we present the results on the large time behaviour of small solutions of (1), which were obtained in [1], and give some remarks. (The detailed proofs of theorems are thus found in [1].)

Theorem 1 ([1]). Let \( n \) be an integer satisfying \( 0 \leq n \leq 3N - 5 \) and let \( r \) be an integer satisfying \( r \geq n + 3N + \frac{3}{2} \). Assume that for initial value \( f_0 \), the quantity

\[ I(f_0) = \|(1 + |x|^2 + |u|^2)^{r/2} f_0\|_{H^m(\mathbb{R}^N \times \mathbb{R}^N)} \]

is finite for some \( m > \left[\frac{N}{2} - 1\right] + N + 1 \). Here \( H^m(\mathbb{R}^N \times \mathbb{R}^N) \) denotes the \( L^2 \)-Sobolev space of order \( m \) and \([q]\) denotes the largest integer less than or equal to \( q \). Then for any \( \varepsilon > 0 \), if \( I(f_0) \) is sufficiently small, there exists a unique global solution \( f(t) \) of (1) in \( C([0, \infty); H^m) \) and \( f(t) \) satisfies

\[ \lim_{t \to \infty} t^{n+1-\varepsilon} \|t^{2N} f(t^{3/2} x, t^{1/2} u, t) - \sum_{k=0}^{n} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha,\beta} g_{\alpha,\beta}(x, u)\|_{L^\infty_{x,u}} = 0. \]
Here $g_{\alpha,\beta}(x,u) = \partial_x^\alpha(\partial_x + \partial_u)^\beta g(x,u)$ and $g(x,u) = e^{-3|x-t^{3/2}|^2 - \frac{1}{4}|t|^2}$; $B_{\alpha,\beta}$ are constants determined by $f_0$ and the nonlinearity. In particular, $B_{0,0} = \int f_0(x,u) \, dx \, du$.

**Remark 2.** In Theorem 1 the range of $n$ is restricted as $0 \leq n \leq 3N - 5$. One can, however, obtain the asymptotics of $f$ with error estimate of order $O(t^{-\frac{n+1}{2}})$ for any nonnegative $n \in \mathbb{Z}$, if the weight is taken large enough in such a way that $r \geq n + 3N + \frac{3}{2}$.

In fact, for $n$ in the range in Theorem 1, the asymptotics is similar to that for the solution of the linear problem (i.e., the problem without the term $E(f) \cdot \nabla_u f$). The only difference from the linear problem appears in the constants $B_{\alpha,\beta}$'s; in the linear case $B_{\alpha,\beta}$'s are given by some moments of $f_0$ only, while in the nonlinear case $B_{\alpha,\beta}$'s also involve some additional terms depending on $f_0$ and the nonlinearity.

If $n$ is beyond the range in Theorem 1, i.e., if $n \geq 3N - 4$, then the effect of the nonlinearity becomes much stronger and the asymptotics is given by not only $t^{-\frac{n}{2}}$ and $g_{\alpha,\beta}$'s but also some terms with $\log t$ and other functions besides $g_{\alpha,\beta}$'s. For example, if $n = 3N - 4$, then we have

$$t^{2N} f(t^{3/2}x, t^{1/2}u, t) \sim \sum_{k=0}^{n-1} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha,\beta} g_{\alpha,\beta}(x,u) + t^{-\frac{n}{2}} \sum_{3|\alpha|+|\beta|=n} (B_{\alpha,\beta} + \tilde{B}_{\alpha,\beta} \log t) g_{\alpha,\beta}(x,u) + h(g_{\alpha,\beta}, t) + O(t^{-\frac{n+1}{2}+\epsilon}),$$

where $B_{\alpha,\beta}$ and $\tilde{B}_{\alpha,\beta}$ are some constants and $h(g_{\alpha,\beta}, t) = O(t^{-\frac{n}{2}})$. See the argument in Section 3 below as for the dependence of $h$ on $g_{\alpha,\beta}$'s. In case $n \geq 3N - 3$, the form of the asymptotics becomes more complicated.

