Finite $p$-groups with two conjugacy length

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1 Introduction

Let $G$ be a finite group. We denote by $cd(G)$ and $ccl(G)$ the sets of numbers which occur as the degrees of irreducible characters of $G$ and as the lengths of conjugacy classes of $G$ respectively. Isaacs and Passman prove in [2] that the commutator subgroup of any finite group $G$ with $cd(G) = \{1, m\} \ (m > 1)$ is abelian. In this paper we will prove more strong analogous result for the set of conjugacy lengths. The class of group $G$ with $ccl(G) = \{1, m\} \ (m > 1)$ has been introduced by Ito in [4] and he has shown that the study of such groups is reduced to that of $p$-groups for some prime $p$. Moreover, a result of Isaacs [1] follows that the central factor of any group of this class is of exponent $p$. Therefore any 2-group of this class is of nilpotent class 2. We show the following theorem:

Main theorem Let $G$ be a finite $p$-group such that $ccl(G) = \{1, p^n\} \ (n \geq 1)$. Then the commutator subgroup of $G$ is an elementary abelian $p$-group.

The result of Heineken in [5] follows that the conclusion of main theorem is equivalent that the nilpotent class of any group of this class is at most 3.

2 Main theorem

Let $G$ be a finite group. We define $[x, y] := x^{-1}y^{-1}xy$ and $[x, y, z] := [[x, y], z]$ for all $x, y, z \in G$. The finite field of $p$ elements will be denoted by $\mathbb{F}_p$. The lower central series of $G$ will be denoted by $G_1 \leq G_2 \leq \ldots$, namely $G_1 := G, G_2 := [G, G]$ and $G_{i+1} := [G_i, G] \ (i \geq 2)$. If $c + 1$ is the least value of $m$ satisfying $G_m = 1$, then $c$ is called the nilpotent class of $H$. The nilpotency class of nilpotent group $H$ will be denoted by $c(H)$.

The following result of Isaacs gives some useful corollaries.

Theorem 2.1 (Isaacs [1]) Let $G$ be a finite group, which contains a proper normal subgroup $N$ such that all of the conjugacy classes of $G$ which lie outside of $N$ have the same lengths. Then either $G/N$ is cyclic, or else every nonidentity element of $G/N$ has prime order.
Corollary 2.2 Let $G$ be a finite $p$-group such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$. Then $G/Z(G)$ is of exponent $p$.

Note that $G_{i-1}Z(G)/G_iZ(G)$ is an elementary abelian $p$-group for $2 \leq i \leq c(G)$.

Corollary 2.3 Let $G$ be a finite 2-group such that $ccl(G) = \{1, 2^n\}$ $(n \geq 1)$. Then $G$ is of nilpotent class 2.

Corollary 2.4 (Verardi, Corollary 2.5[5]) Let $p$ be an odd prime. Let $G$ be a finite $p$-group such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$ and $c(G) \geq 3$. Then $G_{c(G)-1}$ is an elementary abelian $p$-group.

Theorem 2.5 (Heineken, Theorem 2.7[5]) Let $G$ be a finite $p$-group such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$. Then we have

$$\{x \in G; xZ(G) \in Z(G/Z(G))\} = C_G(D(G)) = \{x \in G; C_G(x) \triangleleft G\}.$$

Corollary 2.6 Let $G$ be a finite $p$-group of $ccl(G) = \{1, p^n\}$ $(n \geq 1)$. Suppose that $D(G)$ is abelian. Then $G$ is of nilpotent class at most 3.

Proof. Suppose that $D(G)$ is abelian. By Theorem 2.5, $D(G)Z(G)/Z(G) \leq C_G(D(G))Z(G)/Z(G) \leq Z(G/Z(G))$. Therefore $[D(G), G] \leq Z(G)$. The proof is completed.

Let $G$ be a finite $p$-group such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$. Then corollaries 2.4 and 2.6 say that the commutator subgroup of $G$ is abelian if and only if $c(G) \leq 3$.

