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White Noise Analysis Based on the Lévy Laplacian

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Abstract

Eigenfunctions of the Lévy Laplacian with an arbitrary complex number as an eigenvalue are constructed by means of a coordinate change of white noise distributions. The Lévy Laplacian is diagonalized on the direct integral Hilbert space of such eigenfunctions and the corresponding equi-continuous semigroup is obtained. Moreover, an infinite dimensional stochastic process related to the Lévy Laplacian is constructed from some one-dimensional stochastic process.

1. Introduction

The Lévy Laplacian Δ_L, an infinite dimensional Laplacian introduced by P. Lévy [21], has recently attracted much attention for its peculiar and unexpected characters found in essentially infinite dimensional analysis. For example, harmonic functions with respect to the Lévy Laplacian are related to solutions of the Yang-Mills equations [2]; solutions of the heat equation associated with the Lévy Laplacian is obtained from normal-ordered white noise differential equations with quadratic white noises [26]. In this paper we focus on infinite dimensional stochastic processes associated with the Lévy Laplacian formulated so as to act on a new Hilbert space of functions on a Gaussian space.

There are several natural formulations of the Lévy Laplacian. Originally P. Lévy [21] defined Δ_L as an operator acting on functions on the Hilbert space $L^2(0,1)$, see also [5, 27]. However, for several reasons it seems more natural to consider functions on a Gaussian space or on a nuclear space. Throughout this paper we fix a Gelfand triple:

$$E \equiv \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) \equiv E^*$$

and consider a Gaussian space (E^*, μ), where μ is the Gaussian measure on E^*. It is well known that a certain class of functions on the nuclear space E is characterized as the image of the S-transform of "generalized" functions on the Gaussian space, which are constructed from a Gelfand triple $(E) \subset L^2(E^*, \mu) \subset (E)^*$. Then, taking a complete orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset E$ for $L^2(T)$ with a fixed finite interval $T \subset \mathbb{R}$, we define the Lévy Laplacian by

$$S[\Delta_L \Phi](\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(\zeta_n, \zeta_n), \qquad \Phi \in (E)^*,$$

whenever the limit exists, for a precise definition see Section 3. As the Lévy Laplacian vanishes on $L^2(E^*, \mu)$, its natural domain is to be found in generalized white noise functions (white noise distributions). Along this idea the Lévy Laplacian was first introduced in white noise analysis by T. Hida [6] and has been discussed by many authors, see e.g., [9, 16, 18] for general properties, [4] for Cauchy problems, [7] for related functional equations, [11] for a relation to an infinite dimensional Fourier transform, [29, 30] for a connection with the Itô formula, [31, 32] for stochastic processes generated by powers of the Lévy Laplacian. While, it is also possible to formulate the Lévy Laplacian independent of the Gaussian space [1, 19, 23].

In this paper, extending the ideas in the previous works [20, 22, 34, 35, 36], we introduce a new class of Hilbert spaces based on eigenfunctions of the Lévy Laplacian. In fact, for some continuous function h and any $\lambda \in \mathbb{R}$ we construct a subspace $\mathbf{D}_\lambda^h \subset (E)_{-p} \subset (E)^*$, $p > 5/12$, consisting of eigenfunctions of the Lévy Laplacian with eigenvalue $h(\lambda)$. Those eigenfunctions are constructed by means of an "exponential coordinate change" for white noise functions. Then, for some continuous function h and for any $p > 5/12, N \in \mathbb{N}$ the direct integral Hilbert space:

$$\mathcal{E}_{-p,N}^h = \int_{\mathbb{R}}^{\oplus} \mathcal{D}_{\lambda,-p}^h \, \alpha_N^h(\lambda) \, d\lambda,$$

where $\mathcal{D}_{\lambda,-p}^h$ is the completion of $\mathbf{D}_\lambda^h$ in $(E)_{-p}$ and $\alpha_N^h(\lambda)$ is a certain weight function, becomes a natural domain of the Lévy Laplacian. Thus the Lévy Laplacian is diagonalized:

$$\Delta_L = \int_{\mathbb{R}}^{\oplus} h(\lambda) \, \alpha_N^h(\lambda) \, d\lambda$$

and, thereby, an associated equi-continuous semigroup $\{G_t^h\}$ of class (C_0) is obtained (Theorem 3.4). This idea traces back to [31]. Finally, we obtain a stochastic expression of $\{G_t^h\}$ in terms of an E-valued stochastic process derived from a one-dimensional stochastic process with the function h (Theorem 4.1). It is noteworthy that the stochastic process generated by the Lévy Laplacian depends on the choice of eigenfunctions of the Laplacian.

2. Preliminaries

In this section we assemble some basic notations of white noise analysis following [9, 15, 18, 24].

We take the space $E^* \equiv \mathcal{S}'(\mathbb{R})$ of tempered distributions with the standard Gaussian measure μ which satisfies

$$\int_{E^*} \exp\{i\langle x, \xi\rangle\} \, d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi \in E \equiv \mathcal{S}(\mathbb{R}),$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^* \times E$ and $|\cdot|_0$ is the $L^2(\mathbb{R})$-norm.