**Remark 3.** We mention some related works on large time behaviour of solutions of (1). Carrillo, Soler and Vázquez [4] obtained the asymptotics to the first order for weak solutions of (1) belonging to certain classes. The proof is based on the re-scaling argument. Carrillo and Soler [3] then proved the existence of weak solutions belonging to the classes given in [4] for initial date small in some sense. Carpio [2] obtained the asymptotics to the second order for small solutions by a detailed analysis of the linear problem and using the re-scaling argument. We also mention the work by Ono and Strauss [8], in which sharp decay rates of small solutions were proved and also it was proved...
that small solutions approach to those of the corresponding linear problem
with error estimate of \(O(t^{-\frac{1}{2}})\).

2. Finite Dimensional Invariant Manifolds for the VPFP

We derive the long-time asymptotics given in Theorem 1 and Remark 2 by
constructing finite dimensional invariant manifolds. To construct invariant
manifolds, we change the variables into the "similarity" variables:

\[ \tilde{t} = \log(t + 1), \quad \tilde{x} = x/(t + 1)^{3/2}, \quad \tilde{u} = u/(t + 1)^{1/2}, \]

\[ f(x, u, t) = (t + 1)^{-\gamma} \tilde{f}(x/(t + 1)^{3/2}, u/(t + 1)^{1/2}, \log(t + 1)), \]

where \(\gamma = \frac{N}{2} + 2\). Then the equation for \(\tilde{f}\) is written, after omitting tildes, as

\[ \partial_{t}f - \left(\frac{3}{2}x - u\right) \cdot \nabla_{x}f - \frac{1}{2}u \cdot \nabla_{u}f - \gamma f - E(f) \cdot \nabla_{u}f - \Delta_{u}f = 0, \]

\[ f|_{t=0} = f_{0}. \]

We write the problem (2) in the form

\[ \partial_{t}f = \mathcal{L}f + N(f), \quad f(0) = f_{0}, \]

where \(\mathcal{L}f = \Delta_{u}f + \left(\frac{3}{2}x - u\right) \cdot \nabla_{x}f + \frac{1}{2}u \cdot \nabla_{u}f + \gamma f\) and \(N(f) = E(f) \cdot \nabla_{u}f\).

We first consider the linear problem in the weighted space \(X_{r}^{l,m}\) which is
defined by

\[ X_{r}^{l,m} = \{ f(x, u) \in L^{2}(\mathbb{R}^{N} \times \mathbb{R}^{N}) : (1 + |x|^{2} + |u|^{2})^{r/2} \partial_{x}^{\alpha} \partial_{u}^{\beta}f \in L^{2}(\mathbb{R}^{N} \times \mathbb{R}^{N}), \]

\[ 0 \leq |\alpha| \leq l, 0 \leq |\beta| \leq m \}, \]

where \(l, m\) and \(r\) are nonnegative integers.

We are given a nonnegative integer \(n\) and we fix this \(n\) hereafter. For this
\(n\) we take the weight large enough in such a way that \(r \geq n + 3N + \frac{3}{2}\). Then
as for the spectrum \(\sigma(\mathcal{L})\) of \(\mathcal{L}\) in \(X_{r}^{0,0}\), we have

\[ \sigma(\mathcal{L}) \subset \{ \sigma_{k} : k = 0, 1, \ldots, n \} \cup \{ \text{Re} \sigma \leq \sigma_{n+1} \} \quad (\sigma_{j} = -(2N - \gamma) - \frac{j}{2}). \]

