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Functional Itô Formula for Quantum Semimartingales

UN CIG JI[1]
DEPARTMENT OF BUSINESS ADMINISTRATION
HANYANG UNIVERSITY
SEOUL, 133-791 KOREA
ucji@nuri.net

1 Introduction

The aim of this paper is to give a partial answer to the problem of deriving a functional quantum Itô formula for (unbounded) semimartingales, i.e., an Itô formula for $f(\Xi)$, where Ξ is in a certain class of quantum semimartingales.

Since a quantum stochastic calculus ([21], [27]; also [24] for the white noise approach) of Itô type first formulated by Hudson and Parthasarathy [12], the stochastic integral representations of quantum martingales have been studied by many authors, see [10], [11]. In particular, Parthasarathy and Sinha [28] established a stochastic integral representation of regular bounded quantum martingales in (Boson) Fock space with respect to the basic martingales, namely the annihilation, creation and number processes. A new proof of the Parthasarathy and Sinha representation theorem has been discussed by Meyer in [22] in which he gives the special form of the number operator coefficient. The representation theorem has been extended to regular bounded semimartingales by Attal [1] and the Itô formula for products of regular semimartingales belonging to a certain class has been discussed which yields a quantum Itô formula for polynomial [2]. In [30], by Vincent-Smith, a functional quantum Itô formula for regular bounded semimartingales has been widely studied with closed form of the Itô correction term. For more discussions of functional quantum Itô formula, we refer to [4], [13].

In [16], we extended the quantum stochastic integral to a wider class of adapted quantum stochastic processes on Boson Fock space and a quantum stochastic integral representation theorem has been proved for a class of unbounded semimartingales. Motivated by results in [16] and [30], we discuss a functional quantum Itô formula for $f(\Xi)$, where f is an entire function and Ξ is a (unbounded) semimartingale such that f and Ξ satisfy certain conditions. Our approach is based on riggings of Fock space which are applied in many fields of mathematics and mathematical physics, e.g., [3], [5], [8], [19], [20], [29], we also refer to [9], [18], [23] for nuclear riggings which are the fundamental frameworks of white noise analysis.

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2 Riggings of Fock Space

Let $H_{\mathbb{R}} = L^2(\mathbb{R}_+, dt)$ be the real Hilbert space of square integrable functions on $\mathbb{R}_+ = [0, \infty)$ with norm $|\cdot|_0$ induced by the inner product $\langle \cdot, \cdot \rangle$. The complexification of $H_{\mathbb{R}}$ is denoted by H whose norm is also denoted by $|\cdot|_0$. The (Boson) Fock space $\mathcal{H} \equiv \Gamma(H)$ over H is defined by

$$\mathcal{H} = \left\{ \phi = (f_n)_{n=0}^{\infty} \,\middle|\, f_n \in H^{\widehat{\otimes} n} \text{ for all } n \geq 0 \text{ and } \|\phi\|_0 < \infty \right\},$$

where $H^{\widehat{\otimes} n}$ is the n-fold symmetric tensor power of H and the norm $\|\cdot\|_0$ is defined by

$$\|\phi\|_0^2 = \sum_{n=0}^{\infty} n! \, |f_n|_0^2 < \infty.$$

We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical $\mathbb{C}$-bilinear form on $\mathcal{H}$ defined through $\langle \cdot, \cdot \rangle$.

Let N be the number operator and let $\mathcal{G}_p$ be the $\mathcal{H}$-domain of e^{pN} for each $p \geq 0$. Then $\mathcal{G}_p$ is a Hilbert space with norm $\|\cdot\|_p = \| e^{pN} \cdot \|_0$. More precisely, for any $p \geq 0$

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|_0^2, \qquad \phi = (f_n) \in \mathcal{G}_p. \tag{1}$$

Then we naturally come to

$$\begin{aligned} \mathcal{G} \equiv \operatorname*{proj\,lim}_{p \to \infty} \mathcal{G}_p \subset \cdots \subset \mathcal{G}_q \subset \cdots \subset \mathcal{G}_p \subset \cdots \\ \cdots \subset \mathcal{G}_0 = \mathcal{H} \subset \cdots \subset \mathcal{G}_{-p} \subset \cdots \subset \mathcal{G}_{-q} \subset \cdots \subset \mathcal{G}^*, \end{aligned}$$

where $\mathcal{G}_{-p}$ and $\mathcal{G}^*$ are strong dual spaces of $\mathcal{G}_p$ and $\mathcal{G}$, respectively. Note that $\mathcal{G}$ is a countable Hilbert space equipped with the Hilbertian norms defined in (1) and $\mathcal{G}^* = \operatorname{ind\,lim}_{p \to \infty} \mathcal{G}_{-p}$. The canonical $\mathbb{C}$-bilinear form on $\mathcal{G}^* \times \mathcal{G}$ is also denoted by $\langle\langle \cdot, \cdot \rangle\rangle$, and we have

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, f_n \rangle, \qquad \Phi = (F_n) \in \mathcal{G}^*, \quad \phi = (f_n) \in \mathcal{G}.$$

Moreover, the Schwartz inequality takes the form:

$$|\langle\langle \Phi, \phi \rangle\rangle| \leq \|\Phi\|_{-p} \|\phi\|_p.$$

It is noted that for any $p \in \mathbb{R}$, $e^{pN}\mathcal{H} = \mathcal{G}_{-p}$ and $e^{-pN}\mathcal{G}_{-p} = \mathcal{H}$. Moreover, $e^{pN}\mathcal{G}_q = \mathcal{G}_{q-p}$ for any $p, q \in \mathbb{R}$.

For each $\xi \in H$, we write $\xi_B = \xi \chi_B$, where $B \subset \mathbb{R}_+$ and χ_B is the indicator function on B. For notational convenience, we write $\xi_{t]} = \xi_{[0,t]}$ and $\xi_{[t} = \xi_{[t,\infty)}$ for any $t > 0$. Then we have the decomposition

$$H = H_{s]} \oplus H_{[s,t]} \oplus H_{[t}, \qquad 0 < s < t < \infty,$$

where $H_{s]} = \{\xi_{s]} \,|\, \xi \in H\}$, $H_{[s,t]} = \{\xi_{[s,t]} \,|\, \xi \in H\}$ and $H_{[t} = \{\xi_{[t} \,|\, \xi \in H\}$. Put

$$\mathcal{H}_{s]} = \Gamma(H_{s]}), \quad \mathcal{H}_{[s,t]} = \Gamma(H_{[s,t]}) \quad \text{and} \quad \mathcal{H}_{[t} = \Gamma(H_{[t}).$$