Let $A = -(d/du)^2 + u^2 + 1$. This is a densely defined self-adjoint operator on $L^2(\mathbb{R})$ and there exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for $L^2(\mathbb{R})$ such that $Ae_\nu = 2(\nu+1)e_\nu$. We define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in E$ and $p \in \mathbb{R}$, and let E_p be the completion of E with respect to the norm $|\cdot|_p$. Then E_p is a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space E_p' of E_p is the same as E_{-p} (see [13]). The space E is the projective limit space of $\{E_p; p \geq 0\}$ and E^* is the inductive limit space of $\{E_{-p}; p \geq 0\}$. Then E becomes a nuclear space with the Gel'fand triple $E \subset L^2(\mathbb{R}) \subset E^*$. We denote the complexifications of $L^2(\mathbb{R})$, E and E_p by $L^2_{\mathbb{C}}(\mathbb{R})$, $E_{\mathbb{C}}$ and $E_{p,\mathbb{C}}$, respectively.

The space $(L^2) = L^2(E^*, \mu)$ of complex-valued square-integrable functionals defined on E^* admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbb{N}$ and $H_0 = \mathbb{C}$. Let $L^2_{\mathbb{C}}(\mathbb{R})^{\hat{\otimes} n}$ denote the n-fold symmetric tensor product of $L^2_{\mathbb{C}}(\mathbb{R})$. If $\varphi \in (L^2)$ has the representation $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$, $f_n \in L^2_{\mathbb{C}}(\mathbb{R})^{\hat{\otimes} n}$, then the (L^2)-norm $\|\varphi\|_0$ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n! |f_n|_0^2 \right)^{1/2},$$

where $|\cdot|_0$ is the $L^2_{\mathbb{C}}(\mathbb{R})^{\hat{\otimes} n}$-norm.

For $p \in \mathbb{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p \varphi\|_0$, where $\Gamma(A)$ is the second quantization operator of A. If $p \geq 0$, let $(E)_p$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(E)_p$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(E)_p$, $p \in \mathbb{R}$, is a Hilbert space with the norm $\|\cdot\|_p$. It is easy to see that for $p > 0$, the dual space $(E)_p^*$ of $(E)_p$ is given by $(E)_{-p}$. Moreover, for any $p \in \mathbb{R}$, we have the decomposition

$$(E)_p = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in E_{\mathbb{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $E_{\mathbb{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of $E_{\mathbb{C}}$. We also have $H_n^{(p)} = \{\mathbf{I}_n(f); f \in E_{p,\mathbb{C}}^{\hat{\otimes} n}\}$ for any $p \in \mathbb{R}$, where $E_{p,\mathbb{C}}^{\hat{\otimes} n}$ is also the n-fold symmetric tensor product of $E_{p,\mathbb{C}}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)_p$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n! |f_n|_p^2 \right)^{1/2}, \quad f_n \in E_{p,\mathbb{C}}^{\hat{\otimes} n},$$

where the norm on $E_{p,\mathbb{C}}^{\hat{\otimes} n}$ is denoted also by $|\cdot|_p$.

The projective limit space (E) of spaces $(E)_p$, $p \in \mathbb{R}$ is a nuclear space. The inductive limit space $(E)^*$ of spaces $(E)_p, p \in \mathbb{R}$, is nothing but the strong dual space of (E). The space $(E)^*$ is called the space of *generalized white noise functionals*. We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the canonical bilinear form on $(E)^* \times (E)$. Then we have

$$\langle\langle\Phi,\varphi\rangle\rangle = \sum_{n=0}^{\infty} n!\langle F_n, f_n\rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (E)^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)$, where the canonical bilinear form on $(E_{\mathbb{C}}^{\otimes n})^* \times (E_{\mathbb{C}}^{\otimes n})$ is denoted also by $\langle\cdot,\cdot\rangle$.

Since $\phi_\xi(\cdot) \equiv \exp\left(\langle\cdot,\xi\rangle - \frac{1}{2}\langle\xi,\xi\rangle\right) \in (E)$, we can define the *S-transform* on $(E)^*$ by

$$S[\Phi](\xi) = \langle\langle\Phi,\phi_\xi\rangle\rangle, \qquad \xi \in E_{\mathbb{C}}.$$

A complex-valued function F on $E_{\mathbb{C}}$ is called a *U-functional* if for every $\xi,\eta \in E_{\mathbb{C}}$, the function $z \to F(\xi + z\eta)$, $z \in \mathbb{C}$, is an entire function of z and there exist non-negative constants K, a and p such that

$$|F(\xi)| \leq K\exp\{a|\xi|_p^2\}, \quad \xi \in E_{\mathbb{C}}.$$

Theorem 2.1(see e.g. [9, 18, 24, 28]) *A complex-valued function F on $E_{\mathbb{C}}$ is the S-transform of an element in $(E)^*$ if and only if F is a U-functional.*

3. A semigroup generated by the Lévy Laplacian

Let $F \in S[(E)^*]$. Then, by Theorem 2.1, we see that for any $\xi,\eta \in E_{\mathbb{C}}$ the function $F(\xi + z\eta)$ is an entire function of $z \in \mathbb{C}$. Hence we have the series expansion:

$$F(\xi + z\eta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} F^{(n)}(\xi)(\eta,\dots,\eta),$$

where $F^{(n)}(\xi) : E_{\mathbb{C}} \times \cdots \times E_{\mathbb{C}} \to \mathbb{C}$ is a continuous n-linear functional.