Proof of main theorem. We will show that $c(G) \leq 3$. Suppose that $p$ is an odd prime and $c = c(G) \geq 3$. Then there exist $y \in G$ and $z \in G_{c-2}$ such that $[y, z] \in G_{c-1} - Z(G)$. Now we put $x_1 := [y, z]$, $p^m := |G_{c-1}Z(G)/Z(G)|$ and $G_{c-1}Z(G)/Z(G) = \langle x_1, x_2, \ldots, x_m \rangle Z(G)/Z(G)$. Note that $G_{c-1}Z(G)/Z(G)$ is an elementary abelian $p$-group. We put a coset decomposition of $G$ by $C_G(x_1)$ as the following:

$$G = \bigcup_{\alpha_i=0}^{p-1} u_1^{\alpha_1}u_2^{\alpha_2} \cdots u_n^{\alpha_n}C_G(x_1)$$

and $c_i := [x_1, u_i] \in Z(G)$ for $1 \leq i \leq n$. Since $G_{c-1}$ is a elementary abelian $p$-group, $|\langle c_1, \ldots, c_n \rangle| = p^n$. By Witt's identity, we have

$$[y, z, u_i][u_i, y, z][z, u_i, y] = 1.$$

Then $[y, z, u_i] = c_i$ and we can write

$$[z, u_i] = x_1^{\beta_{i1}}x_2^{\beta_{i2}} \cdots x_m^{\beta_{im}} w \in G_{c-1}.$$
for some $\beta_{ij} \in F_p (1 \leq i \leq n, 1 \leq j \leq m)$ and some $w \in Z(G)$. We put $d_i := [x_i, y]$, then

$$[z, u_i, y] = d_1^{\beta_{1i}} d_2^{\beta_{2i}} \cdots d_m^{\beta_{mi}}$$

for $1 \leq i \leq n$. Therefore we have

$$[u_i, y, z] = c_i^{-1} d_1^{-\beta_{i1}} \cdots d_m^{-\beta_{im}}$$

for $1 \leq i \leq n$.

We put $p^s = |\langle c_i^{-1} d_1^{-\beta_{i1}} \cdots d_m^{-\beta_{im}} ; 1 \leq i \leq n \rangle|$ and $p^t = |\langle d_1, d_2, \ldots, d_m \rangle|$. Then, by $\langle c_1, c_2, \ldots, c_n \rangle \subset \langle d_1, d_2, \ldots, d_m, c_i^{-1} d_1^{-\beta_{i1}} \cdots d_m^{-\beta_{im}} ; 1 \leq i \leq n \rangle$, we have $s + t \geq n$. We choose $I = \{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, n\}$ and $J = \{j_1, j_2, \ldots, j_t\} \subset \{1, 2, \ldots, m\}$ such that $\langle c_i^{-1} d_1^{-\beta_{i1}} \cdots d_m^{-\beta_{im}} ; i \in I \rangle = p^s$ and $\langle d_{j_1}, \ldots, d_{j_t} \rangle = p^t$ respectively.

Now we consider the subset

$$\bigcup_{\gamma = 0}^{p-1} G(y) C_G(y) u_{i_1}^{\gamma_1} u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s} x_{j_1}^{\gamma_{s+1}} x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}$$

of $G$. We claim that this sum is disjoint. We will show that if

$$[u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s} x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] = [u_{i_1}^{\epsilon_1} \cdots u_{i_s}^{\epsilon_s} x_{j_1}^{\epsilon_{s+1}} \cdots x_{j_t}^{\epsilon_{s+t}}, y],$$

then $\delta_i = \epsilon_i$ for $1 \leq i \leq s + t$. This equation is rewritten the following:

$$[u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s}, y] [x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] = [u_{i_1}^{\epsilon_1} \cdots u_{i_s}^{\epsilon_s}, y] [x_{j_1}^{\epsilon_{s+1}} \cdots x_{j_t}^{\epsilon_{s+t}}, y].$$