Here each of \(\sigma_{k}\) \((k = 0, 1, \ldots, n)\) is a semi-simple eigenvalue ; the associated
eigenspace is spanned by functions \(g_{\alpha,\beta}\)'s with \(\alpha\) and \(\beta\) satisfying \(3|\alpha| + |\beta| = k\);
and the eigenprojection \(P_{k}\) is given by

\[ P_{k}f = \sum_{3|\alpha| + |\beta| = k} \langle f, g_{\alpha,\beta}^{*} \rangle g_{\alpha,\beta}. \]
Here \( g_{\alpha,\beta}(x, u) = c_{\alpha,\beta}(\partial_x + 3 \partial_u) \sigma (\partial_x + 2 \partial_u) g(x, u) \) denotes the adjoint eigenfunction \((c_{\alpha,\beta} \text { is a constant})\); and the inner product \( \langle \cdot, \cdot \rangle \) is defined by

\[
\langle f, g \rangle = \int f(x, u)g(x, u)e^{\mu(x, u)} \, dx \, du, \quad \mu(x, u) = 3|x - \frac{u}{2}|^2 + \frac{1}{4}|u|^2.
\]

We denote by \( \mathcal{P}_n = \sum_{k=0}^{n} P_k \) the projection onto the spectral subspace corresponding to discrete eigenvalues \( \{\sigma_k\}_{k=0}^{n} \); and define \( \mathcal{Q}_n \) by \( \mathcal{Q}_n = I - \mathcal{P}_n \).

Then \( X_{r}^{l,m} \) is decomposed into the direct sum:

\[
X_{r}^{l,m} = Y_n \oplus Z, \quad Y_n \equiv \mathcal{P}_n X_{r}^{l,m}, \quad Z \equiv \mathcal{Q}_n X_{r}^{l,m},
\]

and the solution \( e^{tL}f_0 \) of the linear problem is decomposed as

\[
e^{tL}f_0 = y_n(t) + z(t), \quad y_n(t) \in Y_n, \quad z(t) \in Z,
\]

\[
y_n(t) = \sum_{k=0}^{n} e^{\sigma_k t} P_k f_0, \quad z(t) = \mathcal{Q}_n e^{tL} f_0.
\]

As for the part \( z(t) = \mathcal{Q}_n e^{tL} f_0 \) on the subspace \( Z \), the estimate

\[
\| \mathcal{Q}_n e^{tL} f_0 \|_{X_{r}^{l,m}} \leq C(1 + t^{-\frac{j}{2}}) e^{\sigma_{n+1} t} \| f_0 \|_{X_{r}^{l,m-j}}
\]

holds for \( l \geq 0, m \geq j \) and \( j = 0, 1 \). Therefore, the large time behaviour of solutions of the linear problem is described, up to \( O(e^{\sigma_{n+1} t}) \), by the behaviour of solutions on the finite dimensional invariant subspace \( Y_n \).

For the nonlinear problem we have the following theorem, from which the long-time asymptotics given in Theorem 1 and Remark 2 are obtained.

**Theorem 4** ([1]). Let \( n \geq 0 \) be an integer and let \( r \) be an integer satisfying \( r \geq n + 3N + \frac{3}{2} \). Then for any fixed integers \( m \geq 1 \) and \( l \geq \left[\frac{N}{2} - 1\right] + 1 \), there exists a finite dimensional invariant manifold \( \mathcal{M} \) for (2) in a neighborhood of the origin of \( X_{r}^{m,l,m} \), i.e., there exist \( \Phi \in C^1(Y_n; Z) \) and \( R > 0 \) such that \( \Phi(0) = 0, D\Phi(0) = 0 \) and

\[
\mathcal{M} = \{y_n + \Phi(y_n); y_n \in Y_n, \|y_n\| \leq R\},
\]

where \( Y_n = \mathcal{P}_n X_{r}^{m+l,m} \) and \( Z = \mathcal{Q}_n X_{r}^{m+l,m} \); and \( \mathcal{M} \) is invariant under semiflows defined by (2). Furthermore, solutions near the origin approach to \( \mathcal{M} \) at a rate \( O(e^{(\sigma_{n+1} \pm \epsilon)t}) \) as \( t \to \infty \). More precisely, if \( \| f_0 \|_{X_{r}^{m+l,m}} \) is
sufficiently small, then there uniquely exists a solution $\tilde{f}(t)$ of (2) on $\mathcal{M}$ such that

\[(3) \quad \|f(t) - \tilde{f}(t)\|_{X_{r}^{m+l,m}} \leq Ce^{(\sigma_{n+1}+\epsilon)t}.\]

**Remark 5.** Wayne [9] constructed finite dimensional invariant manifolds in Sobolev spaces with polynomial weights for certain semilinear heat equations on whole spaces by using the similarity variables transformation. The method in [9] is then extended to various contexts as in [5, 6, 7, 10].