We fix a finite interval T of $\mathbb{R}$. Take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset E$ for $L^2(T)$ satisfying the equal density and uniform boundedness property (see e.g., [9, 18, 19, 23, 30]). Let D_L denote the set of all $\Phi \in (E)^*$ such that the limit

$$\widetilde{\Delta_L} S[\Phi](\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(\zeta_n,\zeta_n)$$

exists for any $\xi \in E_{\mathbb{C}}$ and is in $S[(E)^*]$. The Lévy Laplacian Δ_L is defined by

$$\Delta_L\Phi = S^{-1}\widetilde{\Delta_L}S\Phi$$

for $\Phi \in D_L$. We denote by D_L^T the set of all functionals $\Phi \in D_L$ such that $S[\Phi](\eta) = 0$ for all $\eta \in E$ with $\mathrm{supp}(\eta) \subset T^c$.

Lemma 3.1. *For $n \geq 1, a_1,\dots,a_n \in \mathbb{C}$ and $f \in L_{\mathbb{C}}^1(T^n)$, let*

$$F(\xi) = \int_{T^n} f(u_1,\dots,u_n)e^{a_1\xi(u_1)+\cdots+a_n\xi(u_n)}\,d\mathbf{u}, \xi \in E_{\mathbb{C}} \tag{3.1}$$

where $d\mathbf{u} = du_1 \cdots du_n$. *Then there exists* $\Phi \in (E)^*$ *such that* $S[\Phi] = F$ *and* Φ *is in* $(E)_{-p}$ *for all* $p > \frac{5}{12}$.

Proof. We can estimate $|F(\xi)|$ as follows:

$$\begin{aligned} |F(\xi)| &\leq \int_{T^n} |f(u_1, \ldots, u_n)| e^{|a_1||\xi(u_1)| + \cdots + |a_n||\xi(u_n)|} d\mathbf{u} \\ &\leq |f|_{L^1} \max\left\{ e^{|a_1||\xi(u_1)| + \cdots + |a_n||\xi(u_n)|}; u_1, \ldots, u_n \in T \right\} \\ &\leq |f|_{L^1} e^{(|a_1| + \cdots + |a_n|)|\xi|_\infty}, \end{aligned}$$

where $|f|_{L^1}$ is the $L^1_{\mathbb{C}}(T^n)$-norm of f and $|\xi|_\infty = \max\{|\xi(u)|; u \in \mathbb{R}\}$. Since for any $p > \frac{5}{12}$ there exists a constant $M_p > 0$ such that $|\xi|_\infty \leq M_p |\xi|_p$ for all $\xi \in E_{\mathbb{C}}$ (see the next Remark), we get

$$|F(\xi)| \leq |f|_{L^1} e^{(|a_1| + \cdots + |a_n|) M_p |\xi|_p}.$$

Hence we have

$$|F(\xi)| \leq |f|_{L^1} e^{2^{-1}[(|a_1| + \cdots + |a_n|)^2 + M_p^2 |\xi|_p^2]}.$$

Similarly we use the same argument as above to show that

$$\begin{aligned} &\sum_{\nu_1, \ldots, \nu_n = 0}^{\infty} \prod_{j=1}^{n} \frac{(|a_j||z|)^{\nu_j}}{\nu_j!} \int_{T^n} |f(u_1, \ldots, u_n)| \prod_{j=1}^{n} \left(|\eta(u_j)|^{\nu_j} \left| e^{a_j \xi(u_j)} \right| \right) d\mathbf{u} \\ \leq \; &\sum_{\nu_1, \ldots, \nu_n = 0}^{\infty} \prod_{j=1}^{n} \frac{(|a_j||z||\eta|_\infty)^{\nu_j}}{\nu_j!} |f|_{L^1} e^{(|a_1| + \cdots + |a_n|)|\xi|_\infty} \\ \leq \; &|f|_{L^1} e^{(|a_1| + \cdots + |a_n|)(|\xi|_\infty + |z||\eta|_\infty)} < \infty. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} F(\xi + z\eta) &= \sum_{\nu_1, \ldots, \nu_n = 0}^{\infty} \int_{T^n} f(u_1, \ldots, u_n) \prod_{j=1}^{n} \left(\frac{(a_j z)^{\nu_j}}{\nu_j!} \eta(u_j)^{\nu_j} e^{a_j \xi(u_j)} \right) d\mathbf{u} \\ &= \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!} F_\ell(\xi; \eta), \end{aligned}$$

where

$$F_\ell(\xi; \eta) = \int_{T^n} f(u_1, \ldots, u_n)(a_1 \eta(u_1) + \cdots + a_n \eta(u_n))^\ell \prod_{j=1}^{n} e^{a_j \xi(u_j)} d\mathbf{u}.$$