Then we have the identification

$$\mathcal{H} = \mathcal{H}_{s]} \otimes \mathcal{H}_{[s,t]} \otimes \mathcal{H}_{[t}$$

via the following decomposition:

$$\phi_\xi = \phi_{\xi_{s]}} \otimes \phi_{\xi_{[s,t]}} \otimes \phi_{\xi_{[t}}, \qquad \xi \in H,$$

where $\phi_\xi = (\xi^{\otimes n}/n!)$ is the exponential vector of $\xi \in H$. Moreover, for any $p \in \mathbb{R}$ and $0 < s < t < \infty$, we have

$$\mathcal{G}_p = \mathcal{G}_{p;s]} \otimes \mathcal{G}_{p;[s,t]} \otimes \mathcal{G}_{p;[t},$$

where $\mathcal{G}_{p;s]} = \mathcal{G}_p \cap \mathcal{H}_{s]}$, $\mathcal{G}_{p;[s,t]} = \mathcal{G}_p \cap \mathcal{H}_{[s,t]}$, $\mathcal{G}_{p;[t} = \mathcal{G}_p \cap \mathcal{H}_{[t}$ and their completion for $p \le 0$.

3 Operators on Fock Space

Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be the space of all bounded linear operators from a locally convex $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$. Let l, m be non-negative integers. Then for each $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$ the *integral kernel operator* $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}, \mathcal{G})$ with kernel $K_{l,m}$ is defined by

$$\Xi_{l,m}(K_{l,m})\phi = \left(\frac{(n+m)!}{n!} (K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}} \right), \qquad \phi = (f_n) \in \mathcal{G}.$$

In this case, we have for any $p \in \mathbb{R}$, $q > 0$ and $\phi \in \mathcal{G}$

$$\| \Xi_{l,m}(K_{l,m})\phi \|_p \le C \left(e^{pl-(p+q)m+q/2} \right) l^{l/2} m^{m/2} \left(\frac{e^{q/2}}{qe} \right)^{(l+m)/2} \| \phi \|_{p+q},$$

where $C \ge 0$ satisfies that $|K_{l,m} f|_0 \le C|f|_0$ for any $f \in H^{\otimes m}$. Moreover, the integral kernel operator $\Xi_{l,m}(K_{l,m})$ has a unique extension to a continuous linear operator from $\mathcal{G}^*$ into itself (see [14], [15]).

Let $\eta \in H$ and let $K_\eta \in \mathcal{L}(H, \mathbb{C})$ be defined by $K_\eta(\xi) = \langle \eta, \xi \rangle$ for any $\xi \in H$. For simple notation, we identify $\eta = K_\eta = K_\eta^*$, where K_η^* is the adjoint operator of K_η, i.e., $K_\eta^*(a) = a\eta$ for all $a \in \mathbb{C}$. For each $t \ge 0$, we put

$$A_t = \Xi_{0,1}(\chi_{t]}), \quad A_t^* = \Xi_{1,0}(\chi_{t]}), \quad \Lambda_t = \Xi_{1,1}(\chi_{t]}), \tag{2}$$

where $\chi_{t]} \equiv \chi_{[0,t]}$ and for the definition of Λ_t, the indicator function is considered as the multiplication operator on H, i.e., $\chi_{t]}(\xi) = \xi_{t]}$ for any $\xi \in H$. For each $t \in \mathbb{R}_+$, A_t and A_t^* are called the *annihilation operator* and the *creation operator*, respectively.

We now mention the following Fock expansion theorem. For the proof, see [17].

Theorem 1 *Let $p, q \in \mathbb{R}$. For any $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ there exists a unique family of operators $K_{l,m} \in \mathcal{L}(H^{\widehat{\otimes} m}, H^{\widehat{\otimes} l})$, $l, m \ge 0$, such that*

$$\Xi = \sum_{l,m,n=0}^{\infty} \frac{(-1)^n}{n!} \Xi_{n+l,n+m}(I^{\otimes n} \otimes K_{l,m}),$$

where the series converges in $\mathcal{L}(\mathcal{G}_{(p\vee q)+r}, \mathcal{G}_{q-s})$ for any $r > 0$ and $s > 0$ satisfying $\rho^r/(-r\log\rho) < 1$ and $\rho^s/(-s\log\rho) < 1$.

For a given entire function f, let $\mathcal{A}_f$ be the class of continuous linear operators Ξ in $\mathcal{L}(\mathcal{G},\mathcal{G})$ satisfying that for any $p \geq 0$ there exist $q \geq 0$, $M \geq 0$ and $0 < \gamma < 1$ such that

$$\frac{|f^{(n)}(0)| \|\Xi^n \phi\|_p}{n!} \leq M\gamma^n \|\phi\|_q, \qquad n \geq 0, \quad \phi \in \mathcal{G}. \tag{3}$$

Proposition 2 *For any $\Xi \in \mathcal{A}_f$, we can define $f(\Xi)$ as a continuous operator on $\mathcal{G}$ by*

$$f(\Xi) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \Xi^n. \tag{4}$$

PROOF. By definition of $\mathcal{A}_f$, for any $p \geq 0$ there exist $q \geq 0$, $M \geq 0$ and $0 < \gamma < 1$ such that (3) holds. Therefore, for any $\phi \in \mathcal{G}$

$$\sum_{n=0}^{\infty} \left\| \frac{f^{(n)}(0)}{n!} \Xi^n \phi \right\|_p \leq \sum_{n=0}^{\infty} M\gamma^n \|\phi\|_q \leq M \left(\sum_{n=0}^{\infty} \gamma^n \right) \|\phi\|_q.$$

Hence the series in the right hand side of (4) converges in $\mathcal{L}(\mathcal{G},\mathcal{G})$. It then follows the proof. ∎

If f is a polynomial, then $\mathcal{A}_f = \mathcal{L}(\mathcal{G},\mathcal{G})$. Also, if f is the exponential function, then an element of $\mathcal{A}_f$ is called an *equicontinuous generator*, see [26].

4 Equicontinuous Generators

Let $GL(\mathcal{G})$ denote the group of all linear homeomorphisms from $\mathcal{G}$ onto itself. In this section we consider a (complex) one-parameter subgroup $\{\Omega_z\}_{z\in\mathbb{C}}$ of $GL(\mathcal{G})$, i.e., for each $z \in \mathbb{C}$, $\Omega_z \in GL(\mathcal{G})$ and

$$\Omega_0 = I \text{ (identity operator)}; \qquad \Omega_{z_1}\Omega_{z_2} = \Omega_{z_1+z_2}, \quad z_1, z_2 \in \mathbb{C}.$$

A one-parameter subgroup $\{\Omega_z\}_{z\in\mathbb{C}}$ is said to be *holomorphic* if there exists a $\Xi \in \mathcal{L}(\mathcal{G})$ such that for any $\phi \in \mathcal{G}$,

$$\lim_{z\to 0} \left\| \frac{\Omega_z \phi - \phi}{z} - \Xi\phi \right\|_p = 0 \quad \text{for all } p \geq 0.$$

Such a Ξ is called the *infinitesimal generator* of $\{\Omega_z\}_{z\in\mathbb{C}}$. A family of operators $\{\Xi_i\}_{i\in I} \subset \mathcal{L}(\mathcal{G},\mathcal{G})$ is said to be *equicontinuous* if for any $p \geq 0$ there exist $q \geq 0$ and $C \geq 0$ such that

$$\|\Xi_i \phi\|_p \leq C \|\phi\|_q, \qquad \phi \in \mathcal{G}, \quad i \in I,$$

see [26].