Then

$$[u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s}, y] [x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] G_3 = [u_{i_1}^{\epsilon_1} \cdots u_{i_s}^{\epsilon_s}, y] [x_{j_1}^{\epsilon_{s+1}} \cdots x_{j_t}^{\epsilon_{s+t}}, y] G_3$$

and

$$[u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s}, y] G_3 = [u_{i_1}^{\epsilon_1} \cdots u_{i_s}^{\epsilon_s}, y] G_3$$

$$[u_{i_1}^{\delta_1}, y] \cdots [u_{i_s}^{\delta_s}, y] G_3 = [u_{i_1}^{\epsilon_1}, y] \cdots [u_{i_s}^{\epsilon_s}, y] G_3$$

By $[G_3, z] = 1$, we have

$$1 = [[u_{i_1}, y]^{\epsilon_1-\delta_1} \cdots [u_{i_s}, y]^{\epsilon_s-\delta_s}, z]$$

and

$$= \prod_{k=1}^{s} (c_{i_k}^{-1} d_1^{-\beta_{i_k1}} d_2^{-\beta_{i_k2}} \cdots d_m^{-\beta_{i_km}})^{\epsilon_k-\delta_k}.$$

By the choice of $I$, we have $\delta_i = \epsilon_i$ for $1 \leq i \leq s$. So it is enough to verify

$$[x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] = [x_{j_1}^{\epsilon_{s+1}} \cdots x_{j_t}^{\epsilon_{s+t}}, y].$$
\[
[x_{j_{1}}^{\delta_{s+1}} \cdots x_{j_{t}}^{\delta_{s+t}}, y] = [x_{j_{1}}^{\epsilon_{s+1}} \cdots x_{j_{t}}^{\epsilon_{s+t}}, y] + t
\]
\[
[x_{j_{1}}, y]^{\delta_{s+1}} \cdots [x_{j_{t}}, y]^{\delta_{s+t}} = [x_{j_{1}}, y]^{\epsilon_{s+1}} \cdots [x_{j_{t}}, y]^{\epsilon_{s+t}}
\]
\[
d_{j_{1}}^{\delta_{s+1}} \cdots d_{j_{t}}^{\delta_{s+t}} = d_{j_{1}}^{\epsilon_{s+1}} \cdots d_{j_{t}}^{\epsilon_{s+t}}
\]
\[
d_{j_{1}}^{\delta_{s+1} - \epsilon_{s+1}} \cdots d_{j_{t}}^{\delta_{s+t} - \epsilon_{s+t}} = 1.
\]

Hence, by the choice of \( J \), we have \( \delta_i = \epsilon_i \) for \( s + 1 \leq i \leq s + t \). Therefore our claim now follows. Then, by \(|G : C_G(y)| = p^n\) and \( s + t \geq n \), we have \( s + t = n \). Therefore

\[
\bigcup_{\gamma_i = 0}^{p-1} C_G(y)u_{i_1}^{\gamma_1}u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s}x_{j_1}^{\gamma_{s+1}}x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}
\]

is a right coset decomposition of \( G \) by \( C_G(y) \). Then we consider the following:

\[
[u_{i_1}^{\gamma_1}u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s}x_{j_1}^{\gamma_{s+1}}x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}, y]
\]

for \( \gamma_i \in \mathbb{F}_p \) \((1 \leq i \leq s + t)\). By the choice of \( I \), if \((\gamma_1, \cdots, \gamma_s) \neq (0, \cdots, 0)\), then this is contained in \( G_2 - G_3 \), or else in \( Z(G) \). Hence \( \{(g, y); g \in G\} \subset (G_2 - G_3) \cup Z(G) \). Therefore, by \([y, z] = x_1 \in G_{c-1} - Z(G)\), we have \( c = 3 \). By Theorem 2.4, the proof is now completed.

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\underline{References}

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