This implies that $F(\xi + z\eta)$ is an entire function of $z \in \mathbb{C}$ for each $\xi, \eta \in E_{\mathbb{C}}$. Thus the assertion follows from Theorem 2.1. □

Remark. (see e.g. [25]) We have $|\delta_t|_{-p}^2 = \sum_{j=0}^{\infty} e_j(t)^2 (2j+2)^{-2p}$. Since $\max_t |e_j(t)| = O(j^{-1/12})$ (see e.g. [12]), we can check that the above series converges if $p > \frac{5}{12}$. Therefore, for any $p > \frac{5}{12}$ there exists a constant $M_p > 0$ such that $|\delta_t|_{-p} \leq M_p$ for all $t \in \mathbb{R}$.

Using the Wick ordering $: \cdot :$, we can write a generalized white noise functional Φ whose S-transform is given as in (3.1) by

$$\Phi(x) = \int_{T^n} f(u_1, \ldots, u_n) : \prod_{\nu=1}^{n} e^{a_\nu x(u_\nu)} : d\mathbf{u}. \tag{3.2}$$

The functional Φ belongs to D_L^T and is important as an eigenfunction of the operator Δ_L. In fact, we have the following result.

Theorem 3.2. [34] *A generalized white noise functional Φ as in (3.2) satisfies the equation*

$$\Delta_L \Phi = \frac{1}{|T|}\left(\sum_{\nu=1}^{n} a_\nu^2\right)\Phi. \tag{3.3}$$

Let $\mathcal{F}$ be the set of all complex-valued continuous functions h satisfying the following conditions:

1) $h(0) = 0$,

2) there exists a stochastic process $\{X_t; t \geq 0\}$ such that $e^{th(z)} = E[e^{izX_t}]$ for all $t \geq 0$ and $z \in \mathbb{R}$,

3) $\mathcal{A}_{\lambda,n}^h \equiv \left\{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n; \sum_{\nu=1}^n a_\nu = \sqrt{|T|}\lambda, \sum_{\nu=1}^n a_\nu^2 = |T|h(\lambda)\right\} \neq \phi$ for all $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

For example, a function $h(z) = -|z|^\gamma, 1 \leq \gamma \leq 2$, is an element of $\mathcal{F}$.

For each $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$, and $h \in \mathcal{F}$, let

$$\begin{aligned}\mathbf{D}_{\lambda,n}^h &= LS\Big\{\int_{T^n} f(\mathbf{u}) :\prod_{\nu=1}^n e^{a_\nu x(u_\nu)}: d\mathbf{u}; \\ &\qquad f \in E_{\mathbb{C}}^{\hat{\otimes} n},\ (a_1, a_2, \ldots, a_n) \in \mathcal{A}_{\lambda,n}^h\Big\},\end{aligned}$$

where LS means the linear span.

From now on, $h \in \mathcal{F}$ will be an arbitrarily fixed function. Set $\mathbf{D}_{\lambda,0}^h = \mathbb{C}$ and let

$$\mathbf{D}_\lambda^h = LS\left\{\mathbf{D}_{\lambda,n}^h; n \in \mathbb{N}\right\}.$$

Then $\mathbf{D}_\lambda^h$ is a linear subspace of $(E)_{-p}$ for all $p > \frac{5}{12}$ by Lemma 3.1, and Δ_L is a linear operator from $\mathbf{D}_\lambda^h$ into itself such that $\Delta_L\Phi = h(\lambda)\Phi$ for any $\Phi \in \mathbf{D}_\lambda^h$. For convenience, we will use the following notation

$$\mathbf{J}_{\mathbf{a}}[f](x) = \int_{T^n} f(\mathbf{u}) :\prod_{\nu=1}^n e^{a_\nu x(u_\nu)}: d\mathbf{u}.$$

Let $p > \frac{5}{12}$ be a number arbitrarily fixed. Define a space $\mathcal{D}_{\lambda,-p}^h$ by the completion of $\mathbf{D}_\lambda^h$ in $(E)_{-p}$ with respect to $\|\cdot\|_{-p}$. Then for each $n \in \mathbb{N} \cup \{0\}$, $\mathcal{D}_{\lambda,-p}^h$ becomes a Hilbert space with the inner product of $(E)_{-p}$ and the Lévy Laplacian Δ_L becomes a continuous linear operator from $\mathcal{D}_{\lambda,-p}^h$ into itself satisfying

$$\Delta_L\Phi = h(\lambda)\Phi \quad \text{for any } \Phi \in \mathcal{D}_{\lambda,-p}^h.$$

The Lévy Laplacian Δ_L is a self-adjoint operator on $\mathcal{D}^h_{\lambda,-p}$ for each $\lambda \in \mathbb{R}$ and $p > \frac{5}{12}$.