Theorem 3 [26] *Every equicontinuous generator $\Xi \in \mathcal{L}(\mathcal{G},\mathcal{G})$ is the infinitesimal generator of some holomorphic one-parameter subgroup $\{\Omega_z\}_{z\in\mathbb{C}} \subset GL(\mathcal{G})$ such that $\{\Omega_z; |z| < R\}$ is equicontinuous for some $R > 0$. In this case,*

$$\Omega_z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Xi^n, \qquad z \in \mathbb{C},$$

where the series converges in $\mathcal{L}(\mathcal{G},\mathcal{G})$.

From Theorem 3, for an equicontinuous generator $\Xi \in \mathcal{L}(\mathcal{G},\mathcal{G})$ the corresponding holomorphic one-parameter subgroup of $GL(\mathcal{G})$ is denoted by $\{\exp(z\Xi)\}_{z\in\mathbb{C}}$.

Lemma 4 [26] *Let $\Xi \in \mathcal{L}(\mathcal{G},\mathcal{G})$. Then the following conditions are equivalent:*

(i) *there exists some $R > 0$ such that $\{(R\Xi)^n/n!; n = 0,1,2,\cdots\}$ is equicontinuous;*

(ii) *$\{(R\Xi)^n/n!; n = 0,1,2,\cdots\}$ is equicontinuous for any $R > 0$.*

Lemma 5 *Let $\zeta, \eta \in H$ and $B \in \mathcal{L}(H,H)$. Then there exists a unique operator $G_{\eta,B,\zeta} \in \mathcal{L}(\mathcal{G},\mathcal{G})$ such that*

$$G_{\eta,B,\zeta}\phi = \left(\sum_{l+m=n}\sum_{k=0}^{\infty}\frac{(l+k)!}{l!k!m!}\zeta^{\otimes m}\widehat{\otimes}\left((e^B)^{\otimes l}\left(\eta^{\otimes k}\widehat{\otimes}_k f_{l+k}\right)\right)\right)_{n=0}^{\infty}$$

for any $\phi = (f_n)_{n=0}^{\infty} \in \mathcal{G}$, where $\widehat{\otimes}_k$ is the right contraction [23].

For the proof, see [15]. For each $\xi \in H$, we can easily see that

$$G_{\eta,B,\zeta}\phi_\xi = \exp\{\langle\eta,\ \xi\rangle\}\phi_{e^B\xi+\zeta}. \tag{5}$$

Motivated by results in [6] and Theorem 3, we now consider a holomorphic one-parameter subgroup of $GL(\mathcal{G})$ with infinitesimal generator $a_1I + a_2A_t + a_3\Lambda_t + a_4A_t^*$ for arbitrary $a_1, a_2, a_3, a_4 \in \mathbb{C}$ and $t > 0$.

For notational convenience, we put

$$G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \alpha_1 G_{\alpha_2\chi_{t]},\alpha_3\chi_{t]},\alpha_4\chi_{t]}}, \qquad \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathbb{C}, \quad t > 0.$$

Let $\mathbb{C}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ be the additive and multiplicative group of complex numbers, respectively.

Theorem 6 *Let $\mathfrak{G}_t = \{G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4} : \alpha_1 \in \mathbb{C}^*,\ \alpha_2,\alpha_3,\alpha_4 \in \mathbb{C}\}$. Then $\mathfrak{G}_t$ forms a subgroup of $GL(\mathcal{G})$.*

PROOF. For any $\xi \in H$ we have, by (5), $G_{t;1,0,0,0}\phi_\xi = \phi_\xi$ and

$$G_{t;\alpha_1',\alpha_2',\alpha_3',\alpha_4'}G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4}\phi_\xi = G_{t;\alpha_1'\alpha_1e^{\alpha_2'\alpha_4t},\alpha_2'e^{\alpha_3}+\alpha_2,e^{\alpha_3'+\alpha_3},e^{\alpha_3'}\alpha_4+\alpha_4'}\phi_\xi,$$

for any $\alpha_1,\alpha_1' \in \mathbb{C}^*$ and $\alpha_2,\alpha_2',\alpha_3,\alpha_3',\alpha_4,\alpha_4' \in \mathbb{C}$. But $\{\phi_\xi : \xi \in H\}$ spans a dense subspace of $\mathcal{G}$ and $G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is continuous. Hence it follows that for any $\phi \in \mathcal{G}$

$$G_{t;1,0,0,0}\phi = \phi \qquad \text{and}$$

$$G_{t;\alpha_1',\alpha_2',\alpha_3',\alpha_4'}G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4}\phi = G_{t;\alpha_1'\alpha_1e^{\alpha_2'\alpha_4t},\alpha_2'e^{\alpha_3}+\alpha_2,e^{\alpha_3'+\alpha_3},e^{\alpha_3'}\alpha_4+\alpha_4'}\phi,$$

Put $\alpha_1' = (1/\alpha_1)\exp\{e^{-\alpha_3}\alpha_2\alpha_4t\}$, $\alpha_2' = -\alpha_2e^{-\alpha_3}$, $\alpha_3' = -\alpha_3$, and $\alpha_4' = -\alpha_4e^{-\alpha_3}$. Then $G_{t;\alpha_1',\alpha_2',\alpha_3',\alpha_4'}$ is the inverse of $G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ in $\mathfrak{G}_t$. This completes the proof. ∎