### 3. Outline of Proof

We here outline how to obtain the long-time asymptotics given in Theorem 1 and Remark 2. (See [1] for the proof of Theorem 4.)

Our starting point is the estimate (3) in Theorem 4. We can rewrite the estimate (3) in the form

\[(4) \quad \|y_n(t) - \tilde{y}_n(t)\|_{X_{r}^{m+l,m}} \leq Ce^{(\sigma_{n+1}+\epsilon)t}\]

and

\[(5) \quad \|z(t) - \Phi(\tilde{y}_n(t))\|_{X_{r}^{m+l,m}} \leq Ce^{(\sigma_{n+1}+\epsilon)t},\]

where

\[f(t) = y_n(t) + z(t), \quad \tilde{f}(t) = \tilde{y}_n(t) + \Phi(\tilde{y}_n(t)), \quad y_n(t), \tilde{y}_n(t) \in Y_n, \quad z(t) \in Z.\]

Thus, to obtain the asymptotics of $f(t)$ up to $O(e^{(\sigma_{n+1}+\epsilon)t})$, it suffices to investigate the behaviour of $\tilde{y}_n(t)$, which is governed by a system of finite number of ordinary differential equations. Since $\tilde{y}_n(t)$ can be written as

\[\tilde{y}_n(t) = \sum_{3|\alpha|+|\beta|\leq n} y_{\alpha,\beta}(t)g_{\alpha,\beta}, \quad y_{\alpha,\beta} \in \mathbb{R},\]

the problem is reduced to the analysis of the behaviour of $y_{\alpha,\beta}$'s.

We now derive a system of ordinary differential equations for $y_{\alpha,\beta}$'s. Since $\tilde{f}(t) = \tilde{y}_n(t) + \Phi(\tilde{y}_n(t))$ is a solution of (2) on $\mathcal{M}$, it satisfies

\[\partial_t \tilde{f} = \mathcal{L}\tilde{f} + \mathcal{N}(\tilde{f}).\]

Taking the inner product of this equation with $g_{\alpha,\beta}^*$, we have

\[\dot{y}_{\alpha,\beta} = \sigma_k y_{\alpha,\beta} + H_{\alpha,\beta}(\tilde{y}_n), \quad 3|\alpha| + |\beta| = k, \quad 0 \leq k \leq n,\]
where \( \dot{y} = \frac{dy}{dt} \) and \( H_{\alpha,\beta}(\tilde{y}_n) = \langle N(\tilde{y}_n + \Phi(\tilde{y}_n)), \Phi'_{\alpha,\beta} \rangle \).

For \( \alpha = \beta = 0 \), one can easily verify that \( H_{0,0}(\tilde{y}_n) = 0 \). Hence,

\[
\dot{y}_{0,0} = \sigma_0 y_{0,0}, \quad i.e., \quad y_{0,0}(t) = e^{\sigma_0 t} y_{0,0}(0).
\]

Recall that \( \sigma_0 = -(2N - \gamma) = -(\frac{3}{2}N - 2) < 0 \).

For \( (\alpha, \beta) \neq (0,0) \), we have, by the variation of constants formula,

\[
y_{\alpha,\beta}(t) = e^{\sigma_k t} y_{\alpha,\beta}(0) + e^{\sigma_k t} \int_0^t e^{-\sigma_k s} H_{\alpha,\beta}(\overline{y}_n(s)) \, ds
\]

with \( k = 3|\alpha| + |\beta|, 1 \leq k \leq n \). Since \( \sigma_k = \sigma_0 - \frac{k}{2} \), one can expect that \( y_{\alpha,\beta}(t) \) decays strictly faster than \( y_{0,0}(t) \). Therefore, the slowest term in \( H_{\alpha,\beta}(\overline{y}_n(s)) \) behaves like \( e^{2\sigma_0 s} \), since the lowest order terms of \( H_{\alpha,\beta}(\overline{y}_n) \) are quadratic in \( \{y_{\alpha,\beta}\} \).