Proposition 3.3. (cf. [34]) *Let $\Phi = \int_{\mathbb{R}} \Phi_\lambda d\lambda$ and $\Psi = \int_{\mathbb{R}} \Psi_\lambda d\lambda$ be generalized white noise functionals such that Φ_λ and Ψ_λ are in $\mathcal{D}^h_{\lambda,-p}$ for each $\lambda \in \mathbb{R}$ and strongly mesurable in λ. If $\Phi = \Psi$ in $(E)^*$, then $\Phi_\lambda = \Psi_\lambda$ in $(E)^*$ for almost all $\lambda \in \mathbb{R}$.*

Proof. Let $\mathcal{A}^h_\lambda = \bigcup_n \mathcal{A}^h_{\lambda,n}$. Then, for almost all $\lambda \in \mathbb{R}$, Φ_λ and Ψ_λ can be expressed in the forms:

$$\Phi_\lambda = \lim_{N\to\infty} \sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} \mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}}], \qquad \Psi_\lambda = \lim_{N\to\infty} \sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} \mathbf{J}_{\mathbf{a}^{[N]}}[g_{\mathbf{a}^{[N]}}],$$

where $\sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda}$ means a sum of finitely many terms on $\mathbf{a}^{[N]} \in \mathcal{A}^h_\lambda$. Suppose $\Phi = \Psi$ in $(E)^*$. Then, taking the S-transform, we have

$$\int_{\mathbb{R}} \lim_{N\to\infty} \sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} S\Big(\mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}} - g_{\mathbf{a}^{[N]}}]\Big)(\xi)\, d\lambda = 0$$

for all $\xi \in E_{\mathbb{C}}$. Take $\xi_T \in E_{\mathbb{C}}$ such that $\xi_T = |T|^{-1/2}$ on T and put $\xi = a\xi_T + \eta$ with $a \in \mathbb{C}$ and $\eta \in E_{\mathbb{C}}$. Then we get

$$\int_{\mathbb{R}} e^{\lambda a} \lim_{N\to\infty} \sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} S\Big(\mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}} - g_{\mathbf{a}^{[N]}}]\Big)(\eta)\, d\lambda = 0$$

for all $a \in \mathbb{C}$ and $\eta \in E_{\mathbb{C}}$. Therefore,

$$\lim_{N\to\infty} \sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} S\Big(\mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}} - g_{\mathbf{a}^{[N]}}]\Big)(\eta) = 0$$

for almost all $\lambda \in \mathbb{R}$ and all $\eta \in E_{\mathbb{C}}$. This implies that $\Phi_\lambda = \Psi_\lambda$ in $(E)^*$ for almost all $\lambda \in \mathbb{R}$. □

For any $N \in \mathbb{N}, p > \frac{5}{12}$, we can define a space $\mathcal{E}^h_{-p,N}$ by the direct integral space $\int^{\oplus}_{\mathbb{R}} \mathcal{D}^h_{\lambda,-p}\alpha^h_N(\lambda)d\lambda$:

$$\mathcal{E}^h_{-p,N} = \left\{(\Phi_\lambda)_\lambda;\ \int_{\mathbb{R}} \|\Phi_\lambda\|^2_{-p}\alpha^h_N(\lambda)\, d\lambda < \infty,\ \Phi_\lambda \in \mathcal{D}^h_{\lambda,-p}\ \forall\lambda \in \mathbb{R}\right\},$$

where $\alpha^h_N(\lambda)$ is given by

$$\alpha^h_N(\lambda) = \sum_{\ell=0}^{N} |h(\lambda)|^{2\ell}. \tag{3.4}$$

Define a norm $|||\cdot|||_{-p,N}$ on $\mathcal{E}^h_{-p,N}$ by

$$|||\Phi|||_{-p,N} = \left(\int_{\mathbb{R}} \|\Phi_\lambda\|^2_{-p}\alpha^h_N(\lambda)\, d\lambda\right)^{1/2}, \quad \Phi = (\Phi_\lambda)_\lambda \in \mathcal{E}^h_{-p,N}.$$

Then the space $\mathcal{E}^h_{-p,N}$ is a Hilbert space with the norm $|||\cdot|||_{-p,N}$ for each $N \in \mathbb{N}$ and $p > \frac{5}{12}$.

Proposition 3.3 implies that $\int_{\mathbb{R}} \Phi_\lambda \, d\lambda$ with $\Phi_\lambda \in \mathcal{D}^h_{\lambda,-p}$ is uniquely determined as an element of $(E)^*$. We note that $\mathcal{E}^h_{-p,N}$ is isomorphic to a Hilbert space

$$\mathbf{E}^h_{-p,N} = \left\{ \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in (E)^*; \int_{\mathbb{R}} \|\Phi_\lambda\|^2_{-p} \alpha^h_N(\lambda) \, d\lambda < \infty, \, \Phi_\lambda \in \mathcal{D}^h_{\lambda,-p} \, \forall \lambda \in \mathbb{R} \right\},$$

with the norm induced from $\mathcal{E}^h_{-p,N}$ by a bijection

$$(\Phi_\lambda)_\lambda \to \int_{\mathbb{R}} \Phi_\lambda d\lambda.$$

We denote the norm on $\mathbf{E}^h_{-p,N}$ by the same notation $|||\cdot|||_{-p,N}$.