For each $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$, we define the functions α_1, α_2, α_3 and α_4 on $\mathbb{C}$ by

$$\begin{cases} \alpha_1(z) = \exp\left\{a_1 z + \frac{a_2 a_4}{a_3} t\left[\frac{1}{a_3}(e^{a_3 z} - 1) - z\right]\right\}, \\ \alpha_2(z) = \frac{a_2}{a_3}(e^{a_3 z} - 1), \\ \alpha_3(z) = a_3 z, \\ \alpha_4(z) = \frac{a_4}{a_3}(e^{a_3 z} - 1) \end{cases} \tag{6}$$

if $a_3 \neq 0$;

$$\begin{cases} \alpha_1(z) = \exp\left\{a_1 z + \frac{a_2 a_4}{2} z^2 t\right\}, \\ \alpha_2(z) = a_2 z, \\ \alpha_3(z) = 0, \\ \alpha_4(z) = a_4 z \end{cases} \tag{7}$$

if $a_3 = 0$. For each $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$, we also define a family of transforms $\{\mathfrak{K}_{\mathbf{a},t;z}\}_{z\in\mathbb{C}}$ by

$$\mathfrak{K}_{\mathbf{a},t;z} = \alpha_1(z)\mathfrak{J}_{\widetilde{\mathbf{a}},t;z} = \alpha_1(z) G_{t;\alpha_2(z),\alpha_3(z),\alpha_4(z)}, \qquad z \in \mathbb{C},$$

where $\widetilde{\mathbf{a}} = (a_2, a_3, a_4)$ and the functions $\alpha_1, \alpha_2, \alpha_3$ and α_4 are given as in (6) or (7). Then, by direct computations using (5), $\{\mathfrak{K}_{\mathbf{a},t;z}\}_{z\in\mathbb{C}}$ is a one-parameter subgroup of $GL(\mathcal{G})$.

Lemma 7 *For each $\widetilde{\mathbf{a}} = (a_2, a_3, a_4) \in \mathbb{C}^3$ and for any $\phi \in \mathcal{G}$, we have*

$$\lim_{z\to 0}\left\|\frac{\mathfrak{J}_{\widetilde{\mathbf{a}},t;z}\phi - \phi}{z} - (a_2 A_t + a_3 \Lambda_t + a_4 A_t^*)\phi\right\|_p = 0, \qquad p \in \mathbb{R}.$$

PROOF. Let $p \in \mathbb{R}$ and $\phi = (f_n) \in \mathcal{G}$ be given. Then by definition of $\mathfrak{J}_{\widetilde{\mathbf{a}},t;z}$, we have

$$\begin{aligned} &\frac{\mathfrak{J}_{\widetilde{\mathbf{a}},t;z}\phi - \phi}{z} - (a_2 A_t + a_3 \Lambda_t + a_4 A_t^*)\phi \\ &= \left(\frac{(e^{a_3 z \chi_{t]}})^{\otimes n} - 1}{z} f_n - a_3 n \left(\chi_{t]} \otimes I^{\otimes(n-1)} f_n\right)_{\mathrm{sym}}\right) \\ &\quad + \left(\frac{\alpha_4(z)}{z}\chi_{t]}\widehat{\otimes}\left((e^{a_3 z \chi_{t]}})^{\otimes n} f_n\right) - a_4 \chi_{t]}\widehat{\otimes} f_n\right) \\ &\quad + \left(\frac{(n+1)!}{n!}\left[\frac{\alpha_2(z)}{z}(e^{a_3 z \chi_{t]}})^{\otimes n} - a_2\right]\chi_{t]}\widehat{\otimes}_1 f_{n+1}\right) \\ &\quad + \left(\frac{1}{z} g_n\right), \end{aligned}$$

where

$$\begin{aligned} g_n = \sum_{l+m=n}\sum_{k+m\geq 2} &\frac{(l+k)!}{l!k!m!} \\ &\times \left(\alpha_4(z)\chi_{t]}\right)^{\otimes m}\widehat{\otimes}\left[(e^{a_3 z \chi_{t]}})^{\otimes l}\left((\alpha_2(z)\chi_{t]})^{\otimes k}\widehat{\otimes}_k f_{l+k}\right)\right]. \end{aligned}$$

Therefore, we obtain that

$$\left\|\frac{\mathfrak{J}_{\widetilde{\mathbf{a}},t;z}\phi - \phi}{z} - (a_2 A_t + a_3 \Lambda_t + a_4 A_t^*)\phi\right\|_p^2 \leq 4\sum_{j=1}^{4} I_j(z),$$

$$
\begin{aligned}
I_1(z) &= \sum_{n=0}^{\infty} n! e^{2pn} \left| \frac{(e^{a_3 z \chi_{t]}})^{\otimes n} - 1}{z} f_n - a_3 n \left(\chi_{t]} \otimes I^{\otimes(n-1)} f_n\right)_{\mathrm{sym}} \right|_0^2, \\
I_2(z) &= \sum_{n=0}^{\infty} n! e^{2pn} \left| \frac{(n+1)!}{n!} \left[\frac{\alpha_2(z)}{z} (e^{a_3 z \chi_{t]}})^{\otimes n} - a_2 \right] \chi_{t]} \widehat{\otimes}_1 f_{n+1} \right|_0^2, \\
I_3(z) &= \sum_{n=0}^{\infty} (n+1)! e^{2p(n+1)} \left| \frac{\alpha_4(z)}{z} \chi_{t]} \widehat{\otimes} \left((e^{a_3 z \chi_{t]}})^{\otimes n} f_n \right) - a_4 \chi_{t]} \widehat{\otimes} f_n \right|_0^2
\end{aligned}
$$

and

$$
I_4(z) = \sum_{n=0}^{\infty} n! e^{2pn} \left| \frac{1}{z} g_n \right|_0^2 .
$$

Then by simple modification of the proof of Proposition 5.4.5 in [23], we can easily see that $\lim_{z \to 0} I_1(z) = 0$. On the other hand, by similar arguments of those used in the proof Lemma 3.4 in [14], we see that $\lim_{z \to 0}(I_2(z) + I_3(z) + I_4(z)) = 0$. The proof follows. ∎

Theorem 8 *For each $t > 0$ and $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}$, $\{\mathfrak{K}_{\mathbf{a},t;z}\}_{z \in \mathbb{C}}$ is a holomorphic one-parameter subgroup of $GL(\mathcal{G})$ with the infinitesimal generator $a_1 I + a_2 A_t + a_3 \Lambda_t + a_4 A_t^*$.*

PROOF. Let $p \in \mathbb{R}$ and $\phi \in \mathcal{G}$ be given. Then we have

$$
\begin{aligned}
& \left\| \frac{\mathfrak{K}_{\mathbf{a},t;z}\phi - \phi}{z} - (a_1 I + a_2 A_t + a_3 \Lambda_t + a_4 A_t^*)\phi \right\|_p \\
\le & \left| \frac{\alpha_1(z) - 1}{z} - a_1 \right| \|\mathfrak{J}_{\tilde{\mathbf{a}},t;z}\phi\|_p + \|a_1 (\mathfrak{J}_{\tilde{\mathbf{a}},t;z} - I)\phi\|_p \\
& + \left\| \frac{\mathfrak{J}_{\tilde{\mathbf{a}},t;z}\phi - \phi}{z} - (a_2 A_t + a_3 \Lambda_t + a_4 A_t^*)\phi \right\|_p .
\end{aligned}
$$