Now let \( n \leq 3N - 5 \). This is just equivalent to \( |\sigma_n| < 2|\sigma_0| \) (and to \( |\sigma_{n+1}| \leq 2|\sigma_0| \)). It then follows that for \( 3|\alpha| + |\beta| = k \), \( 0 \leq k \leq n \),

\[
y_{\alpha,\beta}(t) \sim \text{const.} e^{\sigma_k t} + O(e^{2\sigma_0 t}),
\]

where \( \text{const.} \) depends on \( y_{\alpha,\beta}(0) \) and \( H_{\alpha,\beta} \). We can also obtain

\[
\|z(t)\|_{X_{m+1,m}} \leq Ce^{(\sigma_{n+1} + \epsilon)t}.
\]

Therefore,

\[
\tilde{f}(\tilde{t}) \sim \sum_{k=0}^{n} e^{\sigma_k \tilde{t}} \sum_{3|\alpha| + |\beta| = k} B_{\alpha,\beta} g_{\alpha,\beta} + O(e^{(\sigma_{n+1} + \epsilon)\tilde{t}}).
\]

Here we write the solution of (2) and the time variable with tildes. Since the similarity variables \( \tilde{f} \) and \( \tilde{t} \) are connected with the original variables \( f \) and \( t \) by \( \tilde{t} = \log t \) and \( \tilde{f} = t^\gamma f \), we obtain the asymptotics given in Theorem 1 for \( n \leq 3N - 5 \).

We next consider higher order asymptotics. In higher order cases, the estimates (4), (5) and equations for \( y_{\alpha,\beta}'s \), of course, take the same forms. Let \( n \geq 3N - 4 \). Then \( |\sigma_n| \geq 2|\sigma_0| \) and \( |\sigma_{n+1}| > 2|\sigma_0| \). Therefore, the integrand in (6) does not decay as \( s \rightarrow \infty \) for some \( \alpha \) and \( \beta \), and the effect of the inhomogeneous term is no longer weak. Also, one must take the effect of \( \Phi(\overline{y}_n(t)) \) into account, and, thus, the form of the asymptotics becomes complicated.
For example, if \( n = 3N - 4 \), then we have \( \sigma_n = 2\sigma_0 \) and, therefore, the integrand in (6) with \( 3|\alpha| + |\beta| = n \) is of \( O(1) \). It then follows that for \( 3|\alpha| + |\beta| = n \),

\[
y_{\alpha,\beta}(t) \sim c_1 e^{\sigma_n t} + c_2 t e^{\sigma_n t} + O(e^{\sigma_{n+1} t}),
\]

where \( c_1 \) and \( c_2 \) are some constants. One can also see that \( \Phi_{\alpha,\beta}(\bar{y}_n(t)) = O(e^{\sigma_n t}) \). Combining these with (4) and (5), we see that, in the original variables,

\[
t^{2N} f \sim \sum_{k=0}^{n-1} t^{-\frac{k}{2}} \sum_{3|\alpha|+|\beta|=k} B_{\alpha,\beta} g_{\alpha,\beta} + t^{-\frac{n}{2}} \sum_{3|\alpha|+|\beta|=n} (B_{\alpha,\beta} + \tilde{B}_{\alpha,\beta} \log t) g_{\alpha,\beta} + h(\bar{y}_n(t)) + O(t^{-\frac{n+1}{2} + \epsilon}),
\]

where \( B_{\alpha,\beta} \) and \( \tilde{B}_{\alpha,\beta} \) are some constants and \( h(\bar{y}_n(t)) = O(t^{-\frac{n}{2}}) \). This gives the asymptotics presented in Remark 2 for \( n = 3N - 4 \). For \( n \geq 3N - 3 \), it is possible to obtain the asymptotics in a similar manner as above, but the form of the asymptotics becomes more complicated.

References