Put $\mathbf{E}^h_{-p,\infty} = \bigcap_{N\geq 1} \mathbf{E}^h_{-p,N}$ with the projective limit topology. Then, for any $N \geq 1$, we have the following inclusion relations:

$$\mathbf{E}^h_{-p,\infty} \subset \mathbf{E}^h_{-p,N+1} \subset \mathbf{E}^h_{-p,N} \subset \mathbf{E}^h_{-p,1} \subset (E)_{-p}.$$

The Laplacian Δ_L can be defined on $\mathbf{E}^h_{-p,2}$ and is a continuous linear operator from $\mathbf{E}^h_{-p,2}$ into $\mathbf{E}^h_{-p,1}$ satisfying $|||\Delta_L\Phi|||_{-p,N} \leq |||\Phi|||_{-p,N+1}$ for all $\Phi \in \mathbf{E}^h_{-p,N+1}$ and $N \in \mathbb{N}$. Any restriction of Δ_L is also denoted by the same notation Δ_L.

Let $h \in \mathcal{F}$. For each $t \geq 0$ we consider an operator G^h_t on $\mathbf{E}^h_{-p,\infty}$ defined by

$$G^h_t \Phi = \int_{\mathbb{R}} e^{th(\lambda)} \Phi_\lambda \, d\lambda$$

for $\Phi = \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in \mathbf{E}^h_{-p,\infty}$. Then we have the following:

Theorem 3.4. *For each $h \in \mathcal{F}$ the family $\{G^h_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by Δ_L as a continuous linear operator defined on $\mathbf{E}^h_{-p,\infty}$.*

Proof. Since $e^{th(z)}$ ia a characteristic function, we have $Reh(z) \leq 0$. For any $t \geq 0$, $p > \frac{5}{12}$ and $N \in \mathbb{N}$, the norm $|||G^h_t\Phi|||_{-p,N}$ for $\Phi = \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in \mathbf{E}^h_{-p,\infty}, \Phi_\lambda \in \mathcal{D}^h_{\lambda,-p}, n = 0, 1, 2, \ldots$ can be estimated as follows:

$$\begin{aligned} |||G^h_t\Phi|||^2_{-p,N} &= \int_{\mathbb{R}} \left\| e^{th(\lambda)}\Phi_\lambda \right\|^2_{-p} \alpha^h_N(\lambda) \, d\lambda \\ &\leq \int_{\mathbb{R}} \|\Phi_\lambda\|^2_{-p} e^{2tReh(\lambda)} \alpha^h_N(\lambda) \, d\lambda \\ &= |||\Phi|||^2_{-p,N}. \end{aligned}$$

Hence the family $\{G^h_t; t \geq 0\}$ is equi-continuous in t. It is easily checked that $G^h_0 = I, G^h_t G^h_s = G^h_{t+s}$ for each $t, s \geq 0$. We can also estimate that

$$\begin{aligned} |||G^h_t\Phi - G^h_{t_0}\Phi|||^2_{-p,N} &= \int_{\mathbb{R}} \left| e^{th(\lambda)} - e^{t_0 h(\lambda)} \right|^2 \|\Phi_n\|^2_{-p} \alpha^h_N(\lambda) \, d\lambda \\ &\leq 4 \int_{\mathbb{R}} \|\Phi_\lambda\|^2_{-p} \alpha^h_N(\lambda) \, d\lambda \\ &= 4|||\Phi|||^2_{-p,N} < \infty \end{aligned}$$

for each $t, t_0 \geq 0, N \in \mathbb{N}$ and $\Phi = \int_{\mathbb{R}} \Phi_\lambda d\lambda \in \mathbf{E}^h_{-p,\infty}$. Therefore, by the Lebesgue dominated convergence theorem, we get

$$\lim_{t \to t_0} G^h_t \Phi = G^h_{t_0} \Phi \quad \text{in} \quad \mathbf{E}^h_{-p,\infty}$$

for each $t_0 \geq 0$ and $\Phi \in \mathbf{E}^h_{-p,\infty}$. Thus the family $\{G^h_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0). We next prove that the infinitesimal generator of the semigroup is given by Δ_L. For any $N \in \mathbb{N}$ and $p > \frac{5}{12}$, we see that

$$\left|\left|\left| \frac{G^h_t \Phi - \Phi}{t} - \Delta_L \Phi \right|\right|\right|^2_{-p,N}$$

$$= \int_{\mathbb{R}} \left|\left| \frac{e^{th(\lambda)} - 1}{t} \Phi_\lambda - h(\lambda)\Phi_\lambda \right|\right|^2_{-p} \alpha^h_N(\lambda)\, d\lambda. \tag{3.5}$$