From Lemma 7, the proof follows. ∎

Theorem 9 *The transform $G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ has the following representation:*

$$
G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \alpha_1 e^{\alpha_4 A_t^*} \circ e^{\alpha_3 \Lambda_t} \circ e^{\alpha_2 A_t}.
$$

PROOF. It can be easily shown that for any $\xi \in H$, we have

$$
G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4} \phi_\xi = \alpha_1 e^{\alpha_4 A_t^*} \circ e^{\alpha_3 \Lambda_t} \circ e^{\alpha_2 A_t} \phi_\xi .
$$

We note that $G_{t;\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ and $\alpha_1 e^{\alpha_4 A_t^*} \circ e^{\alpha_3 \Lambda_t} \circ e^{\alpha_2 A_t}$ are continuous linear operators on $\mathcal{G}$. Since $\{\phi_\xi : \xi \in H\}$ spans a dense subspace of $\mathcal{G}$, the proof follows. ∎

By similar arguments of those used in the proof of Lemma 5, $\{\mathfrak{K}_{a,t;z}; |z| < R\}$ is equicontinuous for any $R > 0$. Therefore, by Theorems 8 and 3, $a_1 I + a_2 A_t + a_3 \Lambda_t + a_4 A_t^*$ is an equicontinuous generator for each $t > 0$ and $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$. Hence by Theorem 9 we have

$$
e^{z(a_1 I + a_2 A_t + a_3 \Lambda_t + a_4 A_t^*)} = \alpha_1(z) e^{\alpha_4(z) A_t^*} \circ e^{\alpha_3(z) \Lambda_t} \circ e^{\alpha_2(z) A_t},
$$

where the functions $\alpha_1, \alpha_2, \alpha_3$ and α_4 are given as in (6) or (7).

For each $a, b \in \mathbb{C}$ and $t \in \mathbb{R}_+$, let $Q_{a,b}(t) = aA_t + bA_t^*$. Then by Theorems 8 and 3 we also see that $Q_{a,b}(t)$ is an equicontinuous generator.

5 Quantum Stochastic Processes

A family of operators $\{\Xi_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is called a *quantum stochastic process* if there exists $p, q \in \mathbb{R}$ such that $\Xi_t \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ for all $t \geq 0$ and for each $\phi \in \mathcal{G}_p$ the map $t \mapsto \Xi_t\phi$ is strongly measurable. A quantum stochastic process $\{\Xi_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ $(p \geq q)$ is said to be *adapted* if for each $t \geq 0$ there exists $\Xi_{t]} \in \mathcal{L}(\mathcal{G}_{p;t]}, \mathcal{G}_{q;t]})$ such that $\Xi_t = \Xi_{t]} \otimes I_{[t}$, where $I_{[t} : \mathcal{G}_{p;[t} \hookrightarrow \mathcal{G}_{q;[t}$ is the inclusion map.

For each $t \in \mathbb{R}_+$, the *conditional expectation* $\mathbf{E}_t$ (see [5], [25]) is defined by the second quantization operator $\Gamma(\chi_{t]})$ of $\chi_{t]}$, i.e., for each $t \in \mathbb{R}_+$

$$\mathbf{E}_t\Phi = (\chi_{t]}^{\otimes n} f_n), \qquad \Phi = (f_n) \in \mathcal{G}^*.$$

Then for any $p \in \mathbb{R}$ and $\Phi = (f_n) \in \mathcal{G}_p$, we have

$$\|\mathbf{E}_t\Phi\|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |\chi_{t]}^{\otimes n} f_n|_0^2 \leq \|\Phi\|_p^2.$$

Hence for any $p \in \mathbb{R}$ and $t \in \mathbb{R}_+$, $\mathbf{E}_t \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_p)$ and $\mathbf{E}_t$ is an orthogonal projection. Moreover, $\mathbf{E}_t \in \mathcal{L}(\mathcal{G}, \mathcal{G})$ and $\mathbf{E}_t \in \mathcal{L}(\mathcal{G}^*, \mathcal{G}^*)$.

An adapted process of operators $\{\Xi_t\}_{t\geq 0}$ in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ $(p \geq q)$ is called a *quantum martingale* if for any $0 \leq s \leq t$

$$\mathbf{E}_s \Xi_t \mathbf{E}_s = \mathbf{E}_s \Xi_s \mathbf{E}_s.$$

The processes $\{A_t\}_{t\geq 0}$, $\{A_t^*\}_{t\geq 0}$ and $\{\Lambda_t\}_{t\geq 0}$ defined in (2) are called the *annihilation*, *creation* and *number (or gauge)* processes, respectively. The quantum stochastic process $Q_t = Q_{1,1}(t) = A_t + A_t^*$ is called the *quantum Brownian motion* or the *position process*. For any non-negative integers l, m, the processes $\{(A_t^*)^l A_t^m\}_{t\geq 0}$ and $\{\Lambda_t^{\diamond l}\}_{t\geq 0}$ are quantum martingales, where $\diamond$ is the Wick product or normal-ordered product [7]. In particular, the annihilation process $\{A_t\}_{t\geq 0}$, the creation process $\{A_t^*\}_{t\geq 0}$ and the number process $\{\Lambda_t\}_{t\geq 0}$ are quantum martingales. These martingales are called the *basic martingales*. Also, the basic martingales and the time process are called the *basic processes*.

An adapted process $\{\Xi_t\}_{t\geq 0}$ is called a *regular semimartingale* in $\mathcal{L}(\mathcal{G}, \mathcal{G})$ if for any $p \geq 0$ there exists $q \geq p$ and an absolutely continuous measure m on $\mathbb{R}_+$ such that for any $r < s < t$ and $\phi \in \mathcal{G}_{q;r]}$, $\psi \in \mathcal{G}_{-p;r]}$

$$\begin{aligned}
\|(\Xi_t - \Xi_s)\phi\|_p^2 &\leq \|\phi\|_q^2 m([s,t]); \\
\|(\Xi_t^* - \Xi_s^*)\psi\|_{-q}^2 &\leq \|\psi\|_{-p}^2 m([s,t]); \\
\|(\mathbf{E}_s \Xi_t - \Xi_s)\phi\|_p &\leq \|\phi\|_q m([s,t]).
\end{aligned}$$

Let $L^2_{\mathrm{lb}}(\mathbb{R}_+)$ be the space of all locally bounded square integrable functions on $\mathbb{R}_+$ and $\mathcal{E}_{\mathrm{lb}}$ a dense subspace of $\mathcal{H}$ spanned by all exponential vectors ϕ_ξ, $\xi \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$.