Since $\Phi = \int_{\mathbb{R}} \Phi_\lambda \, d\lambda \in \mathbf{E}^h_{-p,\infty}$, we have

$$\int_{\mathbb{R}} ||\Phi_\lambda||^2_{-p} \alpha^h_{N+1}(\lambda)\, d\lambda < \infty. \tag{3.6}$$

By the mean value theorem, for any $t > 0$ there exists a constant $\theta \in (0,1)$ such that

$$\left| \frac{e^{th(\lambda)} - 1}{t} \right| = |h(\lambda)|\, e^{t\theta Re h(\lambda)} \leq |h(\lambda)|\,.$$

Therefore we can estimate each term in (3.5) as follows:

$$\begin{aligned} & \alpha^h_N(\lambda) \left|\left| \frac{e^{th(\lambda)} - 1}{t} \Phi_\lambda - h(\lambda)\Phi_\lambda \right|\right|^2_{-p} \\ = \; & \alpha^h_N(\lambda) \left| \frac{e^{th(\lambda)} - 1}{t} - h(\lambda) \right|^2 ||\Phi_\lambda||^2_{-p} \\ \leq \; & 4\alpha^h_{N+1}(\lambda) ||\Phi_\lambda||^2_{-p}. \end{aligned}$$

Note that

$$\lim_{t \to 0} \left| \frac{e^{th(\lambda)} - 1}{t} - h(\lambda) \right| = 0.$$

Thus by (3.6) we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{t \to 0} \left|\left|\left| \frac{G^h_t \Phi - \Phi}{t} - \Delta_L \Phi \right|\right|\right|^2_{-p,N} = 0.$$

Hence the proof is completed. □

Remark: For each $N \in \mathbb{N}$, we can write Δ_L and G^h_t acting on $\mathcal{E}^h_{-p,N}$ as

$$\Delta_L = \int^{\oplus}_{\mathbb{R}} h(\lambda) \alpha^h_N(\lambda)\, d\lambda$$

and

$$G_t^h = \int_{\mathbb{R}}^{\oplus} e^{th(\lambda)} \alpha_N^h(\lambda)\, d\lambda,$$

where $\alpha_N^h(\lambda)$ is given in Equation (3.4). These formulations can be regarded as the diagonalizations of the operators Δ_L and G_t^h.

4. A stochastic process generated by the Lévy Laplacian

In this section, we will give a stochastic process generated by the Lévy Laplacian by considering the stochastic expression of the operator G_t^h.

Let $\{X_t^h; t \geq 0\}$ be a stochastic process with the characteristic function of X_t^h given by

$$E[e^{izX_t^h}] = e^{th(z)}, \quad h \in \mathcal{F}.$$

Take a smooth function $\eta_T \in E$ with $\eta_T = \frac{1}{\sqrt{|T|}}$ on T. Define an operator $\widetilde{G_t^h}$ acting on $S[\mathbf{E}_{-p,\infty}^h]$ by

$$\widetilde{G_t^h} = S G_t^h S^{-1}.$$

Here the space $S[\mathbf{E}_{-p,\infty}^h]$ is endowed with the topology induced from $\mathbf{E}_{-p,\infty}^h$ by the S-transform. Then by Theorem 3.4, $\{\widetilde{G_t^h};\, t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the operator $\widetilde{\Delta_L}$.

Let $\{\mathbb{X}_t^h; t \geq 0\}$ be an E-valued stochastic process defined by

$$\mathbb{X}_t^h = \xi + iX_t^h \eta_T, \qquad \xi \in E.$$

Then we have the following theorem.

Theorem 4.1. *Let $h \in \mathcal{F}$. Then for all $F \in S[\mathbf{E}_{-p,\infty}^h]$, the following equality holds*

$$\widetilde{G_t^h} F(\xi) = \mathrm{E}[F(\mathbb{X}_t^h) | \mathbb{X}_0^h = \xi].$$

Proof. First consider the case when $F \in S[\mathbf{E}_{-p,\infty}^h]$ is given by

$$F(\xi) = S\Big(\mathbf{J}_{\mathbf{a}}[f]\Big)(\xi) = \int_{T^n} f(u_1, \ldots, u_n) \prod_{\nu=1}^{n} e^{a_\nu \xi(u_\nu)}\, d\mathbf{u},$$

with $\sum_{\nu=1}^n a_\nu = \sqrt{|T|}\lambda, \sum_{\nu=1}^n a_\nu^2 = |T| h(\lambda)$. Then we have

$$\begin{aligned}
\mathrm{E}[F(\mathbb{X}_t^h) | \mathbb{X}_0^h = \xi] &= \mathrm{E}[F(\xi + iX_t^h \eta_T)] \\
&= \int_{T^n} f(u_1, \ldots, u_n) \prod_{n=1}^{n} e^{a_\nu \xi(u_\nu)} \mathrm{E}[e^{i\lambda X_t^h}]\, d\mathbf{u} \\
&= e^{th(\lambda)} F(\xi) \\
&= \widetilde{G_t^h} F(\xi).
\end{aligned}$$