The space $\mathcal{S}(\mathcal{G})$ of adapted process $\{\Xi_t\}_{t\geq 0}$ in $\mathcal{L}(\mathcal{G}, \mathcal{G})$ admitting the integral representation:

$$\Xi_t = \lambda I + \int_0^t (E d\Lambda + F dA + G dA^* + H ds)$$

on $\mathcal{E}_{\mathrm{lb}}$ with a $\lambda \in \mathbb{C}$ and adapted processes (E, F, G, H) in $\mathcal{L}(\mathcal{G}, \mathcal{G})$ satisfying that for any $p \geq 0$ there exists $q \geq p$ such that $s \mapsto \|F_s\|_{q;p}$ and $s \mapsto \|G_s\|_{q;p}$ are locally square integrable, $s \mapsto \|E_s\|_{q;p}$ is locally bounded and $s \mapsto \|H_s\|_{q;p}$ is locally integrable, where $\|\cdot\|_{q;p}$ is the operator norm on $\mathcal{L}(\mathcal{G}_q, \mathcal{G}_p)$.

Theorem 10 *An adapted process $\{\Xi_t\}_{t\geq 0}$ in $\mathcal{L}(\mathcal{G},\mathcal{G})$ is an element of $\mathcal{S}(\mathcal{G})$ if and only if Ξ is a regular semimartingale.*

For the proof, we refer to [16].

6 Functional Quantum Itô Formula

Let $\mathbf{L}_2(\mathcal{G})$ be the class of quadruples $\mathbf{F} = (E, F, G, H)$ of adapted processes in $\mathcal{L}(\mathcal{G},\mathcal{G})$ satisfying that for any $p \geq 0$ there exists $q \geq p$ such that $s \mapsto \|F_s\|_{q;p}$ and $s \mapsto \|G_s\|_{q;p}$ are locally square integrable, $s \mapsto \|E_s\|_{q;p}$ is locally bounded and $s \mapsto \|H_s\|_{q;p}$ is locally integrable.

Theorem 11 *Let $\{\Xi_t\}_{t\geq 0} \in \mathcal{S}(\mathcal{G})$ and $\{\Xi'_t\}_{t\geq 0} \in \mathcal{S}(\mathcal{G})$ with the following integral representations:*

$$\begin{aligned}
\Xi_t &= \int_0^t (Ed\Lambda + FdA + GdA^* + Hds), \\
\Xi'_t &= \int_0^t (E'd\Lambda + F'dA + G'dA^* + H'ds)
\end{aligned}$$

on $\mathcal{E}_{\rm lb}$ for some $\mathbf{F} \in \mathbf{L}_2(\mathcal{G})$ and $\mathbf{F}' \in \mathbf{L}_2(\mathcal{G})$, respectively. Then both integral representations can be extended to $\mathcal{G}$ and we have

$$\begin{aligned}
\Xi_t\Xi'_t &= \int_0^t (E\Xi'd\Lambda + F\Xi'dA + G\Xi'dA^* + H\Xi'ds) \\
&\quad + \int_0^t (\Xi E'd\Lambda + \Xi F'dA + \Xi G'dA^* + \Xi H'ds) \\
&\quad + \int_0^t (EE'd\Lambda + FE'dA + EG'dA^* + FG'ds). \qquad (8)
\end{aligned}$$

PROOF. By the similar arguments of those used in the proof of Theorems 6.1 and 6.2 in [16], the proof is straightforward. ∎

The equation (8) is sometimes written in the shorter differential form:

$$d(\Xi\Xi') = (d\Xi)\Xi' + \Xi(d\Xi') + (d\Xi)(d\Xi'), \qquad (9)$$

where

$$\begin{aligned}
(d\Xi)\Xi' &= E\Xi'd\Lambda + F\Xi'dA + G\Xi'dA^* + H\Xi'ds, \\
\Xi(d\Xi') &= \Xi E'd\Lambda + \Xi F'dA + \Xi G'dA^* + \Xi H'ds, \\
(d\Xi)(d\Xi') &= EE'd\Lambda + FE'dA + EG'dA^* + FG'ds.
\end{aligned}$$

From now on we consider $\Xi \in \mathcal{S}(\mathcal{G})$ with the integral representation:

$$\Xi_t = \Xi_0 + \int_0^t (Ed\Lambda + FdA + GdA^* + Hds).$$

By similar arguments of those used in the proof of Theorem 5 in [1] with Remark 7.4 in [16], we can easily prove that for each $p \geq 0$ there exists $q \geq 0$ the map $s \to \|\Xi_s\|_{q;p}$ is locally bounded. Therefore, by (8) we see that for each positive integer n, $\Xi^{n+1} \in \mathcal{S}(\mathcal{G})$ and we have

$$d(\Xi^{n+1}) = (d\Xi)\Xi^n + \Xi(d\Xi^n) + (d\Xi)(d\Xi^n). \tag{10}$$

It follows the following lemma. For the proof, see the proof of Lemma 4.1 in [30].

Lemma 12 *We have*

$$\Xi_t^n = \Xi_0^n + \int_0^t (E_n d\Lambda + F_n dA + G_n dA^* + H_n ds), \tag{11}$$

where

$$E_n = (\Xi + E)^n - \Xi^n, \; F_n = \sum_{i+j=n-1} \Xi^i F(\Xi + E)^j, \; G_n = \sum_{i+j=n-1} (\Xi + E)^i G \Xi^j$$

and

$$H_n = \sum_{i+j=n-1} \Xi^i H \Xi^j + \sum_{i+j+k=n-2} \Xi^i F(\Xi + E)^j G \Xi^k.$$

From Lemma 12 we have a quantum Itô formula for $p(\Xi)$, where p is a polynomial and $\Xi \in \mathcal{S}(\mathcal{G})$. Now we consider a quantum Itô formula for $f(\Xi)$, where f is an entire function on $\mathbb{C}$ satisfying certain condition.