Next, let $F \in S[\mathbf{E}^h_{-p,\infty}]$ be represented by $F = \int_{\mathbb{R}} F_\lambda \, d\lambda$ with F_λ being expressed in the following form:

$$F_\lambda(\xi) = \lim_{N\to\infty} \sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} S\Big(\mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}}]\Big)(\xi).$$

Hence we have

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{E}\Big[|F_\lambda(\xi + iX^h_t\eta_T)|\Big]\, d\lambda \\
&= \int_{\mathbb{R}} \mathrm{E}\left[\lim_{N\to\infty}\left|\sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} S\Big(\mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}}]\Big)(\xi + iX^h_t\eta_T)\right|\right] d\lambda \\
&= \int_{\mathbb{R}} \mathrm{E}\left[\lim_{N\to\infty}\left|\sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} S\Big(\mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}}]\Big)(\xi)\prod_{\nu=1}^{n} e^{ia^{[N]}_\nu X^h_t}\right|\right] d\lambda \\
&= \int_{\mathbb{R}} \lim_{N\to\infty}\left|\sum_{\mathbf{a}^{[N]}\in\mathcal{A}^h_\lambda} S\Big(\mathbf{J}_{\mathbf{a}^{[N]}}[f_{\mathbf{a}^{[N]}}]\Big)(\xi)\right| d\lambda \\
&= \int_{\mathbb{R}} |F_\lambda(\xi)|\, d\lambda.
\end{aligned}$$

Since $F_\lambda \in S[\mathbf{E}^h_{-p,\infty}]$, there exists some $\Phi_\lambda \in \mathbf{E}^h_{-p,\infty}$ such that $F_\lambda(\xi) = S[\Phi_\lambda](\xi) = \langle\langle \Phi_\lambda, \phi_\xi\rangle\rangle$ for any $\xi \in E$ and $\lambda \in \mathbb{R}$. By the Schwarz inequality, we see that

$$\begin{aligned}
\int_{\mathbb{R}} |F_\lambda(\xi)|\, d\lambda &\le \int_{\mathbb{R}} \|\Phi_\lambda\|_{-p}\, \|\phi_\xi\|_p\, d\lambda \\
&\le \left\{\int_{\mathbb{R}} \alpha^h_M(\lambda)^{-1}\, d\lambda\right\}^{1/2} \left\{\int_{\mathbb{R}} \|\Phi_\lambda\|^2_{-p}\alpha^h_M(\lambda)\, d\lambda\right\}^{1/2} \|\phi_\xi\|_p \\
&< \infty,
\end{aligned}$$

for all $\xi \in E$ and some $M \ge 1$, where $\alpha^h_N(\lambda)$ is given in Equation (3.4). Therefore by the continuity of $\widetilde{G^h_t}$ we get

$$\begin{aligned}
\mathrm{E}[F(\xi + iX^\gamma_t\eta_T)] &= \int_{\mathbb{R}} \mathrm{E}[F_\lambda(\xi + iX^h_t\eta_T)]\, d\lambda \\
&= \int_{\mathbb{R}} \widetilde{G^h_t}F_\lambda(\xi)\, d\lambda \\
&= \widetilde{G^h_t}F(\xi).
\end{aligned}$$

Thus we obtain the assertion. □

Theorem 4.1 implies that the infinite dimensional stochastic process $\{\mathbb{X}^h_t; t \ge 0\}$ is generated by $\widetilde{\Delta_L}$ defined on $\mathbf{E}^h_{-p,\infty}$ for each $h \in \mathcal{F}$.

The translation $x \mapsto x+\eta$, $x \in E^*$, can be lifted to the space of generalized functions $(E)^*$ whenever $\eta \in E$, i.e., $\tau_\eta\Phi(x) = \Phi(x+\eta)$ is defined for $\Phi \in (E)^*$. More precisely, a continuous linear operator τ_η from $(E)^*$ into itself is uniquely specified by $S[\tau_\eta\Phi](\xi) = S[\Phi](\xi + \eta)$,

$\xi \in E_{\mathbb{C}}$. Then, as is easily verified, $\tau_\eta \phi(x) = \phi(x+\eta)$ for $\phi \in (E)$, which gives a ground for the above formal notation. It is also known that $\tau_\eta \Phi = \phi_{-\eta} \diamond (\phi_\eta \Phi)$, where $\diamond$ is the Wick product, see e.g., [17]. Then, Theorem 4.1 is translated into the language of generalized white noise functionals.

Corollary 4.2. *Let $h \in \mathcal{F}$. Then for all $\Phi \in \mathbf{E}^h_{-p,\infty}$, the following equality holds*

$$G_t^h \Phi = \mathrm{E}[\tau_{iX_t^h \eta_T} \Phi].$$

By Corollary 4.2 we can see that $\{\tau_{iX_t^h \eta_T};\, t \geq 0\}$ is an operator-valued stochastic process and $\{E[\tau_{iX_t^h \eta_T}];\, t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by Δ_L defined on $\mathbf{E}^h_{-p,\infty}$ for each $h \in \mathcal{F}$.

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