Lemma 13 *Let* $\{a_n\}_{n=0}^{\infty} \subset \mathbb{C}$ *such that*

$$|a_n| \leq \frac{1}{n!} M R^n, \qquad n \geq 0$$

for some $M \geq 0$ *and* $R > 0$. *If* Ξ_t *and* $\Xi_t + E_t$ *are equicontinuous generators for all* $t \geq 0$, *then the series*

$$\sum_{n=0}^{\infty} a_n E_n, \quad \sum_{n=0}^{\infty} a_n F_n, \quad \sum_{n=0}^{\infty} a_n G_n, \quad \sum_{n=0}^{\infty} a_n H_n$$

converge in $\mathcal{L}(\mathcal{G}, \mathcal{G})$, *where* E_n, F_n, G_n *and* H_n *are given as in Lemma 12.*

PROOF. We will prove only that the series $\sum_{n=0}^{\infty} a_n H_n$ converges in $\mathcal{L}(\mathcal{G}, \mathcal{G})$ since the proofs of convergence of other series are very similar. For any $n \geq 0$ and $p \geq 0$, we have

$$\|H_n \phi\|_p \leq \left(\sum_{i+j=n-1} \|\Xi^i H \Xi^j \phi\|_p + \sum_{i+j+k=n-2} \|\Xi^i F(\Xi + E)^j G \Xi^k \phi\|_p \right)$$

By the equicontinuity of Ξ and the continuity of H, there exist $C_1, C_2, C_3 \geq 0$ and $q, r, s \geq 0$ such that for any $\epsilon > 0$

$$\begin{aligned}
\sum_{i+j=n-1} \|\Xi^i H \Xi^j \phi\|_p &\leq C_1 \sum_{i+j=n-1} \epsilon^i i! \|H \Xi^j \phi\|_q \\
&\leq C_1 C_2 \sum_{i+j=n-1} \epsilon^i i! \|\Xi^j \phi\|_{q+r} \\
&\leq C_1 C_2 C_3 \epsilon^{n-1} \left(\sum_{i+j=n-1} i! j! \right) \|\phi\|_s \\
&\leq C_1 C_2 C_3 \epsilon^{n-1} n! \|\phi\|_s.
\end{aligned}$$

Similarly, by the equicontinuity of $\Xi + E$ and the continuity of F and G, there exist $C \geq 0$ and $q \geq 0$ such that

$$\sum_{i+j+k=n-2} \|\Xi^i F(\Xi + E)^j G \Xi^k \phi\|_p \leq C\epsilon^{n-2} n! \|\phi\|_q.$$

Therefore, by choosing $\epsilon > 0$ such that $R\epsilon < 1$ we have

$$\sum_{n=0}^{\infty} \|a_n H_n \phi\|_p \leq MR(C_1C_2C_3 + RC) \left(\sum_{n=0}^{\infty} (R\epsilon)^n \right) \|\phi\|_{q'},$$

where $q' = s \vee q$. It proves that the series $\sum_{n=0}^{\infty} a_n H_n$ converges in $\mathcal{L}(\mathcal{G}, \mathcal{G})$. ∎

For the following theorem we assume that there exists $R > 0$ such that $\{(R\Xi_t)^n / n!;\ t \in K,\ n = 0, 1, 2, \cdots\}$ and $\{(R(\Xi_t + E_t))^n / n!;\ t \in K,\ n = 0, 1, 2, \cdots\}$ are equicontinuous families for any bounded interval $K \subset \mathbb{R}_+$, i.e., for any $p \geq 0$ there exists $C, C' \geq 0$ and $q, q' \geq 0$ such that for any bounded interval $K \subset \mathbb{R}_+$

$$\sup_{t \in K} \left\| \frac{(R\Xi_t)^n}{n!} \phi \right\|_p \leq C \|\phi\|_q, \quad \sup_{t \in K} \left\| \frac{(R(\Xi_t + E_t))^n}{n!} \phi \right\|_p \leq C' \|\phi\|_{q'}, \qquad \phi \in \mathcal{G}$$

for all $n = 0, 1, 2, \cdots$.

Theorem 14 *Let f be an entire function with Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \mathbb{C}$$

and $\Xi \in \mathcal{S}(\mathcal{G})$ admit the integral representation

$$\Xi_t = \Xi_0 + \int_0^t (E d\Lambda + F dA + G dA^* + H ds).$$

Assume that there exist $M \geq 0$ and $R > 0$ such that

$$|a_n| \leq \frac{1}{n!} M R^n, \qquad n \geq 0.$$

Then we have

$$f(\Xi_t) = f(\Xi_0) + \int_0^t (E' d\Lambda + F' dA + G' dA^* + H' ds), \tag{12}$$

where

$$E' = \sum_{n=0}^{\infty} a_n E_n, \quad F' = \sum_{n=0}^{\infty} a_n F_n, \quad G' = \sum_{n=0}^{\infty} a_n G_n, \quad H' = \sum_{n=0}^{\infty} a_n H_n.$$

PROOF. By similar arguments of those used in the proof of Lemma 13, we can easily prove that for any $p \geq 0$ there exists $q \geq p$ such that $s \mapsto \|E'_s\|_{q;p}$ is locally bounded, $s \mapsto \|F'_s\|_{q;p}$ and $s \mapsto \|G'_s\|_{q;p}$ are locally square integrable and $s \mapsto \|H'_s\|_{q;p}$ is locally integrable. Therefore, by Lemma 12, for all $t \geq 0$ and $\xi, \eta \in L^2_{\mathrm{lb}}(\mathbb{R}_+)$ we have

$$\begin{aligned}
\langle\langle f(\Xi_t)\phi_\xi, \phi_\eta\rangle\rangle &= \sum_{n=0}^{\infty} a_n \langle\langle \Xi_0^n \phi_\xi, \phi_\eta\rangle\rangle \\
&\quad + \sum_{n=0}^{\infty} a_n \int_0^t \langle\langle (\xi\eta E_n + \xi F_n + \eta G_n + H)\phi_\xi, \phi_\eta\rangle\rangle \, ds \\
&= \langle\langle f(\Xi_0)\phi_\xi, \phi_\eta\rangle\rangle \\
&\quad + \int_0^t \sum_{n=0}^{\infty} a_n \langle\langle (\xi\eta E_n + \xi F_n + \eta G_n + H)\phi_\xi, \phi_\eta\rangle\rangle \, ds,
\end{aligned}$$

where for the last equality we used the dominated convergence theorem. It follows the proof. ∎

Lemma 15 *For each $t \in \mathbb{R}_+$, let $\Xi_t \in \mathcal{L}(\mathcal{G}, \mathcal{G})$ be an equicontinuous generator and let $\{\Omega_{t;z}\}_{z\in\mathbb{C}}$ be the corresponding holomorphic one-parameter subgroup of $GL(\mathcal{G})$. Then the following conditions are equivalent:*

(i) *there exists $R > 0$ such that $\{(R\Xi_t)^n/n!;\ t \in K,\ n = 0, 1, 2, \cdots\}$ is equicontinuous for any bounded interval $K \subset \mathbb{R}_+$;*

(ii) *$\{\Omega_{t;z};\ t \in K,\ |z| < R\}$ is equicontinuous for some $R > 0$ and any bounded interval $K \subset \mathbb{R}_+$.*

PROOF. (i) $\Rightarrow$ (ii) By assumption

$$\Omega_{t;z}\phi = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Xi_t^n \phi, \qquad \phi \in \mathcal{G}, \quad |z| < R$$

converges in $\mathcal{G}$. Moreover, for any $\phi \in \mathcal{G}$ and $|z| < R' < R$ we have

$$\sup_{t\in K} \|\Omega_{t;z}\phi\|_p \leq C\left(\frac{R}{R - |z|}\right)\|\phi\|_q \leq C\left(\frac{R}{R - R'}\right)\|\phi\|_q.$$

(ii) $\Rightarrow$ (i) For each fixed $\Phi \in \mathcal{G}^*$ and $\phi \in \mathcal{G}$, we have

$$\langle\langle \Phi, \Xi_t^n \phi\rangle\rangle = \frac{n!}{2\pi i} \int_{|z|=r} \langle\langle \Phi, \Omega_{t;z}\phi\rangle\rangle \frac{dz}{z^{n+1}}, \qquad r > 0. \tag{13}$$

On the other hand, by assumption for any $p \geq 0$ there exists $C \geq 0$ and $q \geq 0$ such that for any bounded interval $K \subset \mathbb{R}_+$

$$\sup_{t\in K} \|\Omega_{t;z}\phi\|_p \leq C\|\phi\|_q, \qquad |z| < R, \quad \phi \in \mathcal{G}.$$

Therefore, by (13) we have

$$\begin{aligned}|\langle\!\langle \Phi,\, \Xi_t^n \phi \rangle\!\rangle| &\le \frac{n!}{2\pi}\int_{|z|=r} \|\Phi\|_{-p} \|\Omega_{t;z}\phi\|_p \frac{|dz|}{|z|^{n+1}} \\ &\le C\frac{n!}{r^n}\|\Phi\|_{-p}\|\phi\|_q, \qquad t\in K, \quad 0<r<R.\end{aligned}$$

It follows the proof. ∎

Let $t \in \mathbb{R}_+$ and $\mathbf{a}_t = (a_1 t, a_2, a_3, a_4) \in \mathbb{C}^4$. Then by direct computation, we see that $\{\mathfrak{K}_{\mathbf{a}_t,t;z}; t \in K, |z| < R\}$ is equicontinuous for any $R > 0$ and bounded interval $K \subset \mathbb{R}_+$. Therefore, by Lemma 15, $\{(R\Xi_t)^n/n!; t \in K, n = 0,1,2,\cdots\}$ is equicontinuous for any $R > 0$ and bounded interval $K \subset \mathbb{R}_+$, where $\Xi_t = a_1 t + a_2 A_t + a_3 \Lambda_t + a_4 A_t^*$.

Theorem 16 *Let $t \in \mathbb{R}_+$, $(a_1,a_2,a_3,a_4) \in \mathbb{C}^4$ and let $\Xi_t = a_1\Lambda_t + a_2 A_t + a_3 A_t^* + a_4 t$. Assume that $a_1 \neq 0$. Then e^{Ξ_t} is a regular semimartingale and*

$$\begin{aligned}e^{\Xi_t} &= I + \int_0^t \Bigg((e^{a_1}-1)\, e^{\Xi_s} d\Lambda_s + \frac{a_2}{a_1}(e^{a_1}-1)e^{\Xi_s} dA_s \\ &\quad + \frac{a_3}{a_1}(e^{a_1}-1)e^{\Xi_s} dA_s^* + \left(a_4 + \frac{a_2 a_3}{a_1}\left[\frac{1}{a_1}(e^{a_1}-1) - 1\right]\right) e^{\Xi_s} ds \Bigg).\end{aligned}$$

PROOF. By Theorem 14, we have

$$e^{\Xi_t} = I + \int_0^t (E' d\Lambda + F' dA + G' dA^* + H' ds), \tag{14}$$

where

$$E' = \sum_{n=0}^{\infty} \frac{1}{n!}[(\Xi + a_1)^n - \Xi^n] = e^{\Xi + a_1} - e^{\Xi} = (e^{a_1} - 1)\, e^{\Xi},$$

$$F' = a_2 \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{i+j=n} \Xi^i (\Xi + a_1)^j, \; G' = a_3 \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{i+j=n} (\Xi + a_1)^i \Xi^j$$

and

$$\begin{aligned}H' &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(a_4 n \Xi^{n-1} + a_2 a_3 \sum_{i+j+k=n-2} \Xi^i (\Xi + a_1)^j \Xi^k \right) \\ &= a_4 e^{\Xi} + a_2 a_3 \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \sum_{i+j=n} \Xi^i (\Xi + a_1)^j.\end{aligned}$$

By similar arguments of those used in the proof of Proposition 5.1 in [30], we see that

$$F' = \frac{a_2}{a_1}(e^{a_1} - 1)e^{\Xi}, \qquad G' = \frac{a_3}{a_1}(e^{a_1} - 1)e^{\Xi}$$

and

$$H' = \left(a_4 + \frac{a_2 a_3}{a_1}\left[\frac{1}{a_1}(e^{a_1} - 1) - 1\right]\right) e^{\Xi}.$$

This completes the proof from (14). ∎

The following result is immediate from Theorem 16.

Corollary 17 *Let $(a,b,c,d) \in \mathbb{C}^4$. Assume that $a \neq 0, -1$. Then*

$$\begin{aligned}\Xi_t &= \exp\left\{ \ln(a+1)\Lambda_t + \frac{b}{a}\ln(a+1)A_t \right.\\ &\quad \left. + \frac{c}{a}\ln(a+1)A_t^* + \left[\left(d - \frac{bc}{a}\right) + \frac{bc}{a^2}\ln(a+1)\right] t \right\}, \qquad t \in \mathbb{R}_+\end{aligned}$$

is a solution of the following quantum stochastic differential equation:

$$d\Xi_t = \Xi_t\left(a\Lambda_t + bA_t + cA_t^* + d\,dt\right), \qquad \Xi_0 = 1.$$

Theorem 18 *Let $a, b \in \mathbb{C}$ and let f be an entire function with Taylor expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \mathbb{C}.$$

Assume that there exist $M \geq 0$ and $R > 0$ such that

$$|a_n| \leq \frac{1}{n!} M R^n, \qquad n \geq 0.$$

Then we have

$$f(Q_{a,b}(t)) = f(0) + \int_0^t f'(Q_{a,b}(s))dQ_{a,b}(t) + \frac{ab}{2}\int_0^t f''(Q_{a,b}(s))ds.$$

PROOF. Since $Q_{a,b}(t) = \int_0^t (adA_s + bdA_s^*)$, by Lemma 12 we see that

$$E_n = 0, \qquad F_n = anQ_{a,b}^{n-1}, \qquad G_n = bnQ_{a,b}^{n-1}$$

and

$$H_n = ab \sum_{\alpha+\beta+\gamma=n-2} Q_{a,b}^{\alpha+\beta+\gamma} = \frac{ab}{2} n(n-1) Q_{a,b}^{n-2}$$

for each $n = 1, 2, \cdots$. Hence by Theorem 14 we complete the proof. ∎

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